Introduction to stable homotopy theory

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CHAPTER 1

Preliminaries

In these notes, whenever we refer to a topological space we mean a compactly generated topological space (or Kelley space). In particular for us the category of topological spaces will be cartesian closed.

Denis: We need to split this chapter in two: simplicial homotopy theory and homotopy coherent diagrams

1. Simplicial homotopy theory

The standard reference for simplicial homotopy theory is the book by Goerss and Jardine [**GJ09**]. Much of this chapter is modeled on Kan's original papers [**Kan58**] and [**Kan57**].

Let Δ be the category whose objects are finite nonempty totally ordered sets and maps are continuous maps. Concretely the typical object is going to be

$$[n] = \{0 < 1 < \cdots < n\}$$

There's a functor $|-| : \Delta \to \mathrm{Top}$ from Δ to the category of topological spaces sending $[n]$ to the n-dimensional topological simplex

$$|\Delta^n| = \{t \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1,\ t_i \geq 0\}$$

and extend the maps on the vertices linearly.

Important arrows are for every $0 \leq i \leq n$

$$\partial_i : [n] \to [n+1] \qquad j \mapsto \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases}$$

$$s_i : [n] \to [n-1] \qquad j \mapsto \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j \geq i+1 \end{cases}.$$

The first one corresponds under the functor $|-|$ as the inclusion of the i-th face (i.e. the face opposed to the i-th vertex), and correspondingly it is called the ***i*-th face map**. The second one corresponds to the projection onto the $i+1$-th face parallel to the edge $\{i, i+1\}$ and it is called the ***i*-th degeneracy map**.

Exercise 1. *Show that every arrow in Δ can be written as a composition of a sequence of degeneracy maps followed by a sequence of face maps.*

We can visualize a the category Δ as follows

$$[0] \rightleftarrows [1] \ \overset{\longleftarrow}{\underset{\longleftarrow}{\rightleftarrows}}\ [2] \qquad \cdots$$

Definition 1.1. *A simplicial set is a functor $X : \Delta\mathrm{op} \to \mathrm{Set}$.*

The first important example of a simplicial set is the singular complex of a topological space.

Example 1.2. *Let T be a topological space. Then the singular complex* $\operatorname{Sing} T$ *is the simplicial set*

$$[n] \mapsto \mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, T)$$

So for example $(\operatorname{Sing} T)_0$ *is just the set of points of* T*,* $(\operatorname{Sing} T)_1$ *is just the set of paths in* T *and the two maps*

$$\partial_0, \partial_1 : (\operatorname{Sing} T)_1 \to (\operatorname{Sing} T)_0$$

send a path γ *to* $\gamma(1)$ *and* $\gamma(0)$ *respectively.*

Bolstered by this example if X is a simplicial set we will often refer to elements of $X([0])$ as points of X, and elements of $X([1])$ as paths in X.

EXAMPLE 1.3. *There is also a functor* $\Delta \to \mathrm{Cat}$ *sending a poset to the corresponding category. From this we can construct for every category* $\mathcal{C}$ *a simplicial set* $N\mathcal{C}$*, called the* ***nerve*** *of* $\mathcal{C}$*, such that* $(N\mathcal{C})([n])$ *is the set of functors* $[n] \to \mathcal{C}$*, i.e. the set of* n *composable arrows in* $\mathcal{C}$*. Face maps correspond to taking composition of arrows and degeneracies to inserting identity arrows. We will return on this example later.*

EXAMPLE 1.4. *We can take the nerve of the poset* $[n]$ *seen as a category. This is called the* ***standard*** n***-simplex*** *and denoted* $\Delta^n := N[n]$*. The functor* $\Delta \to \mathrm{sSet}$ *sending* $[n]$ *to* Δ^n *is simply the Yoneda embedding for the category* Δ*. In particular* $(\Delta^n)([m])$ *is just the set of maps* $f : [m] \to [n]$*, or equivalently the sequences* $0 \leq i_0 \leq \cdots \leq i_m \leq n$*.*

EXERCISE 2. *Let* P *be a poset. Then* NP *is the colimit of* $\Delta^{\#S}$ *where* S *ranges through the finite non-empty totally ordered subsets of* P*. For example*

$$\Delta^1 \times \Delta^1 = \Delta^{\{(0,0)<(0,1)<(1,1)\}} \cup_{\Delta^{\{(0,0)<(1,1)\}}} \Delta^{\{(0,0)<(1,0)<(1,1)\}}.$$

EXAMPLE 1.5. *In what follows an important example will be the following subcomplexes of* Δ^n*. First, we let the boundary* $\partial\Delta^n$ *be the union of all proper faces of* Δ^n*, i.e. its* m*-simplices are the maps* $f : [m] \to [n]$ *that are not surjective.*

If $0 \leq i \leq n$*, we let the* i*-th horn* Λ^n_i *be the union of all proper faces of* Δ^n *except the* i*-th one. Said differently* $\Lambda^n_i([m])$ *is the set of all maps* $f : [m] \to [n]$ *such that the image does not contain* $\{0, \ldots, i-1, i+1, \ldots, n\}$*.*

EXAMPLE 1.6. *The functor* $\mathrm{Sing} : \mathrm{Top} \to \mathrm{sSet}$ *has a left adjoint called the* ***geometric realization****. It sends a simplicial set* X *to the following topological space*

$$|X| := \left(\coprod_n X([n]) \times |\Delta^n| \right) / \sim$$

where $\sim$ *is the equivalence relation generated by*

$$(\sigma, f_* t) \sim (f^* \sigma, t)$$

for every $\sigma \in X([n])$*,* $t \in |\Delta^m|$ *and* $f : [m] \to [n]$*.*

The following proposition is taken from [**GZ12**, 3.1].

PROPOSITION 1.7. *The geometric realization functor from simplicial sets to compactly generated topological spaces commutes with finite products.*

PROOF. Let us remark that one can verify the special case

$$|\Delta^n \times \Delta^m| \to |\Delta^n| \times |\Delta^m|$$

using exercise 2 (for the details see 3.4 in Gabriel-Zisman).

Now let us fix an m-simplex $\tau \subseteq Y$ and consider the poset of subsets A where $A \subseteq X$ and the map $|A \times \tau| \to |A| \times |\Delta^m|$ is a homeomorphism. This has a maximal element by Zorn, since both sides commute with colimits. Then if σ is a minimal simplex not in A we can write $A' = A \cap_{\partial\Delta^n} \Delta^n$. But then

$$|A' \times B| \cong |(A \cap_{\partial\Delta^n} \Delta^n) \times \Delta^m| \cong |(A \times \Delta^m)| \cup_{|\partial\Delta^n \times \Delta^m|} |\Delta^n$$

$\square$

If X is a topological space $\operatorname{Sing} X$ has an additional property that not all simplicial sets have.

DEFINITION 1.8. *Let X be a simplicial set. We say that X is a* ***Kan complex*** *if every map $f : \Lambda^n_i \to X$ from a horn has an extension to Δ^n.*

LEMMA 1.9. *Let X be a topological space. Then $\operatorname{Sing} X$ is a Kan complex.*

PROOF. By the adjunction $|-||\operatorname{Sing}$ giving a map $f : \Lambda^n_i \to \operatorname{Sing} X$ is the same thing as giving a continuos map $f : |\Lambda^n_i| \to X$, and giving an extension to Δ^n is the same as giving an extension to $|\Delta^n|$. But the inclusion $|\Lambda^n_i| \subseteq |\Delta^n|$ has a retraction pushing the barycenter of the i-th face to the i-th vertex. □

EXAMPLE 1.10. *Let X, Y be two topological spaces. We can define the mapping space as the simplicial set*

$$[n] \mapsto \operatorname{Hom}_{\mathrm{Top}}(X \times |\Delta^n|, Y)\,.$$

This is a Kan complex, where the points are continuous maps and paths are homotopies. Note that this is well defined even when there is no sensible topology on the space of continuous maps.

EXAMPLE 1.11. *Let M, N be smooth manifolds. We can define the subsimplicial set*

$$\operatorname{Emb}(M, N) \subseteq \operatorname{Map}(M, N)$$

whose n-simplices are smooth maps $f : M \times |\Delta^n| \to N$ such that $f|_{M\times\{t\}}$ is an embedding for every $t \in |\Delta^n|$. Its points are smooth embeddings and paths are smooth isotopoies. Then $\operatorname{Emb}(M, N)$ is a Kan complex.

In fact $\operatorname{Emb}(M, N)$ can be realized as the Sing of a certain topological space, but this is not an easy statement to prove at all. We will see that the Kan complex $\operatorname{Emb}(M, N)$ is completely sufficient to talk about the homotopy type of the space of embeddings.

EXAMPLE 1.12. *Let S, T be simplicial sets. Then $\operatorname{Hom}(S, T)$ is the simplicial set given by*

$$[n] \mapsto \operatorname{Hom}_{\mathrm{sSet}}(S \times \Delta^n, T)\,.$$

This has the property that giving a map $A \to \operatorname{Hom}(S, T)$ is the same thing as giving a map $A \times S \to T$.

Our goal in this section is to show that $\operatorname{Sing} X$ contains all the information about the weak homotopy type of X. As a first step we will show that it contains all the information in the homotopy groups.

LEMMA 1.13. *Let X be a Kan complex. Then the relation on X_0 given by*

$$x \sim y \Leftrightarrow \exists \gamma \in X_1\ \partial_1\gamma = x,\ \partial_0\gamma = y$$

is an equivalence relation. The set of equivalence classes will be denoted by $\pi_0 X$ and called the set of connected components of X.

PROOF. We need to check the three properties of an equivalence relation: $\sim$ is reflexive since the existence of the degenerate 1-simplex $s_0 x \in X_1$ implies $x \sim x$. Then it is reflexive since if $x \sim y$, let $\gamma \in X_1$ witnessing the equivalence. Then we can extend γ to a map $f_0 : \Lambda^2_2 \to X$ whose restriction to $\Delta^{1,2}$ is γ and whose restriction to $\Delta^{0,2}$ is the degenerate simplex $s_0 y$. Then we can extend f_0 to $f : \Delta^2 \to X$ and the restriction of f to $\Delta^{0,1}$ is a witness of $y \sim x$.

Similarly, if we have $x \sim y$ and $y \sim z$, we can take $\gamma, \delta \in X_1$ witnessing those relations. Then we build $f_0 : \Lambda^2_1 \to X$ such that $f_0|_{\Delta^{0,1}} = \gamma$ and $f_0|_{\Delta^{1,2}} = \delta$ and extend it to $f : \Delta^2 \to X$. Then $f|_{\Delta^{0,2}}$ witnesses $x \sim z$. □

EXERCISE 3. *Let X be a Kan complex. Then we can write*

$$X \cong \coprod_{\alpha \in \pi_0 X} X^\alpha$$

where $X^\alpha \subseteq X$ is the simplicial subset consisting of simplices of X all whose vertices are in α.

EXERCISE 4. *Show that for X a topological space and Y a Kan complex there are natural bijections*

$$\pi_0 X \cong \pi_0 \operatorname{Sing} X \text{ and } \pi_0 Y \cong \pi_0 |Y|$$

In order to define higher homotopy groups we want to extend the above equivalence relation to higher simplices.

DEFINITION 1.14. *Let X be a simplicial set and $n \geq 0$. Then two n-simplices $\sigma, \tau \in X_n$ are* ***homotopic relative to the boundary*** *if $\sigma|_{\partial\Delta^n} = \tau|_{\partial\Delta^n}$ and there exists an $(n+1)$-simplex η such that*

- *$\partial_n \eta = \sigma$;*
- *$\partial_{n+1} \eta = \tau$;*
- *For every $0 \leq i < n$ we have $\partial_i \eta = s_{n-1}\partial_i \sigma = s_{n-1}\partial_i \tau$.*

EXERCISE 5. *The homotopy relative to the boundary is an equivalence relation on the set of n-simplices.*

EXERCISE 6. *Let Y be a topological space. Then two maps $f, g : |\Delta^n| \to Y$ are homotopic relative to $|\partial\Delta^n|$ if and only if they are homotopic relative to the boundary as elements of $(\operatorname{Sing} Y)([n])$.*

DEFINITION 1.15. *Let X be a Kan complex and $x \in X$ be a point of X. Then the n-th homotopy group is the quotient of the set of n-simplices $\sigma : \Delta^n \to X$ such that $\sigma|_{\partial\Delta^n} = x$ up to homotopy relative to the boundary.*

EXAMPLE 1.16. *If X is a topological space, and $x \in X$ is a point the set $\pi_n(\operatorname{Sing} X, x)$ is just the n-th homotopy group of X.*

CONSTRUCTION 1.17. *By analogy with the topological case we want $\pi_n(X, x)$ to be a group when $n \geq 1$. Let $\alpha, \beta : \Delta^n \to X$ such that $\alpha|_{\partial\Delta^n} = \beta|_{\partial\Delta^n} = x$. Then we can build a map $\eta_0 : \Lambda^{n+1}_n \to X$ such that $\eta_0|_{\partial_{n-1}\Delta^{n+1}} = \alpha$, $\eta_0|_{\partial_{n+1}\Delta^n} = \beta$ and $\eta_0|_{\partial_i \Delta^n} = x$ for $0 \leq i \leq n-2$. Then we can extend it to $\eta : \Delta^{n+1} \to X$ and let $[\alpha] \cdot [\beta] := [\eta|_{\partial_n \Delta^{n+1}}]$.*

LEMMA 1.18. *The multiplication is well-defined up to homotopy and it turns $\pi_n(X, x)$ into a group.*

PROOF. For simplicity we will do only the case $n = 1$. Let us say that a triple of 1-simplices (α, β, γ) is a composition pair if all their faces are the degenerate simplices at x and there is a 2-simplex σ such that $\partial_0 \sigma = \alpha$, $\partial_1 \sigma = \gamma$ and $\partial_2 \sigma = \beta$. Note that α and α' are homotopic if and only if (α, x, α') is a composition pair.

Clearly taking the degenerate 2-simplex $s_0 \alpha$ we see that (α, sx, α) is always a composition pair, therefore the multiplication is unital. We claim that if (α, β, γ), $(\gamma, \delta, \epsilon)$ and (β, δ, θ) are composition pairs, so is (α, δ, θ). In fact we can build $f : \Lambda^3_1 \to X$ such that $\partial_0 f$ is a 2-simplex representing (α, β, γ), $\partial_2 f$ represents $(\gamma, \delta, \epsilon)$ and $\partial_3 f$ represents (β, δ, θ). Then if we extend f to Δ^3 we see that $\partial_1 f$ represents $(\alpha, \theta, \epsilon)$, as required.

In particular if the composition is well-defined it is associative. Moreover if α and α' are homotopic, this means that (α, x, α') is a composition pair. Therefore for every composition pair (α, β, γ), since (β, x, β) is a composition pair, we deduce that (α', β, γ) is also a composition pair, therefore the composition is well-defined in α. A similar argument shows it is also well-defined in β.

Finally, considering the map $f : \Lambda_0^2 \to X$ such that $\partial_1 f = x$ and $\partial_0 f = \alpha$ and taking its extension to Δ^2 we see that there is a β such that (α, β, x) is a composition pair. Therefore every element has a left inverse, and so $\pi_n(X, x)$ is a group. □

Our next step is to define the notion of homotopy a map of Kan complexes and study the behaviour of homotopy groups under it. To do so we will need a technical lemma.

Lemma 1.19 (Homotopy lifting property). *Let K be a simplicial set. Then K is a Kan complex if and only if for every inclusion $A \subseteq B$ and every commutative square*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathrm{Hom}(\Delta^1, K) \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & K \end{tikzcd}$$

where the vertical map is induced by $\{0\} \subseteq \Delta^1$, the dashed lift exists.

