PROSPECTS FOR A THEORY OF NON-ARCHIMEDEAN EXPECTED UTILITY: IMPOSSIBILITIES AND POSSIBILITIES

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EXTENDED ABSTRACT

In this talk I examine the prospects for a theory of probability aspiring to support a decision-theoretic interpretation by which principles of probability derive their normative status in virtue of their relationship with principles of rational decision making. More specifically, I investigate the possibility of a theory of expected utility abiding by distinguished *dominance principles* which have played a pivotal role in the development of subjective probability and critical discussion thereof. I focus on dominance principles that a theory of real-valued expected utility cannot support, motivating proponents of these principles to develop theories admitting a non-Archimedean range to meet the demands which these principles exact. Thus, specifically, in this talk I examine the prospects for a theory of non-Archimedean expected value and more generally, expected utility.

I am certainly not the first to entertain the non-Archimedean possibility. Non-Archimedean representations of uncertainty have captivated the interest of those who wish for rational credal probabilities and expectations either to respect laws supplementing the familiar set of putatively binding laws or to accommodate credal states complementing the familiar set of putatively rational credal states. In other words, at least two distinct considerations have either motivated authors to appeal to a non-Archimedean representation or stirred their excitement about the possibilities such a representation creates. One motivation seeks conformity to a longer list of laws governing credal expressions of uncertainty within a framework facilitating compliance with these laws. A second motivation seeks conformity to the familiar laws possibly along with others within a framework capable of registering putatively rational credal expressions of uncertainty.

For example, some authors, like Shimony [1955], Kemeny [1955], Stalnaker [1970], Carnap [1971], and Skyrms [1980, 1995], have motivated a need to admit a non-Archimedean range to comply with a principle flowing from a decision-theoretic consideration: If credal probabilities and expectations are to track an agent's preferences over gambles, then a gamble the agent evaluates across states as sometimes worse and never better than another gamble ought to receive strictly lower expected value than the other gamble, a quantitative manifestation of an injunction called the *principle of weak dominance*. Consequently, where random quantities are understood as gambles with which the agent may be faced in a decision problem, these authors have in particular demanded that each event an agent judges possible receive positive probability, a constraint called *regularity*. A well-known problem, however, is that an account of probability confined to real numbers cannot meet such a requirement without imposing substantive constraints on admissible credal judgments. To make room for regular probabilities, these authors have suggested that credal probabilities and expectations be permitted to take values in some non-Archimedean extension of the system of real numbers.

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To take another example, de Finetti [1931, 1974b] has motivated a desire to welcome a non-Archimedean range in light of simple measurement-theoretic considerations of comparative credal judgments, de Finetti being among the first to have observed that a representation admitting a non-Archimedean range can countenance a richer set of putatively rational comparisons among random quantities. For example, an agent who judges that a coin is more likely to land heads than tails by a nonzero but minuscule amount is predisposed to exhibit a set of comparisons which cannot be indulged by real-valued expectations. Additionally, while very little came of it, on several occasions de Finetti has suggested introducing infinitesimal numbers not only as a way to pursue a fine-grained analysis of zero probabilities, but also as a way to ensure conformity to decision-theoretic constraints including both the principle of weak dominance and a somewhat less demanding condition called the *principle of simple dominance*, which formulated in this context requires that a gamble an agent evaluates across states as always worse than another gamble receive strictly lower expected value than the other gamble [de Finetti, 1936, 1974a,b, de Finetti et al., 2008].

Enjoying broad expressive power, it is perhaps unsurprising that non-Archimedean representations have found much use in many areas of inquiry, with applications abound not only in philosophy but also in philosophical logic, mathematical statistics, game theory and decision theory, among other areas. To be sure, philosophers have called for a non-Archimedean change, championing a non-Archimedean range for probability, as discussed by, for example, Lewis [1986, 1996], Swineburne [2001], Holder [2002], Herzberg [2007], and Norton [2007], in addition to the aforementioned authors. Lewis and Skyrms, avid supporters of regularity, write approvingly of the mathematical developments due to Berstein and Wattenberg [1969], suggesting that a satisfactory theory of probability would be an outgrowth of their advances. Similarly, in a series of individually and jointly authored articles, Benci et al. advance a theory of probability also enforcing regularity while conceding the Archimedean property [Wenmackers and Horsten, 2010, Benci et al., 2012a,b, Wenmackers, 2012].

