Universität Regensburg, Mathematik SS 2011
Dr. David Gepner 1.6.2011

Introduction to Stable Homotopy Theory
4. Übungsblatt

Aufgabe 1
Let $\{A_n\}$ be a tower (a diagram of the form

$$\cdots \longrightarrow A_{n+1} \xrightarrow{\alpha_n} A_n \xrightarrow{\alpha_{n-1}} A_{n-1} \longrightarrow \cdots \longrightarrow A_0,$$

in other words) of abelian groups. Recall that we have an exact sequence

$$0 \longrightarrow \lim_n A_n \longrightarrow \prod_n A_n \xrightarrow{\text{id}-\text{shift}} \prod_n A_n \longrightarrow \lim^1 A_n \longrightarrow 0.$$

Define $\lim^0 A_n = \lim A_n$ and $\lim^i A_n = 0$ for $i \neq 0, 1$. Show that the category of towers of abelian groups (and compatible group homomorphisms between them) is an abelian category with enough injectives, and that

$$\lim^i A_n \cong \operatorname{Ext}^i(\mathbb{Z}, \{A_n\}),$$

where $\mathbb{Z}$ denotes the constant tower (in which all objects are $\mathbb{Z}$ and all maps are the identity).

Aufgabe 2
Given a tower of abelian groups $\{A_n\}$, then, for a given n and all $m \geq n$, we have subobjects A_n^m of A_n defined by

$$A_n^m = \operatorname{Im}(A_m \xrightarrow{\alpha_{m-1}} \cdots \xrightarrow{\alpha_n} A_n),$$

the image of A_m in A_n. A tower of abelian groups $\{A_n\}$ is said to be *Mittag-Leffler* if, for all n, there exists an $m \geq n$ such that, for $l \geq m$, $A_n^l = A_n^m$. Show that if $\{A_n\}$ satisfies the Mittag-Leffler condition then $\lim_n^1 A_n = 0$.

Aufgabe 3
Let $\mathbb{CP}^n$ denote the n-dimensional complex projective space. Show that, for all integers $n > 0$, there exists a cofibration sequence of pointed spaces

$$S^{2n-1} \to \mathbb{CP}^{n-1} \to \mathbb{CP}^n.$$

[Hint: $\mathbb{CP}^n$ has a closed "hyperplane at ∞" homeomorphic to $\mathbb{CP}^{n-1}$, with open complement homeomorphic to $\mathbb{C}^n$.]

Aufgabe 4
Let $E = \{E^n\}$ be a spectrum. Recall that E represents a generalized cohomology theory by the formula $E^n(X) = [X, E^n]$, where X is any pointed space. Suppose that $E^m(S^0) = 0$ for all odd numbers m, and that $E^m(S^0)$ is a projective $\mathbb{Z}$-module for all even numbers m. Use induction to show that

$$E^0(\mathbb{CP}^n_+) \cong \bigoplus_{m=0}^{n} E^{-2m}(S^0).$$

Use the Mittag-Leffler property to calculate $E^0(\mathbb{CP}^\infty_+)$.

Due: ***8.6.2011*** *in the exercise session.*