Proof. Let us show first that if K satisfies the conditions of the lemma, it is a Kan complex. Let us take $f_0 : \Lambda_i^n \to K$. Let us define the map

$$r : \Delta^n \times \Delta^1 \to \Delta^n \qquad r(j, 0) = \begin{cases} j & \text{if } j \neq i+1 \\ i & \text{if } j = i+1 \end{cases}, \qquad r(j, 1) = j$$

(morally r is a homotopy of the identity to the projection onto the $(i+1)$-th face). This sends $\Lambda_i^n \times \Delta^1$ and $\Delta^n \times \{0\}$ to Λ_i^n. Therefore we can construct a diagram

$$\begin{tikzcd} \Lambda_i^n \arrow[r] \arrow[d] & \mathrm{Hom}(\Delta^1, K) \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "g"] & K \end{tikzcd}$$

where the top face is adjoint to the restriction of $f_0 r$ to $\Lambda_i^n \times \Delta^1$ and the bottom face is the restriction of $f_0 r$ to $\Delta^n \times \{0\}$. If we let g be the lift, we see that $f = g|_{\Delta^n \times \{1\}}$ is the map we were looking for.

Now let us prove the other direction. Suppose K is a Kan complex and let us do first the case where $A = \partial\Delta^n$ and $B = \Delta^n$. Concretely we have a map

$$f_0 : \partial\Delta^n \times \Delta^1 \cup \Delta^n \times \{0\} \to K$$

and we want to extend it to $\Delta^n \times \Delta^1$. For $-1 \leq i \leq n$ let B_i be the subcomplex of $\Delta^n \times \Delta^1$ given by

$$B_i = \partial\Delta^n \times \Delta^1 \cup \Delta^n \times \{0\} \cup H_0 \cup H_1 \cup \cdots \cup H_i \,,$$

where H_i is the $(n+1)$-simplex corresponding to the map of posets

$$H_i(j) = \begin{cases} (j, 0) & \text{if } j \leq i \\ (j-1, 1) & \text{if } j > i \end{cases}.$$

Then it is easy to verify that

$$B_i = B_{i-1} \cup_{\Lambda_i^{n+1}} \Delta^{n+1}$$

and $B_n = \Delta^n \times \Delta^1$. Thus we can extend f_0 by induction to B_i, proving the thesis.

To do the general case let us consider the poset of pairs $(C, h : C \to \mathrm{Hom}(\Delta^1, K)$ such that C is a subset of B containing A and h is a partial lift. We can apply Zorn and deduce that it has a maximal object (C, h). Our goal is to show that in the maximal object $C = B$. Suppose this is not true and let $\sigma \in B([n])$ be a simplex of minimal dimension not in C. Then $\sigma|_{\partial\Delta^n}$ lives in C (because it is composed of simplices of smaller dimension) and we can apply the special case proved above to extend h to $C \cup \sigma$, thus proving a contradiction. □

COROLLARY 1.20. *Let S be a simplicial set and X be a Kan complex. Then* $\mathrm{Hom}(S, X)$ *is a Kan complex (which we will write as* $\mathrm{Map}(S, X)$*).*

PROOF. We verify that $\mathrm{Hom}(S, X)$ satisfies the conditions of lemma 1.19. Indeed let us fix a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathrm{Hom}(\Delta^1, \mathrm{Hom}(S, X)) \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & \mathrm{Hom}(S, X) \end{tikzcd}$$

But providing a lift for this is equivalent to providing a lift for the diagram

$$\begin{tikzcd} A \times S \arrow[r] \arrow[d] & \mathrm{Hom}(\Delta^1, X) \arrow[d] \\ B \times S \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

which exists by lemma 1.19. □

Let X be a Kan complex. Then two maps $f, g : S \to X$ are **homotopic** if they lie in the same connected component of $\mathrm{Map}(S, X)$. This is equivalent to saying that there is $H : S \times \Delta^1 \to X$ such that $H|_{S\times\{0\}} = f$ and $H|_{S\times\{1\}} = g$ or, equivalently, that there's $H : X \to \mathrm{Map}(\Delta^1, X)$ such that $ev_0 \circ H = f$ and $ev_1 \circ H = g$.

LEMMA 1.21. *For any $x \in X$ and $\gamma : \Delta^1 \to X$, the map*

$$(ev_0)_* : \pi_n(\mathrm{Hom}(\Delta^1, X), \gamma) \to \pi_n(X, \gamma 0)$$

is an isomorphism (and analogously for ev_1.

PROOF. Since the constant map $\delta : X \to \mathrm{Hom}(\Delta^1, X)$ adjoint to $X \times \Delta^1 \to X$ is a right inverse, the map is obviously surjective. We need to prove that it is injective. Let $\alpha, \beta : \Delta^n \to \mathrm{Hom}(\Delta^1, X)$ representing two classes in $\pi_n(\mathrm{Hom}(\Delta^1, X), \gamma)$ and let $\eta : \Delta^{n+1} \to X$ be a witness of a homotopy between $ev_0\alpha$ and $ev_0\beta$. Then we can construct the diagram

$$\begin{tikzcd} \partial\Delta^{n+1} \arrow[r] \arrow[d] & \mathrm{Hom}(\Delta^1, X) \arrow[d] \\ \Delta^{n+1} \arrow[r, "\eta"] & X \end{tikzcd}$$

where the top horizontal map is the boundary of a homotopy between α and β. Then lemma 1.19 implies there's a lift. □

Using the lemma we can construct for any path $\gamma : \Delta^1 \to X$ an isomorphism

$$\pi_n(X, \gamma 0) \xrightarrow{(ev_0)_*^{-1}} \pi_n(\mathrm{Hom}(\Delta^1, X), \gamma) \xrightarrow{(ev_1)_*} \pi_n(X, 1)\,.$$

Exercise 7. *Let $H : X \times \Delta^1 \to Y$ be a homotopy between two maps $f = H|_{X\times\{0\}}$ and $g = H|_{X\times\{1\}}$. Then for every $x \in X$ there's a commutative diagram*

$$\begin{tikzcd} \pi_n(X,x) \arrow[r, "f_*"] \arrow[dr, "g_*"'] & \pi_n(Y,fx) \arrow[d, "\gamma_*"] \\ & \pi_n(Y,gx) \end{tikzcd} .$$

In particular homotopy equivalences induce isomorphisms between homotopy groups.

Theorem 1.22. *Let K be a Kan complex. Then the map*

$$\eta : K \to \operatorname{Sing}|K|$$

sending an n-simplex $\Delta^n \to K$ to its geometric realization $|\Delta^n| \to |K|$, is a homotopy equivalence of Kan complexes.

Proof. Let us consider the map

$$\epsilon|\operatorname{Sing}|K|| \to |K|$$

which is adjoint to the identity $\operatorname{Sing}|K| \to \operatorname{Sing}|K|$. Then η is the inclusion of a subcomplex such that $\epsilon \circ |\eta| = \mathrm{id}_{|K|}$. In particular it is a simplicial map, so we can apply theorem 1.91 and obtain a map of simplicial complexes $f : \operatorname{Sing}|K| \to K$ such that $f\eta \cong \mathrm{id}_K$ and a homotopy

$$H : |\operatorname{Sing}|K|| \times |\Delta^1| \to |K|$$

from $|f|$ to ϵ relative to $|K|$. But then $|\operatorname{Sing}|K|| \times |\Delta^1| \cong |\operatorname{Sing}|K| \times \Delta^1|$ by ... and the adjoint map is

$$\tilde{H} : \operatorname{Sing}|K| \times \Delta^1 \to \operatorname{Sing}|K| .$$

One can then immediately verify that $\tilde{H}$ is a homotopy of ηf with the identity. $\square$

Corollary 1.23. *If X is a topological space, the map*

$$|\operatorname{Sing} X| \to X$$

is a weak equivalence.

Proof. Using the fact that $\operatorname{Sing} X$ is a Kan complex we obtain that, by the previous corollary

$$\pi_n(|\operatorname{Sing} X|, x) \cong \pi_n(\operatorname{Sing} X, x) = \pi_n(X, x) .$$

for every $x \in X$. Since $\pi_0|\operatorname{Sing} X| \cong \pi_0 X$ it suffices to check the condition of being a weak equivalence only on the point coming from X. $\square$

Corollary 1.24. *Let $f : X \to Y$ be a map of Kan complexes such that for every $x \in X$ and $n \geq 0$ the map $\pi_n(X,x) \to \pi_n(Y,fx)$ is an isomorphism. Then f is a homotopy equivalence.*

Proof. By example 1.16 it follows that $|f|$ is a weak equivalence of topological spaces. But $|X|$ and $|Y|$ are CW complexes, so $|f|$ is a homotopy equivalence. Then $\operatorname{Sing}|f|$ is a homotopy equivalence of Kan complexes and the thesis follows from theorem 1.22. $\square$

Corollary 1.25. *Let X,Y be topological spaces. Then X and Y are weakly equivalent if and only if $\operatorname{Sing} X$ and $\operatorname{Sing} Y$ are homotopy equivalent.*

Proof. Let $f : X \to Y$ be a weak equivalence between X and Y. But then the previous corollary implies that $\operatorname{Sing} f$ is a homotopy equivalence. $\square$

Putting together the above results, we can deduce the Kan complex $\operatorname{Sing} X$ up to simplicial homotopy knows everything about X up to weak equivalence. In what follows we will refer to $\operatorname{Sing} X$ (as an element of the category of Kan complexes or, later, of the ∞-category of spaces) as the weak homotopy type of X.

EXERCISE 8. *The categories* $\mathbf{Kan}[h.e.^{-1}]$ *and* $\mathrm{Top}[w.e.^{-1}]$ *obtained by respectively inverting the homotopy equivalences and the weak equivalences in the categories of Kan complexes and of topological spaces are equivalent. This category is normally called the* ***homotopy category*** $h\mathscr{S}$.

Moreover $h\mathscr{S}$ *is equivalent to the category whose objects are Kan complexes and whose morphisms are homotopy classes of maps.*

2. ∞-categories

Our goal in this section is to briefly introduce the theory of ∞-categories. The main reason for us to do so is because it will allow us to speak in a very simple way of coherently commutative diagrams. In order to motivate this let us first see a few examples.

Let $\mathbf{C}$ be a category enriched in the category $\mathbf{Kan}$ of Kan complexes. Concretely means that we have a set $\operatorname{ob}\mathbf{C}$ of objects and for every $x, y \in \operatorname{ob}\mathbf{C}$ a Kan complex $\operatorname{Map}_{\mathbf{C}}(x, y)$. Moreover we have a composition operation

$$\circ : \operatorname{Map}_{\mathbf{C}}(x, y) \times \operatorname{Map}_{\mathbf{C}}(y, z) \to \operatorname{Map}_{\mathbf{C}}(x, z)$$

which is associative and unital. We will refer to the 0-simplices of $\operatorname{Map}_{\mathbf{C}}(x, y)$ as maps from x to y and to the 1-simplices as homotopies between maps. We will sometimes refer to higher simplices as "higher homotopies".

EXAMPLE 1.26. *We will leave as an exercise to show that all the simplicial sets we define here are in fact Kan complexes.*

(1) *The category* $\mathbf{Kan}$ *itself has a canonical enrichment such that*

$$\operatorname{Map}_{\mathbf{Kan}}(K, S)_n = \operatorname{Hom}_{\mathbf{Kan}}(K \times \Delta^n, S)\,.$$

(2) *The category* Top *of topological spaces has a canonical enrichment in Kan complexes*

$$\operatorname{Map}_{\mathrm{Top}}(X, Y)_n = \operatorname{Hom}_{\mathrm{Top}}(X \times |\Delta^n|, Y)\,.$$

(3) *The category* Emb *of smooth manifolds and embeddings has a canonical enrichment such that the* n*-simplices in* $\operatorname{Map}_{\mathrm{Emb}}(M, N)$ *are the smooth maps* $F : M \times |\Delta^n| \to N$ *such that* $F|_{M\times\{t\}}$ *is an embedding for every* $t \in |\Delta^n|$. *In particular, homotopies between maps are isotopies.*

Recall that if I is a category, an I-shaped diagram in some category $\mathcal{C}$ is just a functor $F : I \to \mathcal{C}$. Concretely this is the datum of an object $Fi \in \operatorname{ob}\mathcal{C}$ for every $i \in \operatorname{ob} I$ and an arrow $Ff : Fi \to Fj$ for every arrow $i \to j$ in I such that $F(\mathrm{id}_i) = \mathrm{id}_{Fi}$ and $F(gf) = F(g) \circ F(f)$ for every pair of composable arrows f, g. Now if we move to the context of categories enriched in Kan complexes

In order to motivate the definion of coherently commutative diagrams in $\mathbf{C}$ we will first see a few examples:

EXAMPLE 1.27. *We want to say what it means to give a homotopy commutative diagram of the form*

$$\begin{tikzcd} x_0 \arrow[r] \arrow[dr] & x_1 \arrow[d] \\ & x_2 \end{tikzcd}$$

in $\mathbf{C}$*. This should be the datum of* x_0, x_1, x_2 *objects in* $\mathbf{C}$*, arrows* $f_{01} \in \mathrm{Map}_{\mathbf{C}}(x_0, x_1)_0$*,* $f_12 \in \mathrm{Map}_{\mathbf{C}}(x_1, x_2)$ *and* $f_{02} \in \mathrm{Map}_{\mathbf{C}}(x_0, x_2)$ *such that* f_{02} *is homotopic to* $f_{12} \circ f_{01}$*. The big difference with a coherently commutative diagram (or coherent diagram for short) is that in a coherent diagram we remember the homotopy. That is, we add the additional datum of* $f_{012} \in \mathrm{Map}_{\mathbf{C}}(x_0, x_2)_1$ *such that* $\partial_1 f_{012} = f_{02}$ *and* $\partial_0 f_{012} = f_{12} \circ f_{01}$*.*

Example 1.28. *Let's try to see a bigger example: suppose we want to describe* triples *of composable arrows. Then such a diagram should consist in*

- *Four objects* x_0, x_1, x_2, x_3 *in* $\mathbf{C}$*;*
- *Six arrows* $f_{01}, f_{02}, f_{03}, f_{12}, f_{13}, f_{23}$ *where* $f_{ij} \in \mathrm{Map}_{\mathbf{C}}(x_i, x_j)$*;*
- *Four homotopies* $f_{012} \in \mathrm{Map}_{\mathbf{C}}(x_0, x_2)_1$*,* $f_{123} \in \mathrm{Map}_{\mathbf{C}}(x_1, x_3)_1$ *and* $f_{013}, f_{023} \in \mathrm{Map}_{\mathbf{C}}(x_0, x_3)_1$ *such that* $\partial_1 f_{ijk} = f_{ik}$ *and* $\partial_0 f_{ijk} = f_{jk} \circ f_{ij}$
- *A higher homotopy* $f_{0123} : \Delta^1 \times \Delta^1 \to \mathrm{Map}_{\mathbf{C}}(x_0, x_3)$ *representing the following diagram*

$$\begin{tikzcd} f_{03} \arrow[r, "f_{013}"] \arrow[d, "f_{023}"] & f_{13} \circ f_{01} \arrow[d, "f_{123} \circ f_{01}"] \\ f_{23} \circ f_{02} \arrow[r, "f_{23} \circ f_{012}"] & f_{23} \circ f_{12} \circ f_{01} \end{tikzcd}$$

Recall that the nerve of a category I is the simplicial set $N(I)$ such that

$$N(I)_n = \mathrm{ob}\,\mathrm{Fun}([n], I)\,.$$

We will define a coherent diagram in $\mathbf{C}$ of shape I as a map of simplicial sets $N(I) \to N^{\Delta}(\mathbf{C})$ for some suitably defined simplicial set $N^{\Delta}(\mathbf{C})$.

