References on Bifurcation Theory

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1 Static and dynamic bifurcation

Krasnosel'skii [33] contains prototypes of various bifurcation theorems that we discuss in the lectures. General references about (static) Bifurcation Theory include Ize [28, 29], Chow–Hale [11], Deimling [15], Zeidler [50], Smoller [47], Ambrosetti–Prodi [5], Nirenberg [39], Dancer [14], Buffoni–Toland [8], Ma–Wang [37], Hutson–Pym–Cloud [27], Ambrosetti–Malchiodi [4], Ambrosetti–Arcoya [3], Kielhöfer [31], Drábek–Milota [19], Brown [7].

2 Perturbation of linear operators

A thorough understanding of perturbative properties of linear operators is inevitable to attack bifurcation problems.

General references on linear functional analysis include Reed–Simon [43], Kato [30], Yoshida [49], Dieudonné [17], Dunford–Schwartz [20–22], Pedersen [41], Kreyszig [34], Conway [12], Rudin [44], Lang [35], Douglas [18], Abramovich–Aliprantis [1]. For a more concise treatment, see e.g. Carothers [9], Evans [23, Appendix D], Krantz [32]. Hirzebruch–Scharlau [26], Werner [48] are written in German. Helemskii [25] puts emphasis on a categorical aspect.

3 The principle of linearization

For calculus in Banach spaces, see e.g. Cartan [10], Schwartz [45, Chapter I], Dieudonné [16, Chapter VIII], Lang [35, Chapter XIII], Ambrosetti–Prodi [5, §1], Buffoni–Toland [8, Chapter 3], Ambrosetti–Malchiodi [4, §1].

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4 Lyapunov–Schmidt reduction

For Lyapuov–Schmidt reduction of operators whose linearization is Fredholm, see Ambrosetti–Malchiodi [4, §2.2].

5 Brouwer degree

An axiomatic characterization of the Brouwer degree appears in Amann–Weiss [2] at the latest. It might be amusing to read Siegberg [46] for the history of this notion. We mainly follow the treatment of Deimling [15, Chapter 1]. See also Smoller [47, §12.A], Dancer [14, Lectures 1, 2], Nirenberg [39, Chapter 1], O'Regan–Cho–Chen [40, Chapter 1], Drábek–Milota [19, §5.7]. Milnor [38] discusses Brouwer degree for maps of finite-dimensional manifolds. For an exposition of the Brouwer degree in terms of homology, see e.g. Hatcher [24, pp. 134–137], Brown [7, §§8–9].

6 Leray–Schauder degree

The original reference of this notion seems to be Leray–Schauder [36]. We follow Nirenberg [39, Chapter 2] and Deimling [15, Chapter 2]. See also Smoller [47, §12.B], Dancer [14, Lecture 3], O'Regan–Cho–Chen [40, Chapter 2], Drábek–Milota [19, §5.8], Brown [7, §§10–11].

7 Global bifurcation theorem of Rabinowitz

We present Ize's elegant proof (cf. Nirenberg [39, §3.4], Ize [28, 29]) of the celebrated global bifurcation theorem due originally to Rabinowitz [42]. For proofs similar to the original one, see e.g. Deimling [15, §29], Ambrosetti–Malchiodi [4, §4], Ambrosetti–Arcoya [3, §6.3], Brown [7, Chapter 22].

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A Closedness of the range

Definition A.1. A bounded linear map $T : X \to Y$ is called a Fredholm operator if $N(T)$ is of finite dimension and $R(T)$ is of finite codimension. For a Fredholm operator T, define its nullity $n(T) := \dim N(T)$, defect $d(T) := \dim (Y/R(T))$, and Fredholm index $i(T) := n(T) - d(T)$.

The following proposition implies in particular that the range of a Fredholm operator is closed.