In other areas such as philosophical logic, authors have employed non-Archimedean representations in their studies of conditionals and belief revision (e.g., [Lehmann and Magidor, 1992]), while in game theory, authors have appealed to non-Archimedean representations in characterizations of equilibrium refinements (e.g., [Blume et al., 1991]). In mathematical statistics, authors have studied independence within non-Archimedean representations (e.g., [Cozman and Seidenfeld, 2007]). Other applications arise in mathematical economics and physics, to include but two more.

In light of several successful efforts to develop a theory of probability countenancing a non-Archimedean range, it is natural to inquire about whether non-Archimedean expected value, and more generally expected utility, has seen similar encouraging successes. After all, probability contributes to a more comprehensive effort to estimate the values of quantities of interest, whether an interest serves decision making, statistical inference, or mathematical modeling of real-world phenomena. Despite clear reasons for inquiring about the prospects for a theory of non-Archimedean expected value, surprisingly little has been said about the outlook for articulating let alone securing foundations for a theory in which non-Archimedean probabilities and expectations combine to afford an agent the means to express his or her opinions in the face of uncertainty.

In a step toward rectifying this predicament, I introduce a mathematical account of expectation based on a *qualitative criterion of coherence* for *qualitative comparisons* of random quantities. This criterion is reminiscent of de Finetti's quantitative criterion of coherence, yet, as I explain, it does not impose an Archimedean condition on qualitative comparisons. Moreover, unlike de Finetti's criterion of coherence, the qualitative criterion respects a qualitative version of the principle of weak dominance. Just as de Finetti formulated his criterion of coherence for an arbitrary collection of bounded random quantities, free of measurability and closure conditions, I formulate the qualitative criterion of coherence for an arbitrary collection of random quantities. In fact, I do not even require that the quantities be *bounded*, the dispensation of which thereby permits the collection to be truly *any* collection of random quantities of interest. Furthermore, in contrast with many other developments of comparative probability, the qualitative criterion of coherence does not require that the binary relation reflecting an agent's judgments of comparative expectations satisfy *reflexivity, completeness*, or even *transitivity*.

In spite of these weak structural assumptions, I describe a theorem asserting that any coherent system of comparative expectations can be extended to a weakly ordered coherent system of comparative expectations over any collection of random quantities containing the initial collection of random quantities of interest. In addition to the qualitative version of the principle of weak dominance, the extended weakly ordered coherent system of comparative expectations satisfies familiar additivity (independence) and homogeneity postulates when the extended collection forms a linear space.

Next, I describe a numerical representation theorem. I explain how any weakly ordered coherent system of comparative expectations over a linear space including a constant quantity can be represented by a normalized, linear function taking values in a (possibly non-Archimedean) ordered field extension of the system of real numbers while respecting the numerical principle of weak dominance.

Rather than employing an ultraproduct construction to obtain the ordered field extension, my proof of the representation theorem uses a version of the *Hahn Embedding Theorem* to ultimately construct the desired ordered field in which the expectation function is to take values. This route affords an expression of numerical values in terms of *formal power series* in a single infinitesimal ϵ and enables infinitely large numerical differences in expectations of random quantities to be explicitly *traced* to infinitely large qualitative differences in comparisons of these quantities since the ordered collection of the powers of ϵ belonging to the image of the expectation function is order-isomorphic to the ordered collection of Archimedean equivalence classes of random quantities from the system of comparative expectations. Thus, the route by way of the Hahn Embedding Theorem usefully locates the origins of the numbers of interest.

Of course, the foregoing results apply to the special case of qualitative comparative probability. In addition, the associated technical developments have bearing on interesting results due to Dubins [1977] concerning the extendability of real-valued finite order-preserving linear functionals. In the remaining time, I explain how these technical developments also contain key ingredients of a full account of non-Archimedean expected utility in the style of Anscombe and Aumann [1963]. Time permitting, I explain how a general impossibility theorem applying to any theory of non-Archimedean probability and expectation circumscribes the foregoing technical developments. The impossibility theorem shows that a few simple, minimal desiderata for such a theory cannot be jointly satisfied, accordingly placing *a priori* limitations on any theory admitting a non-Archimedean range.

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