Definition 1.29. *Let* $n \geq 0$*. Then we define* $\mathfrak{C}[\Delta^n]$ *to be the category enriched in simplicial sets whose objects are the numbers* $0, \ldots, n$ *and such that*

$$\mathrm{Map}_{\mathfrak{C}[\Delta^n]}(i,j) = \begin{cases} \{A \subseteq \{i, i+1, \ldots, j-1, j\} \mid i, j \in A\} & \textit{if } i \leq j \\ \varnothing & \textit{otherwise} \end{cases}.$$

The composition is given by

One should think of $\mathrm{Map}_{\mathfrak{C}}(i,j)$ *as the poset of monotone paths from* i *to* j *ordered by refinement.*

The following definition is due to Cordier and Porter [**CP86**].

Definition 1.30. *Let* $\mathbf{C}$ *be a category enriched in simplicial sets. Then* $N^{\Delta}\mathbf{C}$ *is the simplicial set whose* n*-simplices are enriched functors*

$$\mathfrak{C}[\Delta^n] \to \mathbf{C}\,.$$

So 0-simplices of $N^{\Delta}\mathbf{C}$ *are just objects of* $\mathbf{C}$*, 1-simplices of* $N^{\Delta}\mathbf{C}$ *are arrows in* $\mathbf{C}$*, 2-simplices of* $N^{\Delta}\mathbf{C}$ *are diagrams as in example 1.27, etc..*

Concretely an n*-simplex of* $N^{\Delta}\mathbf{C}$ *can be thought of* $n+1$ *objects* $c_0, \ldots, c_n$*, for every* $i < j$ *an arrow* $f_{ij} : c_i \to c_j$*, for every triple* $i < j < k$ *a homotopy* $f_{ik} \sim f_{jk} \circ f_{ij}$*, etc..*

Example 1.31. *If* $\mathcal{C}$ *is an ordinary category seen as a category enriched in discrete simplicial sets, then* $N^{\Delta}\mathcal{C} \cong N\mathcal{C}$*. So the two notions of nerve we have coincide in this case.*

Definition 1.32. *Let* I *be a category and* $\mathbf{C}$ *be a category enriched in Kan complexes. Then a coherent diagram of shape* I *in* $\mathbf{C}$ *is a map of simplicial sets* $NI \to N^{\Delta}\mathbf{C}$*.*

Exercise 9. *Let P be a poset and $\mathbf{C}$ a category enriched in Kan complexes. Then a coherent diagram of shape P in $\mathbf{C}$ is the same as an enriched functor*

$$\mathfrak{C}[P] \to \mathbf{C}$$

where $\mathfrak{C}[P]$ is the category enriched in simplicial sets with objects the objects of P and such that $\mathrm{Map}_{\mathfrak{C}[P]}(p, p')$ is the nerve of the poset of chains $\{p = p_0 < p_1 < \cdots < p_r = p'\}$ in P.

Example 1.33. *Let $I = [1] \times [1]$ be the square diagram. Then an I-shaped coherent diagram in $\mathbf{C}$ is the datum of*

- *Four objects x, y, z and t in $\mathbf{C}$;*
- *Five arrows $f : x \to y$, $g : x \to z$, $f' : z \to t$, $g' : y \to t$, $r : x \to t$;*
- *Two homotopies $H : r \sim f'g$, $K : r \sim g'f$.*

We will often draw it as the following

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, "r"] & y \arrow[d, "g'"] \\ z \arrow[r, "f'"] \arrow[ur, phantom, "\Swarrow \quad \Nearrow"] & t \end{tikzcd}$$

When $\mathbf{C}$ is enriched in Kan complexes we can use the composition of paths to get a homotopy $H' = K \circ H^{-1}$, so we disregard part of the data and just write the diagram as

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\overset{H'}{\Swarrow}"] & y \arrow[d, "g'"] \\ z \arrow[r, "f'"] & t \end{tikzcd}$$

As in the case of $\mathrm{Sing}\, Y$ for a topological space Y, the simplicial set $N^{\Delta}\mathbf{C}$ has a special property that makes it easy to work with.

Definition 1.34. *A simplicial set $\mathcal{C}$ is an **∞-category** (also called quasi-category or weak Kan complex) if for every $n \geq 0$ and $0 < i < n$ every map $f_0 : \Lambda^n_i \to \mathcal{C}$ can be extended to $f : \Delta^n \to \mathcal{C}$.*

*A map of simplicial sets between ∞-categories will be called a **functor**.*

Example 1.35. *Let us see what the condition says for $n = 1$. Then a map $f : \Lambda^2_1 \to \mathcal{C}$ is just a pair of composable 1-simplices f_{01} and f_{12}, and finding an extension to Δ^2 is just telling us that we can find a composition f_{02}.*

Exercise 10. *The simplicial set $N^{\Delta}\mathbf{C}$ is an ∞-category. In particular for every category $\mathcal{C}$ the nerve $N\mathcal{C}$ is an ∞-category.*

Example 1.36. *Every Kan complex is an ∞-category.*

Example 1.37. *Let $i : \Delta \to \Delta$ the functor sending a totally ordered set $(T, \leq)$ to the same set with the opposite order $(T, \leq^{\mathrm{op}})$ (that is $t \leq^{\mathrm{op}} t'$ if and only if $t' \leq t$). Then if $\mathcal{C} : \Delta^{\mathrm{op}} \to \mathrm{Set}$ is an ∞-category, the composition*

$$\mathcal{C}^{\mathrm{op}} : \Delta^{\mathrm{op}} \xrightarrow{i^{\mathrm{op}}} \Delta^{\mathrm{op}} \xrightarrow{\mathcal{C}} \mathrm{Set}$$

*is an ∞-category, called the **opposite ∞-category**.*

Note that if $\mathbf{C}$ is a category, $N(\mathbf{C})^{\mathrm{op}} \cong N(\mathbf{C}^{\mathrm{op}})$.

Lemma 1.38. *Let $\mathcal{C}$ be an ∞-category. Then the relation of being homotopic relative to the boundary is an equivalence relation on 1-simplices and it respects the composition.*

Example 1.39. *Let* $\mathbf{C}$ *be a Kan-enriched category. Then two 1-simplices* $f, g \in N^{\Delta}\mathbf{C}([1])$ *with the same boundary* $\{x, y\}$ *are the same thing as two maps* $f, g \in \mathrm{Map}_{\mathbf{C}}(x, y)$*, and they are homotopic if and only if there's a path between* f *and* g *in* $\mathrm{Map}_{\mathbf{C}}(x, y)$*. Therefore the two notions of homotopy of morphisms in* $\mathbf{C}$ *coincide.*

Definition 1.40. *Let* $\mathcal{C}$ *be an* ∞*-category. Then the homotopy category* $h\mathcal{C}$ *of* $\mathcal{C}$ *is the category whose objects are the objects of* $\mathcal{C}$ *and whose morphisms are homotopy classes of maps in* $\mathcal{C}$

$$\mathrm{Hom}_{h\mathcal{C}}(x, y) := \{f \in \mathcal{C}[1] \mid \partial_1 f = x, \partial_0 f = y\}/\sim \ .$$

Example 1.41. *If* $\mathbf{C}$ *is a Kan-enriched category, the homotopy category of the coherent nerve* $hN^{\Delta}\mathbf{C}$ *is the category whose objects are the objects of* $\mathbf{C}$ *and whose morphisms are the paths components in* $\mathrm{Map}_{\mathbf{C}}$*:*

$$\mathrm{Hom}_{hN^{\Delta}\mathbf{C}}(x, y) = \pi_0 \,\mathrm{Map}_{\mathbf{C}}(x, y)\,.$$

This will often be written $h\mathbf{C}$ *for brevity.*

Remark 1.42. *There's a functor* $\mathcal{C} \to N(h\mathcal{C})$ *sending an* n*-simplex* $\sigma \in \mathcal{C}([n])$ *to the sequence of composable arrows* $([\sigma|_{01}], \cdots, [\sigma|_{n-1,n}])$ *in* $N(h\mathcal{C})([n])$*. Therefore every coherent diagram in* $\mathbf{C}$ *produces a diagram in* $h\mathbf{C}$*.*

Definition 1.43. *Let* $\mathcal{C}$ *be an* ∞*-category and* $f : x \to y$ *be an arrow in* $\mathcal{C}$*. Then* f *is an equivalence if it is an isomorphism in* $h\mathcal{C}$*. Equivalently, if there exists another arrow* $g : y \to x$ *in* $\mathcal{C}$ *such that* gf *and* fg *are homotopic to the identity.*

Lemma 1.44. *An* ∞*-category is a Kan complex if and only if all its arrows are equivalences. Therefore Kan complexes are also sometimes called* ∞*-groupoids.*

Proposition 1.45. *Let* $\mathcal{C}$ *be an* ∞*-category and* $\iota\mathcal{C}$ *be the simplicial subset consisting of all* n*-simplices whose 1-dimensional faces are equivalences. Then* $\iota\mathcal{C}$ *is a Kan complex (in fact the biggest simplicial subset of* $\mathcal{C}$ *that is a Kan complex).*

The following lemma is proven in a similar way to lemma 1.19.

Lemma 1.46. *Let* $\mathcal{C}$ *be a simplicial set. Then the following are equivalent:*

(1) $\mathcal{C}$ *is an* ∞*-category;*
(2) *For any* $n \geq 0$ *and any diagram*

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & \mathrm{Hom}(\Delta^2, \mathcal{C}) \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \mathrm{Hom}(\Lambda^2_1, \mathcal{C}) \end{tikzcd}$$

there exists a lift.
(3) *For inclusion of simplicial sets* $A \subseteq B$ *and any diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & \mathrm{Hom}(\Lambda^1_2, \mathcal{C}) \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & \mathrm{Hom}(\Delta^2, \mathcal{C}) \end{tikzcd}$$

there exists a lift.

The intuition behind this result is that $\mathrm{Hom}(\Lambda^2_1, \mathcal{C})$ is the simplicial set parametrizing composable pairs of arrows in $\mathcal{C}$ and $\mathrm{Hom}(\Delta^2, \mathcal{C})$ parametrizes composable pairs of arrows together with a choice of composition. So this is saying that $\mathcal{C}$ is an ∞-category exactly when the composition of two arrows is well-defined up to a contractible space of choices.

Corollary 1.47. *If S is a simplicial set and $\mathcal{C}$ is an ∞-category, then* $\mathrm{Hom}(S,\mathcal{C})$ *is an ∞-category, which will be denoted* $\mathrm{Fun}(S,\mathcal{C})$.

Proof. We will show that $\mathrm{Hom}(S,\mathcal{C})$ satisfies the condition (3) in lemma 1.46. Let us pick a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & \mathrm{Hom}(\Lambda^1_2, \mathrm{Hom}(S,\mathcal{C})) \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & \mathrm{Hom}(\Delta^2, \mathrm{Hom}(S,\mathcal{C})) \end{tikzcd} .$$

Then finding a lift filling the diagram is equivalent, by the adjunction properties of $\mathrm{Hom}(S,-)$, to finding a lift in the diagram

$$\begin{tikzcd} A \times S \arrow[r] \arrow[d, hook] & \mathrm{Hom}(\Lambda^1_2, \mathcal{C}) \arrow[d] \\ B \times S \arrow[r] \arrow[ur, dashed] & \mathrm{Hom}(\Delta^2, \mathcal{C}) \end{tikzcd} .$$

Then we can apply lemma 1.46 to $\mathcal{C}$ and deduce that the required lift exists. □

Let $\mathcal{C}$ be an ∞-category and $x, y \in \mathrm{ob}\,\mathcal{C}$ objects of $\mathcal{C}$. Then we want to associate a Kan complex $\mathrm{Map}_{\mathcal{C}}(x,y)$ whose points are maps $x \to y$. There are many possible equivalent definitions for this object, we will present here one that is particular convenient for the definition of composition.

Definition 1.48. *Let $\mathcal{C}$ be an ∞-category and $x, y \in \mathrm{ob}\,\mathcal{C}$. Then the* ***mapping space*** *is the simplicial set*

$$\mathrm{Map}_{\mathcal{C}}(x,y) := \mathrm{Fun}(\Delta^1, \mathcal{C}) \times_{\mathcal{C}\times\mathcal{C}} \{(x,y)\}$$

where the map $\mathrm{Fun}(\Delta^1,\mathcal{C}) \to \mathcal{C}\times\mathcal{C}$ comes from precomposition along the map $\partial\Delta^1 \to \Delta^1$.

There is a distinguished point $\mathrm{id}_x \in \mathrm{Map}_{\mathcal{C}}(x,x)$ corresponding to the degenerate 1-simplex sx at x.

Example 1.49. *If $\mathcal{C} = \mathrm{Sing}\,Y$ is the singular complex of a space, then $\mathrm{Map}_{\mathcal{C}}(x,y)$ is the singular complex of the space of paths in Y from x to y.*

Lemma 1.50. *The simplicial set $\mathrm{Map}_{\mathcal{C}}(x,y)$ is a Kan complex.*

Proof. By using the criterion of lemma 1.19 we need to construct a lift in every diagram of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & \mathrm{Map}_{\mathcal{C}}(x,y)^{\Delta^1} \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & \mathrm{Map}_{\mathcal{C}}(x,y) \end{tikzcd} .$$

Unwrapping everything and plugging in the definition of $\mathrm{Map}_{\mathcal{C}}(x,y)$, this is equivalent to constructing a dashed arrow in the following diagram

$$\begin{tikzcd} A \times \Delta^1 \times \Delta^1 \cup B \times \{0\} \times \Delta^1 \cup B \times \Delta^1 \times \partial\Delta^1 \arrow[r] \arrow[d, hook] & \mathcal{C} \\ B \times \Delta^1 \times \Delta^1 \arrow[ur, dashed] \end{tikzcd} ,$$

where the map $B \times \Delta^1 \times \partial\Delta^1$ is given by the constant maps $B \times \Delta^1 \to \mathcal{C}$ at x and y respectively. Using the adjunction again this is equivalent to constructing a lift in

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}) \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & \mathrm{Fun}(\{0\} \times \Delta^1 \cup \Delta^1 \times \partial\Delta^1, \mathcal{C}) \end{tikzcd}$$

Using the pushout square in sSet

$$\begin{tikzcd} \Lambda^2_1 \arrow[r] \arrow[d] & \{0\} \times \Delta^1 \cup \Delta^1 \times \partial\Delta^1 \arrow[d] \\ \Delta^2 \arrow[r] & ??? \end{tikzcd}$$

and lemma 1.46 we can reduce to construct a lift

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & \mathrm{Fun}(\Delta^2/\Delta^1, \mathcal{C}) \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & \mathrm{Fun}(\Delta^1, \mathcal{C}) \end{tikzcd}$$

INCOMPLETE! □

Exercise 11. *Two points $f, g \in \mathrm{Map}_{\mathcal{C}}(x, y)$ are in the same path component if and only if they are homotopic relative to the boundary as 1-simplices of $\mathcal{C}$.*

Proposition 1.51. *Let $\mathbf{C}$ be a Kan-enriched category and $x, y \in \mathrm{ob}\,\mathbf{C}$. Then there is a natural equivalence*

$$\mathrm{Map}_{\mathbf{C}}(x, y) \cong \mathrm{Map}_{N^\Delta(\mathbf{C})}(x, y)$$

Proof. MISSING! PROBABLY JUST GIVE A REFERENCE, SINCE THIS IS TRICKY. □

Construction 1.52. *Now we want to construct a composition map*

$$\mathrm{Map}_{\mathcal{C}}(x, y) \times \mathrm{Map}_{\mathcal{C}}(y, z) \to \mathrm{Map}_{\mathcal{C}}(x, z)\,.$$

Unfortunately there won't exist a canonical such map: the construction of the composition involves choices but we can show that the space of such choices is contractible (in particular any two of them are homotopic). Let us contemplate the diagram

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^2, \mathcal{C}) \times_{\mathcal{C}\times\mathcal{C}\times\mathcal{C}} \{(x,y,z)\} \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^2, \mathcal{C}) \arrow[d] \\ \mathrm{Map}_{\mathcal{C}}(x, y) \times \mathrm{Map}_{\mathcal{C}}(y, z) \arrow[r, "\sim"] & \mathrm{Fun}(\Lambda^2_1, \mathcal{C}) \times_{\mathcal{C}\times\mathcal{C}\times\mathcal{C}} \{(x,y,z)\} \arrow[r] & \mathrm{Fun}(\Lambda^2_1, \mathcal{C}) \end{tikzcd}$$

Then by the lemma 1.46 we can find a map

$$\mathrm{Map}_{\mathcal{C}}(x, y) \times \mathrm{Map}_{\mathcal{C}}(y, z) \to \mathrm{Fun}(\Delta^2, \mathcal{C})$$

making the diagram commute, and then the commutativity of the diagram shows that there is a map

$$\mathrm{Map}_{\mathcal{C}}(x, y) \times \mathrm{Map}_{\mathcal{C}}(y, z) \to \mathrm{Fun}(\Delta^2, \mathcal{C}) \times_{\mathcal{C}\times\mathcal{C}\times\mathcal{C}} \{(x,y,z)\}\,,$$

and finally precomposing with the face $\partial_1 : \Delta^1 \to \Delta^2$ we obtain a map

$$\mathrm{Map}_{\mathcal{C}}(x, y) \times \mathrm{Map}_{\mathcal{C}}(y, z) \to \mathrm{Fun}(\Delta^2, \mathcal{C}) \times_{\mathcal{C}\times\mathcal{C}\times\mathcal{C}} \{(x,y,z)\} \to \mathrm{Fun}(\Delta^1, \mathcal{C}) \times_{\mathcal{C}\times\mathcal{C}} \{(x,z)\} = \mathrm{Map}_{\mathcal{C}}(x, z)$$

Exercise 12. *Use lemma 1.46 to show that the simplicial set of lifts in the previous construction is a contractible Kan complex.*

EXERCISE 13. *Show that for $x, y \in \operatorname{ob} \mathcal{C}$ one can choose the section so that the composition map*

$$\operatorname{Map}_{\mathcal{C}}(x,x) \times \operatorname{Map}_{\mathcal{C}}(x,y) \to \operatorname{Map}_{\mathcal{C}}(x,y)$$

sends the pair (id_x, f) to f for every $f \in \operatorname{Map}_{\mathcal{C}}(x,y)$.