Proposition A.2. *Let $T : X \to Y$ be a bounded linear map of Banach spaces. If $R(T)$ has finite codimension, then $R(T)$ is closed.*

Proof. Take a finite-dimensional linear subspace of Y such that $R(T) + V = Y$ and $R(T) \cap V = \{0\}$. Define the bounded linear map $S : X \oplus V \to Y$ by $S(x \oplus v) = T(x) + v$ for all $x \in X$, $v \in V$. Observe that S is bijective, where injectivity of S follows from that of T. By the bounded inverse theorem, S is a closed map. Hence $R(T) = S(X \oplus \{0\})$, being the image of a closed set, is closed. □

Remark A.3. Let V be a linear subspace of a Banach space. If $\dim V < \infty$, then V is closed. In contrast, $\operatorname{codim} V < \infty$ does not necessarily imply closedness of V.

For closedness of range, it might be amusing to take a look at Atiyah [6, pp. 153–154], Kato [30, p. 230]. The proof here is similar to Abramovich–Aliprantis [1, p. 156] but is somewhat simplified.

B The toy case

Let $\Lambda_\star \subset \mathbb{R}$ be an open interval, $I \subset \mathbb{R}$ an open neighborhood of 0, and

$$\begin{aligned} \beta : I \times \Lambda_\star &\to \mathbb{R} \\ (s, \lambda) &\mapsto \beta(s, \lambda) \end{aligned}$$

a C^{k+2} map ($k \geq 0$) such that

$$\beta(0, \lambda) = 0 \tag{B.1}$$

for every $\lambda \in \Lambda_\star$.

Proposition B.1. *If*

$$\frac{\partial \beta}{\partial s}(0, \lambda_\star) = 0, \tag{B.2}$$

$$\frac{\partial^2 \beta}{\partial s \partial \lambda}(0, \lambda_\star) \neq 0, \tag{B.3}$$

then $(0, \lambda_\star)$ *is a bifurcation point for the equation* $\beta(s, \lambda) = 0$ *along the trivial branch* $\{(0, \lambda) \mid \lambda \in \Lambda_\star\}$.

Remark B.2. In the proof, we actually show that the set of all solutions to the equation $\beta(s, \lambda) = 0$ is, locally near $(s, \lambda) = (0, \lambda_\star)$, the union of two C^{k+1} curves intersecting transversely at $(0, \lambda_\star)$; see (B.8). The assumption (B.3) is sometimes called the Crandall–Rabinowitz transversality condition [13].

Proof. Define a C^{k+1} map[1] $h : I \times \Lambda_\star \to \mathbb{R}$ by

$$h(s, \lambda) = \int_0^1 \frac{\partial \beta}{\partial s}(st, \lambda) dt. \tag{B.4}$$

Integrating the equation

$$\frac{\partial}{\partial t}\beta(st, \lambda) = s\frac{\partial \beta}{\partial s}(st, \lambda),$$

we obtain

$$\int_0^1 \frac{\partial}{\partial t}\beta(st, \lambda) dt = s\int_0^1 \frac{\partial \beta}{\partial s}(st, \lambda) dt = sh(s, \lambda).$$

Since

$$\int_0^1 \frac{\partial}{\partial t}\beta(st, \lambda) dt = \beta(s, \lambda) - \beta(0, \lambda) = \beta(s, \lambda)$$

by the fundamental theorem of calculus and (B.1), we conclude

$$h(s, \lambda) = \frac{\beta(s, \lambda)}{s} \tag{B.5}$$

if $s \neq 0$, while

$$h(0, \lambda) = \int_0^1 \frac{\partial \beta}{\partial s}(0, \lambda) dt = \frac{\partial \beta}{\partial s}(0, \lambda)$$

by (B.4). In particular,

$$h(0, \lambda_\star) = \frac{\partial \beta}{\partial s}(0, \lambda_\star) = 0 \tag{B.6}$$

[1] We lose one degree of differentiability here.

by (B.2). Also, again by (B.4),

$$\frac{\partial h}{\partial \lambda}(s,\lambda) = \frac{\partial}{\partial \lambda}\int_0^1 \frac{\partial \beta}{\partial s}(st,\lambda)dt = \int_0^1 \frac{\partial^2 \beta}{\partial s \partial \lambda}(st,\lambda)dt,$$
$$\frac{\partial h}{\partial \lambda}(0,\lambda) = \frac{\partial^2 \beta}{\partial s \partial \lambda}(0,\lambda).$$

