Universität Regensburg, Mathematik SS 2011
Dr. David Gepner 11.5.2011

Introduction to Stable Homotopy Theory
2. Übungsblatt

Aufgabe 1
Recall that the *cofiber* $\mathrm{Cof}(f)$ of a map $f : X \to Y$ of pointed spaces is the pushout $Y \coprod_X X \wedge I$ of f along the inclusion $i : X \to X \wedge I$ of X into the cone on X. Show that, if $f : X \to Y$ is the inclusion of a subcomplex (a cofibration), and $g : X \to X'$ is a homotopy equivalence, then the induced map $Y \to Y' = Y \coprod_X X'$ is a homotopy equivalence. Deduce that, if $f : X \to Y$ is the inclusion of a subcomplex, then the quotient map $\mathrm{Cof}(f) \to Y/X$ (induced by the projection $X \wedge I \to *$) is a homotopy equivalence. Give an explicit example to show that the quotient map $\mathrm{Cof}(f) \to Y/X$ need not be a homotopy equivalence in general.

Aufgabe 2
Dually, the *fiber* $\mathrm{Fib}(f)$ of a map $f : Y \to X$ of pointed spaces is the pullback $Y \times_X X^I$ of f along the projection $p : X^I \to X$ which evaluates a based path $\gamma : I \to X$, $\gamma(0) = * \in X$, at $1 \in I$. Show that, if $f : Y \to X$ a fibration (that is, f has the right lifting property with respect to the inclusion $W \to W \times I$ for any space W), then, given a homotopy equivalence $g : X' \to X$, the projection $Y' = Y \times_X X' \to Y$ is a homotopy equivalence. Deduce that, if $f : Y \to X$ is a fibration, then the inclusion $Y \times_X * \to \mathrm{Fib}(f)$ (induced by the inclusion $* \to X^I$) is a homotopy equivalence. Give an explicit example to show that the inclusion $Y \times_X * \to \mathrm{Fib}(f)$ need not be a homotopy equivalence in general. Lastly, show that if Y and X are pointed spaces, then there is a fiber sequence of the form $Y^X \to Y^{X_+} \to Y$ (these are spaces of *pointed* maps; note that $X_+ \to X$ is a pointed map if we send the disjoint basepoint $*$ of X_+ to the basepoint of X).

Aufgabe 3
Let X be a pointed connected CW complex with basepoint $* \in X$. Show that X is homotopy equivalent to a pointed CW complex $Y = \mathrm{colim}_n\, \mathrm{sk}_n Y$ with basepoint $* \in Y$ such that $\mathrm{sk}_0 Y = *$ and, inductively, $\mathrm{sk}_n Y$ is homotopy equivalent to the cofiber of an "attaching map" of the form

$$f_n : \bigvee_{\lambda \in \Lambda_n} S^{n-1} \longrightarrow \mathrm{sk}_{n-1} Y.$$

In other words, connected CW complexes can be built up from the point as cofibers of pointed maps from spheres. [Hint: use cellular approximation.]

Aufgabe 4
Suppose given a commutative diagram

$$\begin{array}{ccccc} Z & \xrightarrow{g} & Y & \xrightarrow{f} & X \\ \downarrow & & \downarrow & & \downarrow \\ Z' & \xrightarrow{g'} & Y' & \xrightarrow{f'} & X' \end{array}$$

of CW complexes such that X and X' are connected, the rows are fiber sequences (for some, and hence any, choice of basepoint $x \in X$), the map $X \to X'$ is an $(n+1)$-equivalence, and the map $Z \to Z'$ is an n-equivalence. Show that the map $Y \to Y'$ is also an n-equivalence. [Hint: use the long exact sequence on homotopy groups associated to a fiber sequence; for the π_0 case, show that $\pi_1(X)$ acts on $\pi_0(Z)$ with quotient $\pi_0(Y)$.]

Due: ***18.5.2011*** *in the exercise session.*