EXERCISE 14. *Show that for any choice of composition map and three arrows $f \in \operatorname{Map}_{\mathcal{C}}(x,y)$, $g \in \operatorname{Map}_{\mathcal{C}}(y,z)$ and $h \in \operatorname{Map}_{\mathcal{C}}(x,z)$, there is a homotopy $h \sim g \circ f$ if and only if there exists a 2-simplex $\sigma : \Delta^2 \to \mathcal{C}$ such that $\partial_0 \sigma = g$, $\partial_1 \sigma = h$ and $\partial_2 \sigma = f$.*

DEFINITION 1.53. *Let $\mathcal{C}$ be an ∞-category. Its **homotopy category** $h\mathcal{C}$ is the ordinary category with set of objects the set of 0-simplices $\operatorname{ob} \mathcal{C}$ of $\mathcal{C}$ and with morphisms the homotopy classes of morphisms in $\mathcal{C}$, that is*

$$\operatorname{Hom}_{h\mathcal{C}}(x,y) := \pi_0 \operatorname{Map}_{\mathcal{C}}(x,y)\,.$$

Composition is defined by the map induced on π_0 by the map of construction 1.52.

EXAMPLE 1.54. *If $\mathcal{C} = \operatorname{Sing} Y$ for a topological space Y, then $h\mathcal{C}$ is the fundamental groupoid of Y. More generally we will call hX the fundamental groupoid of X for X a Kan complex.*

EXAMPLE 1.55. *Let $\mathscr{S} := N^{\Delta}(\mathbf{Kan})$ be the coherent nerve of the Kan-enriched category of Kan complexes. We will refer to $\mathscr{S}$ as the **∞-category of spaces**. Then $h\mathscr{S}$ is equivalent to the category of CW-complexes and homotopy classes of maps. Similarly we can define the ∞-category of pointed spaces $\mathscr{S}_*$.*

DEFINITION 1.56. *Let $\mathcal{C}$ be an ∞-category and $f : x \to y$ be an arrow in $\mathcal{C}$. Then we say that f is an equivalence when its class in $h\mathcal{C}$ is an isomorphism. Equivalently, f is an equivalence when there exists an arrow $g : y \to x$ such that $fg \sim \mathrm{id}_y$ and $gf \sim \mathrm{id}_x$.*

PROPOSITION 1.57. *Let $\mathcal{C}$ be an ∞-category and $f : \Delta^1 \to \mathcal{C}$ be an arrow in $\mathcal{C}$. Then the following are equivalent:*

- *f is an equivalence;*
- *f can be extended to a map $N(\bar{\Delta}^1) \to \mathcal{C}$ where $\bar{\Delta}^1$ is the groupoid with two objects and four arrows;*
- *For every map $\sigma_0 : \Lambda^n_0 \to \mathcal{C}$ with $\sigma_0|_{\Delta^{0,1}} = f$, there exists an extension $\sigma : \Delta^n \to \mathcal{C}$.*

PROOF. MISSING! $\square$

COROLLARY 1.58. *Let $\mathcal{C}$ be an ∞-category. Then $\mathcal{C}$ is a Kan complex if and only if $h\mathcal{C}$ is a groupoid, that is if all arrows in $\mathcal{C}$ are equivalences.*

Due to the previous corollary the Kan complexes are also sometimes called **∞-groupoids**.

CONSTRUCTION 1.59. *If $\mathcal{C}$ is an ∞-category we can construction its **interior** as the subsimplicial set $\iota\mathcal{C} \subseteq \mathcal{C}$ given by all n-simplices $\sigma : \Delta^n \to \mathcal{C}$ such that every 1-face of σ is an equivalence. This is the biggest Kan complex contained in $\mathcal{C}$ (i.e. every other simplicial subset of $\mathcal{C}$ is a Kan complex, then it is contained in $\iota\mathcal{C}$.*

DEFINITION 1.60. *Let $\mathcal{C}$ be an ∞-category and $F, G : S \to \mathcal{C}$ be two functors from a simplicial set S. Then a natural transformation $F \Rightarrow G$ is just an arrow in the ∞-category $\operatorname{Fun}(S, \mathcal{C})$ of corollary 1.47.*

EXERCISE 15. *Show that if $\mathcal{C}, \mathcal{D}$ are categories there's a natural isomorphism of simplicial sets*

$$N\,\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \cong \mathrm{Fun}(N\mathcal{C}, N\mathcal{D})$$

where $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is the category of functors and natural transformations. In particular the notion of natural transformation we have given here recovers the classical one.

DEFINITION 1.61. *Let* **qCat** *be the simplicial category whose objects are the ∞-categories and such that the mapping spaces are*

$$\mathrm{Map}_{\mathbf{qCat}}(\mathcal{C}, \mathcal{D}) = \iota\,\mathrm{Fun}(\mathcal{C}, \mathcal{D})\,,$$

i.e. we take natural equivalences as our homotopies. Then its coherent nerve is the ∞-category of ∞-categories $\mathrm{Cat}_\infty := N^\Delta(\mathbf{qCat})$.

REMARK 1.62. *Unwrapping the previous definition, a functor $F : \mathcal{C} \to \mathcal{D}$ between ∞-categories is an equivalence if and only if there is a functor $G : \mathcal{D} \to \mathcal{C}$ such that FG is naturally equivalent to $\mathrm{id}_\mathcal{D}$ and GF is naturally equivalent to $\mathrm{id}_\mathcal{C}$. One can show that this is equivalent to $hF : h\mathcal{C} \to h\mathcal{D}$ being essentially surjective and F inducing homotopy equivalences on mapping spaces.* **Denis:** elaborate?

REMARK 1.63. *One could consider also a variant of* **qCat** *where we take the full functor category, without restricting to the interior. Then its coherent nerve would not be an ∞-category anymore but rather the $(\infty, 2)$-category of ∞-categories. For more details on these ideas see XXX.*

DEFINITION 1.64. *Let $\mathcal{C}$ be an ∞-category and $x \in \mathrm{ob}\,\mathcal{C}$ be an object of $\mathcal{C}$. Then we say that*

- *x is **initial** if for every $y \in \mathrm{ob}\,\mathcal{C}$ the Kan complex $\mathrm{Map}_\mathcal{C}(x, y)$ is contractible;*
- *x is **terminal** if for every $y \in \mathrm{ob}\,\mathcal{C}$ the Kan complex $\mathrm{Map}_\mathcal{C}(y, x)$ is contractible.*

EXAMPLE 1.65. *Let $\mathcal{C}$ be a category. Then an object of $\mathcal{C}$ is initial (resp. terminal) in the ∞-category $N\mathcal{C}$ if and only if it is initial (resp. terminal) in $\mathcal{C}$.*

EXAMPLE 1.66. *Let X be a Kan complex. Then X has an initial (resp. terminal) object if and only if it is contractible.*

DEFINITION 1.67. *Let S be a simplicial set. Then its right cone $S^\triangleright$ is the simplicial set*

$$[n] \mapsto \{(f, \sigma) \mid f : \Delta^n \to \Delta^1,\ \sigma : f^{-1}(0) \to S\}$$

EXERCISE 16. *If $\mathcal{C}$ is an ∞-category, then the objects of $\mathcal{C}^\triangleright$ are the set $\mathrm{ob}\,\mathcal{C} \cup \{\infty\}$ obtained by adding an object ∞ to the set of objects of $\mathcal{C}$. The mapping spaces can be computed as*

$$\mathrm{Map}_{\mathcal{C}^\triangleright}(x, y) = \begin{cases} \mathrm{Map}_\mathcal{C}(x, y) & \text{if } x, y \in \mathrm{ob}\,\mathcal{C}; \\ \Delta^0 & \text{if } y = \infty \\ \varnothing & \text{if } x = \infty, y \neq \infty \end{cases}.$$

EXAMPLE 1.68. *If $S = \Lambda^2_0$ (i.e. the nerve of the poset $\{b < a > c\}$), then $S^\triangleright \cong \Delta^1 \times \Delta^1$. More generally if S is the nerve of a poset, then $S^\triangleright$ is the nerve of the poset obtained by adding an element ∞ which is bigger than all the others.*

DEFINITION 1.69. *Let $\mathcal{C}$ be an ∞-category and $p : S \to \mathcal{C}$ be a diagram. A **colimit** of p is a diagram $\bar{p} : S^\triangleright \to \mathcal{C}$ such that $\bar{p}|_S = p$ and it is an initial object of the ∞-category*

$$\mathrm{Fun}(S^\triangleright, \mathcal{C}) \times_{\mathrm{Fun}(S, \mathcal{C})} \{p\}\,.$$

That is it is such that for every diagram $\bar{q} : S^{\triangleright} \to \mathcal{C}$ such that $\bar{q}|_S = p$, then the space of natural transformations $\bar{p} \Rightarrow \bar{q}$ that restrict to the identity on S is contractible.

EXAMPLE 1.70. *If $\mathcal{C}$ is the nerve of a category, this notion of colimit coincides with the notion of colimit in ordinary category theory.*

EXAMPLE 1.71. *If $S = \Lambda^2_0$, this notion of colimit in $\mathscr{S}$ coincides with the notion of homotopy colimit discussed in algebraic topology II■.*

Denis: Insert details!

REMARK 1.72. *By considering the left cone $S^{\triangleleft}$ instead there is a dual notion of* ***limit*** *in ∞-categories.*

DEFINITION 1.73. *Let $\mathcal{C}$ an ∞-category. Then a* ***zero object*** $0 \in \operatorname{ob}\mathcal{C}$ *is an object that is both initial and terminal. If $\mathcal{C}$ has a zero object we say that $\mathcal{C}$ is* ***pointed****.*

DEFINITION 1.74. *Let $\mathcal{C}$ be a pointed ∞-category. Then the* ***suspension*** Σx *of an object $x \in \operatorname{ob}\mathcal{C}$ is the pushout of the diagram*

$$\begin{tikzcd} x \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma x \end{tikzcd} .$$

Dually the ***loopspace*** Ωx *is the pullback of the diagram*

$$\begin{tikzcd} \Omega x \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & x \end{tikzcd}$$

EXAMPLE 1.75. *Let $X : N\Delta^{\mathrm{op}} \to \mathcal{C}$ be a functor into an ∞-category $\mathcal{C}$. Then we call the colimit of X (if it exists) the* ***geometric realization*** *of X. This terminology is due to the following example: suppose we have a functor*

$$Y : \Delta^{\mathrm{op}} \to \mathrm{Top}$$

into the category of topological spaces such that for every face map $\partial_i : [n-1] \to [n]$ the induced map $Y(\partial_i)$ is a closed cofibration. Then we can construct the composition

$$X : N\Delta^{\mathrm{op}} \xrightarrow{NY} N^{\Delta}\mathrm{Top} \xrightarrow{N^{\Delta}\,\mathrm{Sing}} \mathbf{Kan} \cong \mathscr{S}\,.$$

The colimit of X is represented by the following topological space

$$\coprod_{n \geq 0} (Y([n]) \times |\Delta^n|)/\{(f^*x, t) \sim (x, f_*t) \mid f : [n] \to [m], x \in X([m]), t \in |\Delta^n|\}$$

Compare with example 1.6.

EXAMPLE 1.76. *Suppose $\mathbf{C}$ is a Kan-enriched category with all coproducts. Then $N^{\Delta}(\mathbf{C})$ has all coproducts and they are given by the coproducts in $\mathbf{C}$. This example works also if we restrict to a smaller set of coproducts.*

EXAMPLE 1.77. *Suppose $\mathbf{C}$ is a Kan enriched category that is tensored over simplicial sets. That is for every simplicial set S and object $x \in \operatorname{ob}\mathbf{C}$ there is an object $S \otimes x$ such that there exists a natural equivalence*

$$\mathrm{Map}_{\mathbf{C}}(S \otimes x, y) \cong \mathrm{Map}(S, \mathrm{Map}_{\mathbf{C}}(x, y))\,.$$

For example Top *is such a Kan enriched category, with the tensoring $S \otimes x \cong |S| \times x$. Suppose moreover that $\mathbf{C}$ has pushouts. Then for every pushout diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \\ C & \end{tikzcd}$$

in $N^\Delta\mathbf{C}$, the (homotopy) pushout exists and it is given by

$$B \amalg^h_A C \cong B \amalg_A (\Delta^1 \otimes A) \amalg_A C\,.$$

Theorem 1.78 (Bousfield-Kan formula). *Let $\mathcal{C}$ be an ∞-category admitting coproducts and geometric realizations and let $X : S \to \mathcal{C}$ be a diagram where S is a simplicial set. Then $\operatorname{colim}_S X$ exists and it is given by the geometric realization of the simplicial object in $\mathcal{C}$*

$$[n] \mapsto \coprod_{\sigma \in S([n])} X(\sigma(0))\,.$$

Proof. This is [**Sha18**, Corollary 12.3]. □

Denis: Worth adding the proof in an appendix? It's not long but it requires saying what a Kan extension is

Remark 1.79. *One can show that an ∞-category has all (finite) colimits if and only if it has all pushouts and all (finite) coproducts. Therefore by example 1.76 and example 1.77 one can show that the ∞-categories $\mathscr{S}$ and Cat_∞ have all colimits (and, by a dual argument, all limits).*

Definition 1.80. *Let $\mathcal{C}, \mathcal{D}$ be two ∞-categories. An adjunction between $\mathcal{C}$ and $\mathcal{D}$ is a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ plus a natural equivalence*

$$\mathrm{Map}_{\mathcal{D}}(Fx, y) \cong \mathrm{Map}_{\mathcal{C}}(x, Gy)$$

of functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathscr{S}$.

