# Quantification over events in probability logic and its applications to elementary analysis

Stanislav O. Speranski

Probably the most popular formal language in the philosophy of mathematics is that of the second order arithmetic: in particular, employing certain coding techniques, many classical objects of analysis (and topology) can be defined in its terms (e. g., see [3]). Due to these reasons, the second order theory of the standard model of natural numbers $\langle \mathbb{N}; +, \times \rangle$ is often referred to as *elementary analysis*. Also, this language and its prefix fragments play an important role in describing computational complexity in both computability theory and theoretical computer science. We denote the computational complexity of elementary analysis by $\Pi^1_\infty$ (expanding the classical notation for complexity of its fragments, namely $\Pi^1_0, \Pi^1_1, \Pi^1_2, \ldots$, that is, the *analytical hierarchy*—cf. [2]).

However, as was mentioned above, coding is required to express notions of analysis (or topology) in the language of the second order arithmetic, which is clearly not designed to speak about them directly; in effect, this language seems to be not as natural from the viewpoint of actual practice in number theory as its first order counterpart, though certainly very attractive in providing foundations for mathematics (e. g., it's often more convinient to work within this language rather than in that of axiomatic set theory, where a lot of coding is needed to represent $\mathbb{N}$ already).

Thus it is interesting to search for various languages that enable us to reason directly about objects of analysis (or topology) and have virtually the same complexity as elementary analysis itself. To be more precise, since probability plays one of the leading roles in the philosophy of science (not only in that of mathematics), we are aiming to find such languages among those for reasoning about probabilities (after A. N. Kolmogorov, probability theory is viewed as a branch of functional analysis), meeting the following substantial conditions:

- the quantifier-free fragment of the language is simple enough from the viepoint of computability theory, namely the validity problem for that fragment is algorithmically decidable;
- only two quantifiers, $\forall$ and $\exists$, are available in the language and these range over the same sort of objects (this allows us to introduce prefix classification in a standard way and avoid going deep into analysis of relationships between different sorts of objects);
- no quantifiers may occur in the scope of the probability sign, i. e., probability is distibuted over events that are defined by means of quantifier-free expressions;
- the above quantification must be intuitively appealing from the viewpoint of probability theory (and statistics), and the syntax and semantics of the language should be easily described.

We provide a natural example of a language of this kind, consider some modifications of it, and discuss its connections to other probabilistic formalisms. Let us briefly describe the language itself.

Assume $\mathscr{X} := \{x_i\}_{i=1}^\infty$ is the set of *variables*. The collection of *e-terms* is defined inductively as the smallest set $Term_e$ containing $\mathscr{X}$ and such that if $t_1$ and $t_2$ are *e*-terms, then $\overline{t_1}$ and $t_1 \cap t_2$ are *e*-terms as well. Next, by a $\mathscr{L}_\mu$*-atom* we mean an expression of the sort

$$f\,(\mu\,(t_1), \ldots, \mu\,(t_n)) \;\leqslant\; g\,(\mu\,(t_{n+1}), \ldots, \mu\,(t_{n+m}))\,,$$

where $f$ and $g$ are polynomials with rational coefficients, and $\{t_1, \ldots, t_{n+m}\} \subset Term_e$. Finally, $\mathscr{L}_\mu$*-formulas* are obtained from $\mathscr{L}_\mu$-atoms by closing under $\neg$, $\wedge$ and applications of $\forall x$, with $x \in \mathscr{X}$. Let

$$\Phi_1 \vee \Phi_2 \;:=\; \neg\,(\neg\Phi_1 \wedge \neg\Phi_2)\,, \quad \Phi_1 \to \Phi_2 \;:=\; \neg\Phi_1 \vee \Phi_2 \quad \text{and} \quad \exists x\,\Phi_1 := \neg\forall x\,\neg\Phi_1\,.$$

Suppose that $\mathfrak{A} := \langle \Omega, \mathscr{A}, \mathsf{P} \rangle$ is a discrete probability space (i. e., $\mathsf{P}$ is a discrete measure over a $\sigma$-algebra $\mathscr{A}$ of subsets of a non-empty $\Omega$). Given a $\mathscr{L}_\mu$-formula $\Phi$ and a valuation $v : \mathscr{X} \to \mathscr{A}$, we define $\mathfrak{A} \Vdash \Phi[v]$ recursively (in a first-order fashion) by interpreting $\overline{t_1}$ as the complement of $t_1$, $t_1 \cap t_2$ as the intersection of $t_1$ and $t_2$, and $\mu$ as $\mathsf{P}$, while quantifiers are viewed as ranging over all the events from $\mathscr{A}$. The language $\mathscr{L}_\mu$ is referred to as *probability logic with quantifiers over events*.

The quantifier-free fragment of $\mathscr{L}_\mu$ is, essentially, a variant of a standard probabilistic language from [1, Section 5], for which the validity is known to be decidable (cf. [1]). In addition, $\mathscr{L}_\mu$ is closely connected to the language $\mathfrak{QPL}$ suggested in [4] (which is, in turn, closely related to formalisms previously introduced by H. J. Keisler and D. N. Hoover, J. Paris, etc—cf. [5, Section 2.3]). In fact, quantification over events naturally corresponds to quantification over Bernoulli random variables (i. e., characteristic functions of the foregoing events), whence these quantifiers are very appealing from the viewpoint of probability theory and statistics (see [7] for a bunch of theorems of probability theory that involve quantification over Bernoulli random variables).

We prove that the computational complexity of the validity problem for $\mathscr{L}_\mu$ is $\Pi^1_\infty$ (using alternative characterisation of the analytical hierarchy obtained in [6]). Moreover, this result can be carried over to some extentions and reducts of $\mathscr{L}_\mu$: e. g., adding quantifiers over the reals to $\mathscr{L}_\mu$ does not effect the complexity, as well as removing $+$ from $\mathscr{L}_\mu$.

In some cases, it is convinient to enrich $\mathscr{L}_\mu$ with a set of constants $C$ (obtaining the language $\mathscr{L}^C_\mu$) —that allows us to transfer the (un-)decidability results from [4, 5], concerning the validity for prefix fragments (i. e., for $\forall$-sentences, $\exists$-sentences, $\forall\exists$-sentences, $\exists\forall$-sentences, and so on):

- for any computable $C$, the validity problem for $\forall\exists$-$\mathscr{L}^C_\mu$-sentences is decidable;
- for any non-empty $C$, the validity problem for $\exists\forall$-$\mathscr{L}^C_\mu$-sentences is undecidable.

Notice that, though $\mathscr{L}_\mu$ and $\mathfrak{QPL}$ are closely connected, the (general) validity for the latter is only $\Pi^1_1$; therefore, we discuss what causes such a huge difference between the two.

## References


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*Novosibirsk State University, Novosibirsk, Russia*
*E-mail address:* `katze.tail@gmail.com`