# Empiricism, Probability, and Knowledge of Arithmetic

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The topic of this paper is the tenability of a certain type of empiricism about our knowledge of the Peano axioms. The Peano axioms constitute the standard contemporary axiomatization of arithmetic, and they consist of two parts, a set of eight axioms called *Robinson's Q*, which ensure the correctness of the addition and multiplication tables, and the principle of *mathematical induction*, which says that if zero has a given property and $n+1$ has it whenever $n$ has it, then all natural numbers have this property (cf. Hájek & Pudlák [HP98] p. 28, or Simpson [Sim09] p. 4). The type of empiricism about the Peano axioms which I want to consider holds that arithmetical knowledge is akin to the knowledge by which we infer from the past to the future, or from the observed to the unobserved. It is not uncommon today to hold that such inductive inferences can be rationally sustained by appeal to informed judgments of probability. The goal of this paper is to evaluate an empiricism which contends that judgements of probability can help us to secure knowledge of the Peano axioms.

This empiricism merits our attention primarily because standard accounts of our knowledge of the Peano axioms face difficult problems, problems going above and beyond skepticism about knowledge of abstract objects. Logicism, for example, suggests that knowledge of the Peano axioms may be based on knowledge of ostensibly logical principles– such as Hume's Principle– and the knowledge that the Peano axioms are representable within these logical principles (see Wright [Wri83] p. xiv, p. 131). The success of logicism thus hinges upon identifying a concept of representation which can sustain this inference, and as I have argued elsewhere, it seems that we presently possess no such concept. Alternatively, some structuralists have suggested that knowledge of the Peano axioms may be based on our knowledge of the class of finite structures. However, this account then owes us an explanation of why the analogues of the Peano axioms hold on the class of finite structures: why, for example, there is no finite structure which is larger than all the other finite structures (see Shapiro [Sha00] p. 112, MacBride [Mac08] p. 159).

The second reason that this kind of empiricism about the Peano axioms merits our attention is that it has been suggested in different ways by both historical and contemporary sources. For instance, prior to Frege, a not uncommon view seems to have been that mathematical induction was an empirical truth akin to enumerative induction. This is why Kästner thought that mathematical induction was not fit to be an axiom ([Käs90] pp. 426-428), and this is part of the background to Reid's begrudging concession that "necessary truths may sometimes have probable evidence" ([Rei85] VII.ii.1). However, some contemporary authors writing on the epistemology of arithmetic and arithmetical cognition have also suggested views related to this. For instance, Rips and Asmuth– two cognitive scientists who work on mathematical cognition– have recently considered the suggestion that "the theoretical distinction between math[ematical] induction and empirical induction" is not as clear as has been claimed, and that "even if the theoretical difference were secure, it wouldn't follow that the psychological counterparts of these operations are distinct" ([RA07] p. 205). Finally, in the course of their work on the epistemic propriety of randomized algorithms, Gaifman and Easwaran have both suggested the possibility of extending the notion of probability which they employ to broader issues in the epistemology of arithmetic ([Gai04] pp. 107-108, [Eas09] p. 347).

I present a preliminary defense of empiricism about arithmetical knowledge by responding to several objections to the importation of probability into the epistemology of mathematics. In the philosophy of arithmetic, one resilient topic has been the status of the $\omega$-rule (cf. Isaacson [Isa92] § III). This is a proof-theoretic rule which licenses the inference to a universal arithmetic hypothesis on the basis of the (infinite) set of premises expressing that the hypothesis holds of each specific natural number. One probabilistic analogue of the $\omega$-rule would read as follows, wherein $P$ is the probability assignment and $S^n(0)$ is the term in the formal language of arithmetic for the $n$-th natural number:

$$P(\forall\, x\; \varphi(x)) = \lim_N P(\bigwedge_{n=1}^{N} \varphi(S^n(0))) \tag{P$\omega$}$$

However, an advocate of the type of empiricism countenanced here *must* articulate principled reasons for rejecting P$\omega$. For, it is not difficult to see that P$\omega$ forces the alignment of the true and the highly probable, at least if the axioms of Robinson's Q are given high probability. In the paper, I suggest that the advocate of empiricism proceed by distinguishing between P$\omega$ and another version of the $\omega$-rule, which I call P$\omega_1$, based on the infinitary language $L\omega_1\omega$ (see Keisler [Kei71], Nadel [Nad85]). This distinction is relevant for two reasons. First, one can show that P$\omega_1$ does not force the alignment of the the true and the highly probable. Second, one can show that P$\omega$ does not follow from Dutch book arguments, whereas P$\omega_1$ does follow from Dutch book arguments (see Williamson [Wil99] pp. 411-412).

Another objection at the intersection of logic and probability concerns the computational features of the probability assignments. Any probability assignment which gives the axioms of Robinson's $Q$ a non-zero probability computes a complete consistent extension of Robinson's $Q$. This thus raises the specter of a deep philosophical objection to the type of empiricism developed here, since it suggests that arithmetical probability is no more accessible to us than arithmetical truth itself. In response to this, I suggest adverting to the notion of an atomless probability assignment which is computationally tractable. A probability assignment is atomless if there is no complete consistent extension $T$ of Robinson's $Q$ such that $\widehat{P}(\{T\}) > 0$, where $\widehat{P}$ is the natural extension of $P$ to the Polish space of complete consistent extensions of Robinson's $Q$. By a forcing argument, one can show that there are atomless probability assignments which are computationally tractable in that they are the computable limit of a computable function (cf. Limit Lemma in Soare [Soa87] p. 57, Kelly [KS97] § 3, [Kel00] § 3). I thus suggest that the advocate of empiricism maintain that the probability assignments in question are from a probability assignment of this type: for, since they are computationally tractable they are presumably accessible to us, and since they are atomless they do not "hone in" on any one particular complete consistent extension of Robinson's $Q$. Obviously this response raises many further questions. For instance, a side-effect of computational tractability is arithmetical definability, and accordingly there are liar-like sentences in the language, i.e. sentences $\lambda$ such that $Q \vdash [(\lambda \leftrightarrow (P(\lambda) = 0)]$. If $\neg\lambda$ is true on the standard model and $P(\lambda) < 1$, then this has the consequence that $P(\lambda | P(\lambda) = 0) = 1 > P(\lambda)$, which seems to undermine the idea that confirmation is always a source of justification in this setting. In response to this, I examine the circumstances under which $\lambda$ can be true on the standard model of arithmetic. The mathematical tools here in question are very similar to the methods we use to show that the sentence obtained from diagonalizing against provability is true on the standard model of arithmetic.

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