Remark 1.81. *As in classical category theory, giving an adjunction is the same as giving two natural transformations*

$$\eta : 1_{\mathcal{C}} \to GF \qquad \epsilon : FG \to 1_{\mathcal{D}}$$

together with two homotopies implementing the triangular identities:

$$\epsilon F \circ F\eta \cong 1_F \qquad \eta G \circ \epsilon G \cong 1_G$$

(note that now however the triangular identities are additional data). This can be proven using the ∞-categorical Yoneda lemma.

Discuss colimits in functor categories (w/out proof)

Definition 1.82. *A simplicial set S is filtered if for any finite simplicial set K and every map $f : K \to S$ there is an extension $\bar{f} : K^\triangleright \to S$.*

Example 1.83. *If $S = NP$ for P a poset, S is filtered if and only if for every $p, p' \in P$ there exists $q \in P$ such that $q \geq p$ and $q \geq p'$.*

Example 1.84. *If $S = N\mathcal{C}$ for $\mathcal{C}$ a category, S is filtered if and only if the following two conditions are satisfied:*

- *For every $x, x' \in \operatorname{ob}\mathcal{C}$ there exists $y \in \mathcal{C}$ and maps $f : y \to x$ and $f' : y \to x'$;*
- *For every two maps $f, f' : x \to x'$ in $\mathcal{C}$ there exists a map $g : z \to x$ such that $fg = f'g$.*

Exercise 17. *Let S be a filtered simplicial set. Then the space $\mathrm{Sing}\,|S|$ is contractible.*

Proposition 1.85. *In the ∞-category $\mathscr{S}$ filtered colimits commute with pullbacks. That is, for S a filtered simplicial set the subcategory of colimit diagrams in $\mathrm{Fun}(S^\triangleright, \mathscr{S})$ is closed under pullbacks.*

Proof. This follows from [**Lur09**, Proposition 5.3.3.3]. □

Denis: Can we perhaps find a direct argument deducing it from some other property of $\mathscr{S}$ (perhaps Whitehead)? Ultimately this should follow from the compact generation of $\mathscr{S}$

Proposition 1.86. *The forgetful functor $U : \mathscr{S}_* \to \mathscr{S}$ commutes with filtered colimits.*

PROOF. MISSING! In the end this comes from the fact that U is a right adjoint and that filtered colimits of contractible diagrams at the point are contractible. $\square$

COROLLARY 1.87. *The functor $\pi_0 : \mathscr{S} \to \mathrm{Set}$ commutes with all colimits, and the functor $\pi_n : \mathscr{S}_* \to \mathrm{Set}$ commutes with filtered colimits.*

PROOF. The functor π_0 commutes with filtered colimits because it is the left adjoint of the fully faithful inclusion $\mathrm{Set} \subseteq \mathscr{S}$. Then the functor π_n can be written as the composition

$$\mathscr{S}_* \xrightarrow{\Omega^n} \mathscr{S}_* \xrightarrow{U} \mathscr{S} \xrightarrow{\pi_0} \mathrm{Set}$$

where $\Omega : \mathscr{S}_* \to \mathscr{S}_*$ commutes with filtered colimits by proposition 1.85. $\square$

ADD THAT SEQUENTIAL COLIMITS ARE COMPUTED AS A PUSHOUT OF COPRODUCTS

A1. Simplicial approximation (INCOMPLETE!)

Let P be a poset. Its barycentric subdivision $\mathrm{sd}\, P$ is the poset of non-empty totally ordered subsets of P. The last vertex map is the map of posets $\mathrm{sd}\, P \to P$ sending a subset to its maximum.

If $S \subseteq N(P)$ is a simplicial subset, we let $\mathrm{sd}\, S \subseteq \mathrm{sd}\, P$ to be the subposet of those simplices that are contained in S

LEMMA 1.88. *There's a natural homeomorphism*

$$|P| \cong |\,\mathrm{sd}\, P|$$

sending a vertex of $|\,\mathrm{sd}\, P|$ corresponding to a subset $A \subseteq P$ to the baricenter of $|\Delta^{\#A}| \to |P|$.

LEMMA 1.89. *Let X be a Kan complex and $f : N(\mathrm{sd}\, P) \to X$ be a map of simplicial sets. Then f is homotopic to a map factoring through the last vertex map. Moreover if $S \subseteq N(P)$ is such that $f|_{N(\mathrm{sd}\, S)}$ factors through the last vertex map, we can choose the homotopy so that it is relative to $N(\mathrm{sd}\, S)$.*

PROOF. Let us consider the poset $\tilde{\mathrm{sd}}(P)$ whose elements are either pairs $(A, 0)$ with A a finite totally ordered subset of P, or $(p, 1)$ where $p \in P$. We equip it with the ordering

$$(A, 0) \le (B, 0) \Leftrightarrow A \subseteq B \qquad (p, 1) \le (q, 1) \Leftrightarrow p \le q \qquad (A, 0) \le (p, 1) \Leftrightarrow \max A \le p$$

Then it is clear that there is a map $N(\mathrm{sd}(P)) \times \Delta^1 \to N(\tilde{\mathrm{sd}}(P))$ which is the identity on $N(\mathrm{sd}(P)) \times \{0\}$ and the last vertex map on $N(\mathrm{sd}(P)) \times \{1\}$. Let $f : N(\mathrm{sd}\, P) \to X$ be a map such that $f|_{N(\mathrm{sd}\, S)}$ factors through the last vertex map. Then we can build a map

$$N(\mathrm{sd}\, P) \cup N(\tilde{\mathrm{sd}} S) \to X$$

and our goal is to extend it to $N(\tilde{\mathrm{sd}} P)$. $\square$

PROPOSITION 1.90.

THEOREM 1.91 (Simplicial approximation). *Let X be a simplicial set, $A \subseteq X$ be a simplicial subset, and K be a Kan complex. If $g_0 : A \to K$ is a map of simplicial sets and $f : |X| \to |K|$ be a map of topological spaces such that $g|_{|A|} = |g_0|$, then there exists $g : X \to K$ map of simplicial sets such that $g|_A = g_0$ and a homotopy $H : |g| \sim f$ relative to $|A|$.*

PROOF. $\square$

$\boxed{\textbf{UNFINISHED}}$

A2. Existence results for homotopy colimits

LEMMA 1.92. *Let S, T be two simplicial sets and $\mathcal{C}$ be an ∞-category. Let $F : S \times T \to \mathcal{C}$ be a diagram such that for every $s \in S$ the colimit of $F|_{\{s\}\times T}$ exists. Then there exists a diagram $\operatorname{colim}_T F : S \to \mathcal{C}$ sending $s \in S$ to $\operatorname{colim}_{\{s\}\times T} F$ and*

$$\operatorname*{colim}_{S\times T} F \cong \operatorname*{colim}_{S} \operatorname*{colim}_{T} F$$

in the sense that either side exists if the other does and they coincide.

LEMMA 1.93. *Let S, T, R be simplicial sets with maps $R \to S$ and $R \to T$. Then there is a natural isomorphism of simplicial sets*

$$(S \amalg_T R)^{\triangleright} \cong S^{\triangleright} \amalg_{T^{\triangleright}} R^{\triangleright} .$$

LEMMA 1.94. *An object is initial (terminal) in $\mathcal{C} \times_{\mathcal{C}'} \mathcal{C}''$ if and only if ...*

PROPOSITION 1.95. *Let S, T, R be simplicial sets and $F : S \amalg_R T \to \mathcal{C}$ be a diagram. Suppose $\operatorname{colim}_S F$, $\operatorname{colim}_T F$ and $\operatorname{colim}_R F$ exist and that $\mathcal{C}$ has all pushouts. Then $\operatorname{colim}_{S\amalg_R T} F$ exists and the following square*

$$\begin{tikzcd}
\operatorname{colim}_R F \arrow[r] \arrow[d] & \operatorname{colim}_S F \arrow[d] \\
\operatorname{colim}_T F \arrow[r] & \operatorname{colim}_{S\amalg_R T} F
\end{tikzcd}$$

is a pushout square.

THEOREM 1.96. *Let $\mathcal{C}$ be an ∞-category with all coproducts and pushouts. Then $\mathcal{C}$ has all colimits.*

PROOF. Let S be a simplicial set and $F : S \to \mathcal{C}$ be a diagram. For every $n \geq 0$, let $S^{(n)} \subseteq S$ be the n-skeleton of S: it is the simplicial subset of S generated by all non-degenerated n-simplices. We say that S is n-dimensional if $S = S^{(n)}$. We will show by induction that $\mathcal{C}$ has colimits for all n-dimensional diagrams for every n. The base case $n = 0$ follows from the coproduct. Suppose that $\mathcal{C}$ has all colimits for all n-dimensional diagrams, let us show that it has colimits for all $(n+1)$-dimensional diagrams

Then there's a pushout diagram in simplicial sets

$$\begin{tikzcd}
S([n+1]) \times \Delta^{n+1} \arrow[r] \arrow[d] & S^{(n)} \arrow[d] \\
S([n+1]) \times \Delta^{n} \arrow[r] & S^{(n+1)}
\end{tikzcd}$$

so by proposition 1.95 it suffices to show that it has colimits for all the other vertices of the square. But the top right and top left vertices are n-dimensional, so we can apply the induction hypothesis. $\square$

EXERCISE 18. *Let X be a normal space, and $U, V \subseteq X$ open subsets such that $U \cup V = X$. Then the square*

$$\begin{tikzcd}
U \cap V \arrow[r] \arrow[d] & U \arrow[d] \\
V \arrow[r] & X
\end{tikzcd}$$

is a homotopy pushout square. (Hint: use Urysohn lemma to construct a section s of the projection $p : U \cup_{(U\cap V)\times\{0\}} (U \cap V) \times [0,1] \cup_{(U\cap V)\times\{1\}} V \to X$ and do a straight-line homotopy to show that $sp \sim \mathrm{id}$.

$\boxed{\textbf{UNFINISHED}}$

CHAPTER 2

The ∞-category of spectra

This chapter is mostly based on chapter 1 of [**Lur17**]. Another classical reference for these ideas is Adams' blue book [**Ada74**].

1. Cohomology theories and Brown representability

Let $\mathscr{S}_*$ be the ∞-category of spaces. Our goal is to describe axiomatically what a cohomology theory is

Definition 2.1. [1] *A* ***cohomology theory*** *is a pair* (E^*, ∂) *where*

$$E^* : h\mathscr{S}_*^{\mathrm{op}} \to \mathrm{grAb}$$

is a functor from the homotopy category of pointed spaces to graded abelian groups and

$$\partial : E^* \cong E^{*+1} \circ \Sigma$$

is a natural isomorphism, satisfying the following conditions

(1) *For every small collection of spaces* $\{X_\alpha\}_{\alpha \in A}$ *the natural map*

$$E^*\left(\coprod_{\alpha \in A} X_\alpha\right) \to \prod_{\alpha \in A} E^*(X_\alpha)$$

is an isomorphism. In particular $E^*(*) \cong 0$.

(2) *For any cofiber sequence*

$$X' \to X \to X''$$

the sequence

$$E^*(X'') \to E^*(X) \to E^*(X')$$

is exact.

Example 2.2. *The functor sending a pointed space* X *to its reduced ordinary cohomology* $\tilde{H}^*(X)$ *is a cohomology theory.*

In this class we will see many more examples of cohomology theories. To name a few, topological K-theory, cobordism etc.

Remark 2.3. *If* E^* *is a cohomology theory and*

$$X' \to X \to X''$$

is a cofiber sequence, we know that

$$E^n(X'') \to E^n(X) \to E^n(X')$$

is exact in the middle. But we can "shift" the cofiber sequence to

$$X \to X'' \to \Sigma X'$$

and so we get that the sequence

$$E^n(\Sigma X') \to E^n(X'') \to E^n(X)$$

[1] see Eilenberg-Steenrod, [**Lur17**, Definition 1.4.1.6]

is exact in the middle. Using $\partial : E^n(\Sigma X') \cong E^{n+1}(X')$ *we obtain an exact sequence*

$$E^{n+1}(X') \to E^n(X'') \to E^n(X) \to E^n(X')$$

and inducting and letting n *vary we recover a long exact sequence*

$$\cdots \to E^{n+1}(X) \to E^{n+1}(X') \to E^n(X'') \to E^n(X) \to E^n(X') \to E^{n-1}(X'') \to \cdots .$$

Remark 2.4. *From a cohomology theory in the sense above we can recover a cohomology theory on pairs by setting*

$$E^*(X, A) := E^*(X/A)\,,$$

where X/A *is pointed by* A/A*. Recall that* $X/\varnothing = X_+$*. So the previous construction applied to the cofiber sequence of pointed spaces*

$$A_+ \to X_+ \to X/A$$

recovers the long exact sequence of a pair.

Our goal in this section is to classify all possible cohomology theories. In order to do so we will use the Brown representability theorem. Recall that a functor is called representable

Theorem 2.5 (Brown representability theorem). *Let* $h\mathscr{S}_*^{\geq 0}$ *be the homotopy category of connected pointed spaces. Then a functor* $F : (h\mathscr{S}_*^{\geq 0})^{\mathrm{op}} \to \mathrm{Set}$ *is representable if and only if it has the following two properties:*

(1) *For every collection* $\{X_\alpha\}_{\alpha \in A}$ *of connected pointed spaces the map*

$$F\left(\bigvee_{\alpha \in A} X_\alpha\right) \to \prod_{\alpha \in A} F(X_\alpha)$$

is a bijection.

(2) *For every pushout square*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Y' \end{array}$$

in $\mathscr{S}_*^{\geq 0}$*, the map*

$$F(Y') \to F(X') \times_{F(X)} F(Y)$$

is surjective.

Proof. One direction is clear, by the universal property of the wedge and of the pushout. We will then prove the other.

We will first prove the following property of the functor F:

Claim: For every X connected pointed space and $\xi \in F(X)$ there exists and arrow $f_X : X \to Z_X$ and a class $\tilde{\xi} \in F(Z_X)$ such that $f_X^* \tilde{\xi} = \xi$ and for every $m \geq 1$

$$[S^m, Z_X]_* \to F(S^m)$$

sending f to $f^* \xi_X$ is a bijection.

First let us define

$$Z_0 = X \vee \bigvee_{m \geq 0,\ \gamma \in F(S^m)} S^m$$

by condition 1 we can pick the element

$$\xi_0 := (\xi, \gamma) \in F(X) \times \prod_{m \geq 0,\ \gamma \in F(S^m)} F(S^m) \cong F(Z_0)$$

and the map $f_0 : X \to Z_0$ including the summand has the property that $f_0^*\xi_0 = \xi$ by construction.

By Yoneda this induces a map

$$[-, Z_0]_* \to F(-)$$

sending f to $f^*\xi_0$ which is surjective on every sphere and sends f_0 to ξ. Our goal is to fix it so that it is bijective on every sphere. We will do so by constructing a sequence of spaces

$$Z_0 \xrightarrow{r_0} Z_1 \xrightarrow{r_1} Z_2 \xrightarrow{r_2} \cdots$$

and classes $\eta_n \in F(Z_n)$ such that $r_i^*\eta_{i+1} = \eta_i$. We will do this by induction on n. Suppose Z_n has been constructed and let K_n be the functor sending X to the kernel of

$$[X, Z_n]_* \to F(X)\,.$$

Then we let Z_{n+1} be the pushout

$$\begin{tikzcd} \bigvee_{m\geq 0,\gamma\in K_n(S^m)} S^m \arrow[r] \arrow[d] & * \arrow[d] \\ Z_n \arrow[r] & Z_{n+1} \end{tikzcd}$$

Note that pulling back along the left vertical arrow sends ξ_n to 0 by definition and so by condition (ii) we can find a lift $\xi_{n+1} \in F(Z_{n+1})$. Let Z_X be the colimit of Z_n. Since we can write it as a pushout

$$\begin{tikzcd} \bigvee_n Z_n \arrow[r] \arrow[d] & \bigvee_n Z_{2n} \arrow[d] \\ \bigvee_n Z_{2n+1} \arrow[r] & Z_X \end{tikzcd}$$

by condition (ii) we can lift $\{\xi_n\}$ to a class $\tilde{\xi} \in F(Z)$. This induces a natural transformation

$$[-, Z_X]_* \to F(-)$$

that sends the composition $X \to Z_0 \to Z_X$ to ξ. We want to show that it is bijective on every sphere. It is clearly surjective, since the composition

$$[S^m, Z_0]_* \to [S^m, Z]_* \to F(S^m)$$

is surjective by construction. Now let $\psi \in [S^m, Z]_* = \pi_m Z$ in the kernel. Since homotopy groups commute with filtered colimits (REF!) we have that ψ lifts to some $\pi_m Z_n$ for finite n. But then $\psi \in K_n(S^m)$ and so it is sent to 0 to $\pi_m Z_{n+1}$. Therefore $\psi = 0$ in $\pi_m Z_X$. Thus we have proven the claim.