In particular,

$$\frac{\partial h}{\partial \lambda}(0,\lambda_\star) = \frac{\partial^2 \beta}{\partial s \partial \lambda}(0,\lambda_\star) \neq 0 \tag{B.7}$$

by (B.3). Thanks to (B.6) and (B.7), we may uniquely solve the equation $h(s,\lambda) = 0$ near $(s,\lambda) = (0,\lambda_\star)$. That is, by the implicit function theorem, there exists a C^{k+1} function $\varphi : J \to \Lambda_{\star\star}$, where $J \subset I$ is an open neighborhood of $0 \in \mathbb{R}$ and $\Lambda_{\star\star} \subset \Lambda_\star$ is an open interval containing $\lambda_\star$, such that $\varphi(0) = \lambda_\star$ and

$$\{(s,\lambda) \in J \times \Lambda_{\star\star} \mid h(s,\lambda) = 0\} = \{(s,\varphi(s)) \mid s \in J\}.$$

We observe from (B.5) that, for $s \in I \setminus \{0\}$ and $\lambda \in \Lambda_\star$, $h(s,\lambda) = 0$ if and only if $\beta(s,\lambda) = 0$, whence

$$\{(s,\lambda) \in I \times \Lambda_\star \mid h(s,\lambda) = 0,\ s \neq 0\} = \{(s,\lambda) \in I \times \Lambda_\star \mid \beta(s,\lambda) = 0,\ s \neq 0\}.$$

Therefore,

$$\{(s,\lambda) \in J \times \Lambda_{\star\star} \mid \beta(s,\lambda) = 0\} = \{(0,\lambda) \mid \lambda \in \Lambda_{\star\star}\} \cup \{(s,\varphi(s)) \mid s \in J\}, \tag{B.8}$$

where $\{(0,\lambda) \mid \lambda \in \Lambda_{\star\star}\} \cap \{(s,\varphi(s)) \mid s \in J\} = \{(0,\lambda_\star)\}$. In particular, $(0,\lambda_\star)$ is a bifurcation point. □

Remark B.3. Implicit differentiation gives

$$\frac{d\varphi}{ds}(0) = -\frac{1}{2}\frac{\partial^2 \beta}{\partial s^2}(0,\lambda_\star) \Big/ \frac{\partial^2 \beta}{\partial s \partial \lambda}(0,\lambda_\star).$$

Indeed,

$$0 = h(s,\varphi(s)),$$
$$0 = \frac{\partial h}{\partial s}(s,\varphi(s)) + \frac{\partial h}{\partial \lambda}(s,\varphi(s))\frac{d\varphi}{ds}(s),$$
$$\frac{d\varphi}{ds}(0) = -\frac{\frac{\partial h}{\partial s}(0,\lambda_\star)}{\frac{\partial h}{\partial \lambda}(0,\lambda_\star)} = -\frac{\frac{1}{2}\frac{\partial^2 \beta}{\partial s^2}(0,\lambda_\star)}{\frac{\partial^2 \beta}{\partial s \partial \lambda}(0,\lambda_\star)}.$$

Here, the last equality follows from (B.7) for the denominator and

$$\frac{\partial h}{\partial s}(s,\lambda) = \frac{\partial}{\partial s}\int_0^1 \frac{\partial \beta}{\partial s}(st,\lambda)dt = \int_0^1 \frac{\partial^2 \beta}{\partial s^2}(st,\lambda)tdt,$$
$$\frac{\partial h}{\partial s}(0,\lambda) = \frac{\partial^2 \beta}{\partial s^2}(0,\lambda)\int_0^1 tdt = \frac{1}{2}\frac{\partial^2 \beta}{\partial s^2}(0,\lambda)$$

for the numerator.

The bifurcation in Proposition B.1 is said to be transcritical if

$$\frac{\partial^2 \beta}{\partial s^2}(0,\lambda_\star) \neq 0,$$

in which case the equation $\beta(s,\lambda) = 0$ has a nontrivial solution for every $\lambda \in \Lambda_\star \setminus \{\lambda_\star\}$ close to $\lambda_\star$.