Let now $Z := Z_*$ (where $\xi = 0 \in F(*) = 0$) and let us denote with η the corresponding class. We want to show that the map

$$[X, Z] \to F(X)$$

sending $f : X \to Z$ to $f^*\eta$ is bijective. We know this is the case for $X = S^m$ by the claim. Let us show that it is surjective for every X. Let $\xi \in F(X)$ we need to show that it is pulled back from Z. We can consider the class $(\xi, \eta) \in F(X) \times F(Z) = F(X \amalg Z)$ and apply the claim to this class. Then we find $\widetilde{Z}$ with a map $X \amalg Z \to \tilde{Z}$ and a lift $\tilde{\xi}$ of (ξ, η). But notice that the map $Z \to \tilde{Z}$ induces a commutative diagram

$$\begin{tikzcd} {[S^m, Z]_* \cong \pi_m Z} \arrow[r] \arrow[d] & {[S^m, \tilde{Z}]_* \cong \pi_m \tilde{Z}} \arrow[dl] \\ F(S^m) & \end{tikzcd}$$

where the vertical arrows are bijections. Therefore the map $Z \to \tilde{Z}$ is an equivalence and the composite $X \to \tilde{Z} \xleftarrow{\sim} Z$ pulls back η to ξ.

Let us now show injectivity. Let $f, g : X \to Z$ such that $\xi := f^*\eta = g^*\eta$. We can then form the pushout diagram

$$\begin{tikzcd} X \amalg X \arrow[r, "{(f,g)}"] \arrow[d] & Z \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

and by condition (ii) we can find $\psi \in F(Y)$ such lifting η and ξ. Let us apply the claim to (Y, ψ). Then we can find $(\tilde{Z}, \tilde{\psi})$ with a map $Y \to \tilde{Z}$ lifting ψ such that

$$\pi_m Y \to F(S^m)$$

is a bijection. But proceeding as before we see that the map $Z \to \tilde{Z}$ is an equivalence and therefore f and g are homotopic (since $X \amalg X \to Z$ factors through the fold map). $\square$

Corollary 2.6. *Let $E^* : \mathscr{S}_*{}^{\mathrm{op}} \to \mathrm{grAb}$ be a cohomology theory. Then there exist a (unique) collection of pointed spaces $E_n \in \mathscr{S}_*$ and homotopy equivalences*

$$\delta_n : E_n \xrightarrow{\sim} \Omega E_{n+1}$$

such that there are natural isomorphisms

$$\varphi_n : E^n(X) \cong [X, E_n]_*$$

such that the isomorphisms $\partial : E^n(X) \cong E^{n+1}(X)$ are given by

$$E^n(X) \cong [X, E_n]_* \cong [X, \Omega E_{n+1}]_* \cong [\Sigma X, E_{n+1}]_* \cong E^{n+1}(\Sigma X)\,.$$

Proof. First we need to show that the functor $X \mapsto E^n(X)$ satisfies the hypotheses of the Brown representability theorem. The first hypothesis is obvious, so we need to show the second. Let

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

be a pushout square. Then the vertical map induces an equivalence of cofibers $Y/X \xrightarrow{\sim} Y'/X'$ and we can write a diagram of long exact squences

$$\begin{tikzcd} E^n(Y'/X') \arrow[r] \arrow[d, "\wr"] & E^n(Y') \arrow[r] \arrow[d] & E^n(X') \arrow[r] \arrow[d] & E^{n-1}(Y'/X') \arrow[d, "\wr"] \\ E^n(Y/X) \arrow[r] & E^n(Y) \arrow[r] & E^n(X) \arrow[r] & E^{n-1}(Y/X) \end{tikzcd}$$

Let us take $\alpha \in E^n(X')$ and $\beta \in E^n(Y)$ such that $(g')^*\alpha = f^*\beta$. Our goal is to find a class $\gamma \in E^n(Y')$ such that $(f')^*\gamma = \alpha$ and $g^*\gamma = \beta$. Now first notice that $\partial\beta = 0$ (since it becomes $\partial(g')^*\alpha = 0$ downstairs), so we can find $\gamma_0 \in E^n(Y')$ such that $(f')^*\gamma_0 = \beta$. Now if we let $\epsilon = \alpha = (f')^*\gamma_0$ we have that $\partial\epsilon = 0$, therefore we can lift it to $\tilde{\epsilon} \in E^n(Y/X) \cong E^n(Y'/X')$. But then it suffices to let $\gamma = \gamma_0 + (p')^*\tilde{\epsilon}$.

So there exist connected pointed spaces $\tilde{Z}_n$ and natural equivalences

$$E^n(X) \cong [X, \tilde{Z}_n]_*$$

for all *connected* pointed spaces X. But notice that for every pointed space X, the space ΣX is connected, so we have

$$E^n(X) \cong E^{n+1}(\Sigma X) \cong [\Sigma X, \tilde{Z}_{n+1}]_* \cong [X, \Omega \tilde{Z}_{n+1}]_* .$$

Therefore, letting $Z_n := \Omega \tilde{Z}_{n+1}$ we have that E^n is represented by Z_n on all pointed spaces.

Finally the existence and uniqueness of δ_n follows from the Yoneda lemma applied to the isomorphisms ∂. □

The corollary allows us to give the most important definition of this course.

DEFINITION 2.7. *A* ***spectrum*** *E is a pair $(\{E_n\}_{n\in\mathbb{Z}}, \delta_n)$ where $\{E_n\}_{n\in\mathbb{Z}}$ is a collection of pointed spaces and $\delta_n : E_n \xrightarrow{\sim} \Omega E_{n+1}$ is a family of equivalence of each space with the loopspace of the following one.*

More precisely, we can define the ∞-category of spectra as

$$\mathrm{Sp} \cong \lim \left(\mathscr{S}_* \xleftarrow{\Omega} \mathscr{S}_* \xleftarrow{\Omega} \mathscr{S}_* \xleftarrow{\Omega} \cdots \right)$$

Warning: while every cohomology theory is represented by a spectrum (and in fact every map of cohomology theories is represented by a map of spectra – exercise?) there are differnet maps of spectra that induce the same map on cohomology theories. These are the famous *phantom maps*. It'd be cool to add an example.

Warning: let E, F be two spectra. Then we can describe the mapping space as

$$\mathrm{Map}_{\mathrm{Sp}}(E, F) \cong \lim_n \mathrm{Map}_{\mathscr{S}_*}(E_n, F_n) .$$

That is a map of spectra is the datum of a family of maps of pointed sets $f_n : E_n \to F_n$ and homotopies $H_n : f_n \sim \Omega f_{n+1}$. On the other hand, Yoneda tells us that a map of cohomology theories $E^* \to F^*$ is given by a collection of homotopy classes $[f_n] \in [E_n, F_n]_*$ such that $[f_n] = [\Omega f_{n+1}]$.

Therefore every map of cohomology theories induces a map of spectra, but this is not injective: different choices of homotopies might induce different maps of spectra. We will see later an example where this happens: a non-zero map of spectra inducing the zero map on cohomology theories.

REMARK 2.8. *So far we have only talked about the relationship between spectra and* cohomology *theories. There is a similar story about* homology theories *on which we will return after we have discussed the symmetric monoidal structure on spectra. For now let us limit to note that if E is a spectrum we can define the E-homology groups of a pointed space X as follows*

$$E_n(X) := \operatorname*{colim}_{k\to\infty} \pi_{n+k}(E_k \wedge X) .$$

2. Examples of spectra

The spectrum 0 such that $0_n := *$ with the obvious bounding maps is a zero object for Sp. In fact

$$\mathrm{Map}_{\mathrm{Sp}}(0, E) = \lim_n \mathrm{Map}_{\mathscr{S}_*}(*, E_n) = * \qquad \mathrm{Map}_{\mathrm{Sp}}(E, 0) = \lim_n \mathrm{Map}_{\mathscr{S}_*}(E_n, *) = *$$

Applying the Brown representability theorem to ordinary homology $\tilde{H}^*(-; M)$, we obtain a representing spectrum HM, which is called the **Eilenberg-MacLane spectrum**. This name is because

$$\pi_n(HM_m) = [S^n, HM_m]_* = \tilde{H}^m(S^n; M) = \begin{cases} M & \text{if } n = m \\ 0 & \text{otherwise.} \end{cases}$$

Denis: when?

Therefore HM_m is the Eilenberg-MacLane space $K(M,m)$. We will see later that $H-$ is a fully faithful functor from abelian groups to spectra.

If $E = (\{E_n\}_{n\in\mathbb{Z}}, \partial)$ is a spectrum we write

$$\Omega^\infty E := E_0$$

This induces a functor $\Omega^\infty : \mathrm{Sp} \to \mathscr{S}_*$, which is sometimes called the 0-th space of the spectrum. Let us try to construct the left adjoint.

If $X \in \mathscr{S}_*$ is a pointed space, we let

$$QX := \operatorname*{colim}_{m\in\mathbb{Z}} \Omega^m \Sigma^m X\,.$$

Since $\Omega : \mathscr{S}_* \to \mathscr{S}_*$ commutes with filtered colimits (REF!), we have a natural equivalence

$$QX \xrightarrow{\sim} \Omega Q \Sigma X$$

therefore we can construct a spectrum $\Sigma^\infty X$ whose n-th space is $Q\Sigma^n X$. We call this the **suspension spectrum** of X. There is a map of pointed spaces

$$X \to QX \cong \Omega^\infty \Sigma^\infty X\,.$$

Lemma 2.9. *Let $X \in \mathscr{S}_*$ be a pointed space and $E \in \mathrm{Sp}$ a spectrum. Then the transformation*

$$\mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty X, E) \to \mathrm{Map}_{\mathscr{S}_*}(\Omega^\infty\Sigma^\infty X, \Omega^\infty E) \to \mathrm{Map}_{\mathscr{S}_*}(X, \Omega^\infty E)$$

is an equivalence. Therefore the functor $\Omega^\infty : \mathrm{Sp} \to \mathscr{S}_*$ has a left adjoint sending X to $\Sigma^\infty X$.

Proof. We can write the mapping space

$$\mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty X, E) \cong \lim_n \mathrm{Map}_{\mathscr{S}_*}(Q\Sigma^n X, E_n)$$

and the map we claim is an equivalence is the projection onto the $n = 0$ component. Let us consider the following diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathscr{S}_*}(X, E_0) & & & \\
\mathrm{Map}_{\mathscr{S}_*}(\Omega\Sigma X, E_0) \arrow[u] & \mathrm{Map}_{\mathscr{S}_*}(\Sigma X, E_1) \arrow[l, "\Omega"] \arrow[ul, "\sim"'] & & \\
\mathrm{Map}_{\mathscr{S}_*}(\Omega^2\Sigma^2 X, E_0) \arrow[u] & \mathrm{Map}_{\mathscr{S}_*}(\Omega\Sigma^2 X, E_1) \arrow[l, "\Omega"] \arrow[u] & \mathrm{Map}_{\mathscr{S}_*}(\Sigma^2 X, E_2) \arrow[l, "\Omega"] \arrow[ul, "\sim"'] & \\
\vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[ul, "\sim"']
\end{tikzcd}\,.$$

Then taking the limit first in the vertical direction and then in the horizontal direction shows that the limit of the whole diagram is precisely $\mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty X, E)$, and the map we investigating is the projection on the top left corner. But all the diagonal maps going one step to the left and one step up are equivalences by the $\Sigma \dashv \Omega$ adjunction, and the leftmost diagonal map is cofinal. Therefore projecting onto the first coordinate is an equivalence, as required. $\square$

Probably the most important spectrum of all is the suspension spectrum of S^0. This is called the sphere spectrum and it is normally indicated as $\mathbb{S}$. By the previous argument we have

$$\mathrm{Map}_{\mathrm{Sp}}(\mathbb{S}, E) \cong \mathrm{Map}_{\mathscr{S}_*}(S^0, \Omega^\infty E) \cong \Omega^\infty E\,.$$

PROPOSITION 2.10. *Let $E : I \to \mathrm{Sp}$ be a diagram. Then $\lim_{i\in I} E(i)$ exists and it is given by the spectrum*

$$\left(\{\lim_{i\in I} E(i)_n\}_{n\in\mathbb{Z}}, \lim_{i\in I} \delta(i)_n \right) .$$

PROOF. First we need to check that the formula gives a spectrum, that is that $\lim_{i\in I} \delta(i)$ is an equivalence. But this is because we can write it as

$$\lim_{i\in I} E(i)_n \to \lim_{i\in I} \Omega E(i)_{n+1} \cong \Omega \lim_{i\in I} E(i)_{n+1} .$$

and all arrows in the diagram are equivalences.

Now let us see that this formula actually gives a limit. Let F be a spectrum. Then we can write

$$\mathrm{Map}_{\mathrm{Sp}}(F, \lim_{i\in I} E(i)) \cong \lim_n \mathrm{Map}_{\mathscr{S}_*}(F_n, \lim_{i\in I} E(i)_n) \cong \lim_n \lim_{i\in I} \mathrm{Map}_{\mathscr{S}_*}(F_n, E(i)_n) \cong$$
$$\cong \lim_{i\in I} \lim_n \mathrm{Map}_{\mathscr{S}_*}(F_n, E(i)_n) \cong \lim_{i\in I} \mathrm{Map}_{\mathrm{Sp}}(F, E(i)) .$$

□

COROLLARY 2.11. *Suspension and loop in* Sp *exist and are computed by shifting. In particular they are inverse equivalences.*

PROOF. By proposition 2.10 taking loops is done pointwise, and therefore it is equivalent to shifting. Then shifting in the opposite direction is obviously an inverse. In particular it has to be the left adjoint to Ω (since inverses are always both left and right adjoints to the original functor). Therefore the suspension exists. □

Due to the previous corollary we will often write Σ^{-n} to denote Ω^n for uniformity of notation.

PROPOSITION 2.12. *Let I be a filtered simplicial set and $E : I \to \mathrm{Sp}$ be a diagram. Then $\mathrm{colim}_{i\in I} E(i)$ exists and it is given by the spectrum*

$$\left(\{\mathrm{colim}_{i\in I} E(i)_n\}_{n\in\mathbb{Z}}, \mathrm{colim}_{i\in I} \delta(i)_n \right)$$

PROOF. The proof of this is the same as the proof of proposition 2.10 using the fact that Ω commutes with filtered colimits by proposition 1.85 and proposition 1.86. □

DEFINITION 2.13. *A prespectrum is a sequence $\{E_n\}_{n\in\mathbb{Z}}$ of spaces and maps $\delta_n : \Sigma E_n \to E_{n+1}$. If $\{E_n\}$ is a prespectrum the* ***associated spectrum*** *is the spectrum given by the formula*

$$E := \mathrm{colim}\left(\Sigma^\infty E_0 \cong \Omega\Sigma\Sigma^\infty E_0 \cong \Omega\Sigma^\infty \Sigma E_0 \to \Omega\Sigma^\infty E_1 \to \Omega^2\Sigma^\infty E_2 \to \cdots\right) .$$

REMARK 2.14. *One can construct an ∞-category of prespectra and see that* Sp *is a full subcategory. Then the associated spectrum is the left adjoint to the inclusion of spectra into prespectra. Informally we think of prespectra as "presentations" of our spectrum by generators and relations (where the spaces E_n play the role of the generators and the maps δ_n those of the relations).*

LEMMA 2.15 (Standard presentation). *Let E be a spectrum. Then the natural map*

$$\mathrm{colim}_n \Sigma^{-n}\Sigma^\infty E_n \to E$$

is an equivalence.

Proof. Let F be any other spectrum. We will show that $\mathrm{Map}_{\mathrm{Sp}}(-,F)$ turns the above map into an equivalence.

$$\mathrm{Map}_{\mathscr{S}_*}(\operatorname*{colim}_n \Sigma^{-n}\Sigma^\infty E_n, F) \cong \lim_n \mathrm{Map}_{\mathscr{S}_*}(\Sigma^{-n}\Sigma^\infty E_n, F) \cong \lim_n \mathrm{Map}_{\mathscr{S}_*}(\Sigma^\infty E_n \Sigma^n F) \cong$$
$$\cong \lim_n \mathrm{Map}_{\mathscr{S}_*}(E_n, \Omega^\infty\Sigma^n F) \cong \lim_n \mathrm{Map}_{\mathscr{S}_*}(E_n, F_n) \cong \mathrm{Map}_{\mathrm{Sp}}(E,F)\,.$$

□

Corollary 2.16. *The ∞-category of spectra has all colimits.*

Proof. Let $E : I \to \mathrm{Sp}$ be a diagram of spectra. Then for every n we get a diagram of spaces $E_n : I \to \mathscr{S}_*$. We claim that the following

$$\operatorname*{colim}_n \Sigma^{-n}\Sigma^\infty \left(\operatorname*{colim}_i E_n(i)\right)\,,$$

which exists because Sp has all filtered colimits and $\mathscr{S}_*$ is cocomplete, is the colimit of E. In fact for every $F \in \mathrm{Sp}$

$$\mathrm{Map}_{\mathrm{Sp}}\left(\operatorname*{colim}_n \Sigma^{-n}\Sigma^\infty(\operatorname*{colim}_i E_n(i)), F\right) \cong \lim_n \lim_i \mathrm{Map}_{\mathscr{S}_*}(E_n(i), \Sigma^n F) \cong \lim_i \mathrm{Map}_{\mathrm{Sp}}(E(i), F)\,.$$

□

Proposition 2.17 (Stability). *Let*

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_1 \arrow[d] \\ X_2 \arrow[r] & X_{12} \end{tikzcd}$$

be a square of spectra. Then it is a pushout square if and only if it is a pullback square.

Proof. We will show that if it is a pushout square then it is a pullback square. The other direction is similar. Let us consider the following diagram as the following diagram

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d] & 0 \arrow[d] & \\ X_2 \arrow[r] \arrow[d] & X_{12} \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \arrow[ddl] \\ 0 \arrow[r] & Z \arrow[r] \arrow[d] & \Sigma X_0 \arrow[r] \arrow[d] & \Sigma X_2 \arrow[d] \\ & 0 \arrow[r] \arrow[urr] & \Sigma X_1 \arrow[r] & \Sigma X_{12} \end{tikzcd}$$

where all the squares are pushout squares (so that Y is the cofiber of $X_1 \to X_{12}$ etc.). Looking at the maps induced on the limits of the highlighted diagrams we obtain a factorizations

$$\begin{tikzcd} X_0 \arrow[r] \arrow[dr] & X_1 \times_{X_{12}} X_2 \arrow[d] \arrow[dr] & \\ & \Omega\Sigma X_0 \arrow[r] & \Omega\Sigma X_1 \times_{\Omega\Sigma X_{12}} \Omega\Sigma X_2 \end{tikzcd}$$

where the diagonal arrows are equivalences by corollary 2.11. Therefore all the other arrows in the diagram are equivalences as well. □

EXERCISE 19. *The functor $Q : \mathscr{S}_* \to \mathscr{S}_*$ sends the point to the point and cocartesian squares to cartesian squares (such a functor is called a* ***linear functor****, the ∞-category of spectra is in fact equivalent to the ∞-category of linear functors that preserve filtered colimits).*

COROLLARY 2.18. *If E and F are two spectra, the canonical maps*

$$E \cong E \times 0 \to E \times F,\ F \cong 0 \times F \to E \times F$$

exhibit the product $E \times F$ as the coproduct. For this reason we will often write

$$E \oplus F := E \times F$$

and call it the ***direct sum*** *of E and F.*

PROOF. Let us consider the two cartesian squares

$$\begin{array}{ccc} 0 & \longrightarrow & E \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & E \end{array} \quad \text{and} \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ F & \longrightarrow & F \end{array} .$$

Then their product is also a cartesian square

$$\begin{array}{ccc} 0 & \longrightarrow & E \\ \downarrow & & \downarrow \\ F & \longrightarrow & E \times F \end{array}$$

But then it is also a cocartesian square by proposition 2.17, which proves the thesis. ☐

DEFINITION 2.19. *An* ***exact sequence*** *of spectra is a square*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & X'' \end{array}$$

that is both cartesian and cocartesian.

REMARK 2.20. *If one take the same definition of exact sequence in abelian groups, one recovers the notion of short exact sequence.*

DEFINITION 2.21. *Let E be a spectrum. Then its n-th homotopy group is the set*

$$\pi_n E := [\Sigma^n \mathbb{S}, E] \cong \pi_{n+m}\Omega^\infty \Sigma^m E \text{ for } n + m \geq 0\,.$$

LEMMA 2.22 (Whitehead's theorem for spectra). *A map $f : E \to F$ of spectra is an equivalence if and only if it is an equivalence on every homotopy group.*

PROOF. It is enough to show that the map $f_n : E_n \to F_n$ is an equivalence for every n. Note that since $\pi_* f_n$ is an equivalence, then f_n is an equivalence on the connected component of the basepoint. But then $\Omega f_n = f_{n-1}$ is an equivalence, this implies the thesis. ☐

DEFINITION 2.23. *Let E, F be two spectra. Then we can define a spectrum $\mathrm{map}_{\mathrm{Sp}}(E, F)$ such that*

$$\mathrm{map}_{\mathrm{Sp}}(E, F)_n := \mathrm{Map}_{\mathrm{Sp}}(E, \Sigma^n F)$$

and the bounding maps are the canonical equivalences

$$\Omega\,\mathrm{Map}_{\mathrm{Sp}}(E, \Sigma^{n+1} F) \cong \mathrm{Map}_{\mathrm{Sp}}(E, \Omega\Sigma^{n+1} F) \cong \mathrm{Map}_{\mathrm{Sp}}(E, \Sigma^n F)\,.$$

REMARK 2.24. *Note that if E is a spectrum the associated cohomology theory is given by*

$$E^n(X) = [X, E_n]_* \cong \pi_0 \operatorname{Map}_{\mathrm{Sp}}(\Sigma^\infty X, \Sigma^n E) \cong \pi_{-n} \operatorname{map}_{\mathrm{Sp}}(\Sigma^\infty X, E)\,.$$

By analogy with this, when F is a spectrum we define the E-cohomology of the spectrum as

$$E^n(F) := \pi_{-n} \operatorname{map}(F, E)\,.$$

LEMMA 2.25. *Let $E' \to E \to E''$ be an exact sequence of spectra, and let F be an arbitrary spectrum. Then*

$$[E'', F] \to [E, F] \to [E', F]$$

is an exact sequence of abelian groups.

PROOF. Note that $[-, F] = \pi_0 \operatorname{map}(-, F)$ by definition and $\operatorname{map}(-, F)$ preserves exact sequences because it is a right adjoint functor (right adjoint to $\operatorname{map}(-, F)^{\mathrm{op}}$). Therefore it is enough to show that $\pi_0(-) := \pi_0\Omega^\infty(-)$ sends exact sequences to exact sequences of pointed sets. But $\Omega^\infty(-)$ sends exact sequences to fiber sequences, since it's a right adjoint, so the thesis follows from the long exact sequence of a fibration. □

DEFINITION 2.26 (Tensor product of spectra). *If E and F are two spectra, we define their tensor product, or smash product, as*

$$E \otimes F := \operatorname*{colim}_{n,m} \Sigma^{-n-m}\Sigma^\infty(E_n \wedge F_m)$$

where $- \wedge -$ is the smash product of pointed spaces.

DEFINITION 2.27. *Let E be a spectrum and X be a pointed space. Then the E-homology of X is the graded abelian group*

$$E_*(X) := \pi_*(E \otimes \Sigma^\infty X)$$

EXERCISE 20. *Let A be an abelian group. Then the homology theory $HA_*(-)$ associated to the Eilenberg-MacLane spectrum HA coincides with singular homology. (Hint: pick a CW-complex structure on a pointed space X and use the skeletal filtration to show that HA_*X is given by the homology of the cellular chain complex).*

PROPOSITION 2.28. *Let E, F and K be three spectra. Then there is a natural equivalence*

$$\operatorname{Map}_{\mathrm{Sp}}(E \otimes F, K) \cong \operatorname{Map}_{\mathrm{Sp}}(E, \operatorname{map}(F, K))$$

PROOF. It suffices to plug in the definition of $E \otimes F$:

$$\begin{aligned}
&\operatorname{Map}_{\mathrm{Sp}}(E \otimes F, K) \cong \operatorname{Map}_{\mathrm{Sp}}\left(\operatorname*{colim}_{n,m} \Sigma^{-n-m}\Sigma^\infty(E_n \wedge F_m), K\right) \cong \\
&\cong \lim_{n,m} \operatorname{Map}_{\mathrm{Sp}}\left(\Sigma^\infty(E_n \wedge F_m), \Sigma^{n+m}K\right) \cong \lim_{n,m} \operatorname{Map}_{\mathscr{S}_*}\left(E_n \wedge F_m, \Omega^\infty\Sigma^{n+m}K\right) \cong \\
&\cong \lim_{n,m} \operatorname{Map}_{\mathscr{S}_*}\left(E_n, \operatorname{Map}_{\mathscr{S}_*}(F_m, \Omega^\infty\Sigma^{n+m}K)\right) \cong \lim_{n} \operatorname{Map}_{\mathscr{S}_*}\left(E_n, \lim_{m} \operatorname{Map}_{\mathscr{S}_*}(F_m, \Omega^\infty\Sigma^{n+m}K)\right) \cong \\
&\cong \lim_{n} \operatorname{Map}_{\mathscr{S}_*}\left(E_n, \operatorname{Map}_{\mathrm{Sp}}(F, \Sigma^n K)\right) \cong \operatorname{Map}_{\mathrm{Sp}}(E, \operatorname{map}(F, K))\,.
\end{aligned}$$

□

COROLLARY 2.29. *There's a natural equivalence $E \otimes \mathbb{S} \cong E$ for every spectrum E.*

EXERCISE 21 (Brown representability for spectra). *A functor* $F : h\mathrm{Sp}^{\mathrm{op}} \to \mathrm{Ab}$ *is representable if and only if it sends coproducts to products and exact sequences to exact sequences. (Hint: show that the functor* $F^n X = F(\Sigma^{-n}\Sigma^{\infty} X)$ *is a cohomology theory and use the pushout formula for the sequential colimit plus the canonical presentation of a spectrum to deduce the result for general spectra).*

A1. A counterexample to Brown representability for non connected spaces

WRITE A PROOF FOLLOWING FREYD-HELLER.

CHAPTER 3

Commutative monoids

This chapter is modeled on Segal's foundational paper [**Seg74**]. A useful resource for more details on this story is Adams' book on infinite loop spaces [**Ada78**].

1. Classifying spaces of monoids

Let $T \in \Delta$. Then a **gap** in T is a pair of elements (t, t') of T such that $t < t'$ and there is no element t'' such that $t < t'' < t'$. For example the set of gaps of the standard object $[n] = \{0 < 1 < 2 < \cdots < n\}$ is

$$(0,1),\ (1,2),\ \ldots,\ (n-1,n)\,.$$

For every gap (t,t') there is a map $g_{(t,t')} : [1] \to [n]$ sending 0 to $\sup T'$ and 1 to $\inf T''$. In the case of the gap $(i-1, i)$ of $[n]$ we will write simply $g_i : [1] \to [n]$. We will write $\mathrm{Gap}(T)$ for the set of gaps of T. Note that $\mathrm{Gap}(T)$ is naturally totally ordered by the relation $(t,t') < (s,s') \Leftrightarrow t < s$.

Definition 3.1. *Let $\mathcal{C}$ be an ∞-category with finite products. Then an associative monoid is a functor*

$$M : \Delta^{\mathrm{op}} \to \mathcal{C}$$

such that for every $[n] \in \Delta$ the map

$$\prod_{i=1}^{n} g_i : M([n]) \to \prod_{i=1}^{n} M([1])$$

is an equivalence. In particular, by choosing $n = 0$, we see that $M([0])$ is contractible.

We will write $\mathrm{Mon}(\mathcal{C})$ *for the full subcategory of* $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$ *spanned by the associative monoids.*

An associative monoid in the ∞-category $\mathscr{S}$ of spaces is called an E_1-space.

Remark 3.2. *The forgetful functor* $\mathrm{Mon}(\mathcal{C}) \to \mathcal{C}$ *sending M to $M([1])$ is conservative.*

Proposition 3.3. *Let $\mathcal{C}$ be a category with finite products. Then the category of associative monoids is equivalent to the category whose objects are triples $(M, \eta : 1 \to M, \mu : M \times M \to M)$ satisfying*

$$\mu \circ (\eta, \mathrm{id}_M) \cong \mu \circ (\mathrm{id}_M, \eta) \cong \mathrm{id}_M \qquad \mu \circ (\mathrm{id}_M, \mu) \cong \mu \circ (\mu, \mathrm{id}_M)\,.$$

and whose morphisms are arrows $f : M \to M'$ in $\mathcal{C}$ such that $f \circ \eta = \eta'$ and $f \circ \mu = \mu' \circ (f, f)$.

Proof. Let us denote the category of triples in the statement as the category of "classical" monoids $\mathrm{Mon}_{cl}(\mathcal{C})$. Then for every $M \in \mathrm{Mon}_{cl}(\mathcal{C})$ and every finite (possibly empty) totally ordered set T we have a map $\mu_T : M^T \to M$ defined inductively as

$$\mu_\varnothing = \eta,\ \mu_T := \mu(\mathrm{id}_M, \mu_{T \smallsetminus \{\max T\}})\,.$$

If $\mathcal{C} = \mathrm{Set}$ this is the map sending $(m_1, \ldots, m_n)$ to $m_1 \cdots m_n$. If $f : T \to S$ is a map in Δ and $(t,t') \in \mathrm{Gap}(T)$ then we can define $f(t,t')$ as the ordered subset of

$\mathrm{Gap}(S)$ given by those gaps (s, s') such that $ft \le s < s' \le ft'$ (this is empty if $ft = ft'$). Then we can construct a functor

$$\mathrm{Mon}_{cl}(\mathcal{C}) \to \mathrm{Mon}(\mathcal{C})$$

sending a triple (M, η, μ) to the functor

$$T \mapsto M^{\mathrm{Gap}(T)}$$

such that if $f : T \to S$ is a map in Δ the map $f^* : M^{\mathrm{Gap}(S)} \to M^{\mathrm{Gap}(T)}$ is given by

$$f^* : M^{\mathrm{Gap}(S)} \cong \prod_{(t,t')\in \mathrm{Gap}(T)} M^{f(t,t')} \xrightarrow{\prod \mu_{f(t,t')}} M^{\mathrm{Gap}(T)} .$$

It is then an easy exercise to see that this is in fact an associative monoid and that this construction is functorial.

To construct a functor in the opposite direction

$$\mathrm{Mon}(\mathcal{C}) \to \mathrm{Mon}_{cl}(\mathcal{C})$$

sending M to the triple

$$\left(M([1]), \eta : * \xrightarrow{g_0^{-1}} M([0]) \to M([1]), \mu : M([1]) \times M([1]) \xrightarrow{(g_1,g_2)^{-1}} M([2]) \to M([1]) \right)$$

where η is induced by the map $[1] \to [0]$ in Δ and μ is induced by the map $[1] \to [2]$ in Δ sending 0 to 0 and 1 to 2.

It is easy now to check that these two functors are inverse to each other □

Remark 3.4. *If $F : \mathcal{C} \to \mathcal{D}$ is a functor preserving finite products, then post-composition with F induces a functor* $\mathrm{Mon}(\mathcal{C}) \to \mathrm{Mon}(\mathcal{D})$*. In particular if M is an E_1-space, the set $\pi_0 M$ has a canonical associative monoid structure.*

Definition 3.5. *Let M be an E_1-space. We say that M is **group-like** or an E_1**-group** if $\pi_0 M$ is a group.*

Exercise 22. *Let* Cat *be the ∞-category of categories (i.e. the subcategory of* Cat_∞ *spanned by nerves of categories). Then a monoid in* Cat *is the same thing as a monoidal category and the space of maps of monoids is equivalent to the nerve of the groupoid of monoidal functors.*■

Denis: Expand! Use the functor sending a mon cat $\mathcal{C}$ to the monoid sending T to monoidal functors from $\mathrm{Fin}_{/T}$ to $\mathcal{C}$

Example 3.6. *Let $X \in \mathscr{S}_*$ be a pointed space. We want to put an E_1-space structure on ΩX which is given, informally speaking, by the composition of loops. To do so let for every $T \in \Delta$ $T^\triangleright$ be the cone on T, that is the poset whose objects are $T \amalg \{\infty\}$ equipped with the partial order such that $x \ge y$ iff $x = y$ or $y = \infty$.*

Let $X_T : T^\triangleright \to \mathscr{S}_*$ the functor sending T to $*$ and ∞ to X. Then we have a functor

$$\Omega X : \Delta^{\mathrm{op}} \to \mathscr{S}_* \qquad T \mapsto \lim_{T^\triangleright} X_T .$$

It is now easy see that $\Omega X([1]) = \Omega X$ and that it is an E_1-space: for example

$$(\Omega X)([2]) \cong (\Omega X)(0 < 1) \times_{(\Omega X)(1)} (\Omega X)(1 < 2) \cong \Omega X \times \Omega X$$

and the map $(\Omega X)([2]) \to (\Omega X)([1])$ is exactly given by the composition of paths (in fact one can construct a point-set model of ΩX where $(\Omega X)([n])$ is the space of maps $\Delta^n \to X$ sending the vertices to the basepoint).

Note that $\pi_0 \Omega X \cong \pi_1 X$ is a group, therefore ΩX is an E_1-group. Our goal in this section is to show that all E_1-groups are of this form.

Definition 3.7. *Let M be an E_1-space. Then its* ***classifying space*** *is simply its colimit seen as a functor*

$$BM := \operatorname*{colim}_{\Delta^{\mathrm{op}}} M\,.$$

Note that the canonical map $* = M([0]) \to BM$ gives BM a canonical basepoint. Therefore we have a functor

$$B : \mathrm{Mon}(\mathscr{S}) \to \mathscr{S}_*$$

Note that the commutative diagram

$$\begin{tikzcd}
M = M([1]) \arrow[r, "d_1"] \arrow[d, "d_0"] & M([0]) = * \arrow[d] \\
* = M([0]) \arrow[r] & BM = \operatorname{colim} M
\end{tikzcd}$$

induces a natural transformation $M \to \Omega BM$. We want to extend it to a natural transformation of E_1-spaces. The definition of colimits induces a functor $(\Delta^{\mathrm{op}})^{\triangleright} \to \mathscr{S}$. There is a functor $\{0, \ldots, n\}^{\triangleleft} \to \Delta^{\mathrm{op}}$ sending the cone point to $[n]$, all the other points to $[0]$ and the i-th arrow to d_i. Therefore we have constructed a diagram, natural in $[n]$

$$\begin{tikzcd}
 & & M([n]) \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
M([0]) = * \arrow[drr] & * \arrow[dr] & \cdots \arrow[d] & * \arrow[dl] & * \arrow[dll] \\
 & & \operatorname{colim}_{\Delta^{\mathrm{op}}} M = BM & &
\end{tikzcd}$$

and therefore a natural transformation $M([n]) \to (\Omega BM)([n])$.

On the other hand there is a natural transformation of pointed spaces

$$B\Omega X \to X$$

induced on colimits by the natural transformation

$$(\Omega X)(T) \cong \lim_{T^{\triangleright}} X_T \to \lim_{T^{\triangleright}} X \cong X\,.$$

where the second limit is the constant diagram at X.

Lemma 3.8. *The two natural transformations above satisfy the triangular identities, and so exhibit $B : \mathrm{CMon}(\mathscr{S}) \to \mathscr{S}_*$ as a left adjoint of $\Omega : \mathscr{S}_* \to \mathrm{CMon}(\mathscr{S})$.*

Proof. Let us verify only one of them, the other is analogous. We need to show that the composition

$$BM = \operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} M([n]) \to \operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} \lim_{[n]^{\triangleright}} BM[n] \to \operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} \lim_{[n]^{\triangleright}} BM$$

is homotopic to the identity. But this is equivalent to the transformation

$$\operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} M([n]) \to \operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} BM$$

induced at every $[n]$ by the inclusion $M([n]) \to BM$ into the colimit. Therefore it is homotopic to the identity. $\square$

Our goal is to prove the following theorem:

Theorem 3.9 (Recognition theorem for loop spaces). *Let M be an E_1-space and X be a pointed space.*

- *The unit $M \to \Omega BM$ is an equivalence if and only if M is an E_1-group.*
- *The pointed space BM is connected.*

- *The counit $B\Omega X \to X$ is equivalent to the inclusion of the connected component of the basepoint. In particular it is an equivalence if and only if X is connected.*

Therefore the adjunctions $B \dashv \Omega$ restricts to an equivalence between the ∞-category of E_1-groups $\mathrm{Mon}(\mathscr{S})^{\mathrm{gp}}$ and the ∞-category of connected pointed spaces $\mathscr{S}_*{}^{\geq 0}$.

In order to prove the theorem we will need an important property of the ∞-category of spaces.

Definition 3.10. *Let $F, G : I \to \mathcal{C}$ be two functors and $\alpha : F \Rightarrow G$ be a natural transformation. Then α is* ***cartesian*** *if for every arrow $i \to j$ in I the square*

$$\begin{tikzcd} Fi \arrow[r] \arrow[d] & Fj \arrow[d] \\ Gi \arrow[r] & Gj \end{tikzcd}$$

is cartesian.

Denis: Ideally all properties of colimits in spaces I left unproven in the past should be deduced from this, with perhaps the exception of compact generation

Theorem 3.11 (Descent property for colimits). *Let I be a small ∞-category, $F, G : I^{\triangleright} \to \mathscr{S}$ be two diagrams in the ∞-category of spaces and $\alpha : F \Rightarrow G$ be a natural transformation such that $\alpha_I : F|_I \Rightarrow G|_I$ is cartesian. Suppose G is a colimit diagram. Then F is a colimit diagram if and only if α is cartesian.*

Proof. This is [**Lur09**, Theorem 6.1.3.9]. □

Remark 3.12. *Using results about the limits in the ∞-category Cat_∞, we can rephrase the following theorem as stating that for any diagram $X : I \to \mathscr{S}$ the natural transformation*

$$\mathscr{S}_{/\operatorname{colim}_{i\in I} X} \to \lim_{i \in I} \mathscr{S}_{/X_i}$$

is an equivalence. This is sometimes stated as "colimits in spaces are van Kampen". Using the straightening-unstraightening theorem ([**Lur09**, Theorem 3.2.0.1]*) this is a simple consequence of the fact that the functor $\mathrm{Fun}(-, \mathscr{S})$ preserves limits.*

Lemma 3.13 (Extra degeneracy argument). *Let $X : \Delta^{\mathrm{op}} \to \mathcal{C}$ be a simplicial object. Then the map*

$$X([0]) \to \operatorname*{colim}_{n \in \Delta^{\mathrm{op}}} X([n+1])$$

is an equivalence.

Proof. Let $\Delta_{+\infty}$ be the category of finite non-empty totally ordered sets with maps preserving the maximum. Then $\Delta_{+\infty}$ has an initial object given by $[0]$. Moreover there is a map

$$\Delta \to \Delta_+ \qquad T \mapsto T_+ := T \amalg \{\infty\}$$

which is a left adjoint. In particular if $Y : \Delta_{+\infty}^{\mathrm{op}} \to \mathcal{C}$ is a functor, it induces an equivalence

$$\operatorname*{colim}_{T \in \Delta^{\mathrm{op}}} Y(T_+) \cong \operatorname*{colim}_{\Delta_+^{\mathrm{op}}} Y \cong Y([0])\,.$$

The thesis now follows from the fact that the map $\Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}}$ sending $[n]$ to $[n+1]$ factors through Δ_+. □

Armed with the above result let us prove the first part of theorem 3.9.

Proposition 3.14. *Let $M : \Delta^{\mathrm{op}} \to \mathscr{S}$ be an associative monoid. The natural map*

$$M([1]) \to \Omega B M$$

is an equivalence if and only if M is group-like.

PROOF. Since $\pi_0\Omega X \cong \pi_1 X$, the E_1-space ΩBM is always an E_1-group. Therefore if the map $M \to \Omega BM$ is an equivalence, then M must be an E_1-group. Let us prove the other direction. Let

$$PM : \Delta^{\mathrm{op}} \to \mathscr{S} \qquad [n] \mapsto M([n+1])$$

we have a natural transformation $\alpha : PM \Rightarrow M$ whose components are the maps $d_{n+1} : M([n+1]) \to M([n])$. We claim that if M is group-like, α is cartesian. Indeed for an arrow $f : [m] \to [n]$ we need to check that the square

$$\begin{tikzcd} M([1])^{m+1} \arrow[r] \arrow[d] & M([1])^{n+1} \arrow[d] \\ M([1])^m \arrow[r] & M([1])^n \end{tikzcd}$$

is cartesian. But, if we let $fm = n - d$, this is the base change of the square for the map $f : [0] \to [d]$ sending 0 to 0, and this is cartesian if and only if the map

$$M([1])^{d+1} \to M([1])^{d+1} \qquad (m_1, \dots, m_{d+1}) \mapsto (m_1, \dots, m_d, m_1 \cdots m_{d+1})$$

is an equivalence, which it is since M is an E_1-group.

Therefore the square

$$\begin{tikzcd} PM([0]) = M([1]) \arrow[r] \arrow[d] & \operatorname{colim} PM \cong * \arrow[d] \\ M([0]) \cong * \arrow[r] & \operatorname{colim} M \cong BM \end{tikzcd}$$

is cartesian by the descent property, which is what we wanted to prove. □

LEMMA 3.15. *Let $M \in \mathrm{Mon}(\mathscr{S})$ be an E_1-space. Then BM is connected.*

PROOF. This follows immediately from the fact that π_0 commutes with colimits and so $\pi_0 BM$ is a quotient of $\pi_0(M[0]) = *$. □

PROPOSITION 3.16. *Let X be a pointed space. Then the counit*

$$B\Omega X \to X$$

is equivalent to the inclusion of the connected component of the basepoint.

PROOF. Since $B\Omega X$ is connected, the image of the counit lands in the connected component of the basepoint. Now we need to prove that it induces an equivalence on π_i for $i \geq 1$. But this is the same as checking that the map

$$\Omega B\Omega X \to \Omega X$$

is an equivalence. By the triangular identities this follows from the fact that

$$\Omega X \to \Omega B\Omega X$$

is an equivalence since ΩX is an E_1-group. □

COROLLARY 3.17. *The adjunction $B \dashv \Omega$ restricts to an equivalence between E_1-groups and connected pointed spaces.*

DEFINITION 3.18. *The functor $\mathrm{Mon}(\mathscr{S}) \to \mathrm{Mon}(\mathscr{S})^{\mathrm{gp}}$ sending M to ΩBM is the left adjoint to the inclusion $\mathrm{Mon}(\mathscr{S})^{\mathrm{gp}} \subseteq \mathrm{Mon}(\mathscr{S})$. It is called the **group completion** and written $M^{\mathrm{gp}} := \Omega BM$.*

EXERCISE 23. *Show that $\pi_0 : \mathrm{Mon}(\mathscr{S}) \to \mathrm{Mon}(\mathrm{Set})$ is the left adjoint to the inclusion $\mathrm{Mon}(\mathrm{Set}) \subseteq \mathrm{Mon}(\mathscr{S})$. Deduce that if M is an E_1-space, the group $\pi_0(M^{\mathrm{gp}})$ is the group obtained by formally inverting all elements of the monoid $\pi_0 M$.*

REMARK 3.19. *It is not true that if $M \in \mathrm{Mon}(\mathrm{Set})$, then its group completion M^{gp} is necessarily discrete. We will see later that this is true for commutative monoids or, more generally, when $\pi_0 M$ satisfies the so-called* Ore conditions. *However* every *E_∞-group can be realized as the group completion of some discrete monoid $M \in \mathrm{Mon}(\mathrm{Set})$ (see* [**McD79**]*).*

Denis: I should add at least a reference to a counterexample

2. Commutative monoids and the recognition principle

Let Fin_* be the category of finite pointed sets. We will often write I_+ for an element of Fin_*, where I is the set of non-basepoint elements. We will refer to I as the underlying set of I_+. For any $i \in I$ there is a map

$$\chi_i : I_+ \to \{i\}_+$$

sending i to i and all other elements to the basepoint. We will call it the **characteristic map** at i.

For brevity we will often write n_+ for the pointed set $\{1, \ldots, n\}_+$.

DEFINITION 3.20. *Let $\mathcal{C}$ be an ∞-category with finite products. Then a commutative monoid is a functor*

$$M : \mathrm{Fin}_* \to \mathcal{C}$$

such that for every pointed finite set I_+ the product

$$M(I_+) \to \prod_{i \in I} M(\{i\}_+)$$

is an equivalence. We write the subcategory of $\mathrm{Fun}(\mathrm{Fin}_*, \mathcal{C})$ spanned by the commutative monoids as $\mathrm{CMon}(\mathcal{C})$.

*An E_∞**-space** is a commutative monoid in the ∞-category $\mathscr{S}$ of spaces.*

EXAMPLE 3.21. *There is a map $e : \Delta^{\mathrm{op}} \to \mathrm{Fin}_*$ sending $T \in \Delta$ to the pointed set of gaps $\mathrm{Gap}(T)_+ = \mathrm{Gap}(T) \amalg \{*\}$. If $f : T \to S$ is a monotone map the map $f^* : \mathrm{Gap}(S)_+ \to \mathrm{Gap}(T)_+$ is given by*

$$f^*(t, t') = \begin{cases} (\sup_{s < ft} s, \inf_{ft' < s'} s') & \text{if } \exists s < ft', f't < s \neq \varnothing \\ \infty & \text{otherwise} \end{cases}$$

Let $M : \mathrm{Fin}_* \to \mathcal{C}$ be a commutative monoid. Then the precomposition with the map $\Delta^{\mathrm{op}} \to \mathrm{Fin}_*$ turns a commutative monoid into an associative monoid.

PROPOSITION 3.22. *Let $\mathcal{C}$ be a category. Then the functor $\mathrm{CMon}(\mathcal{C}) \to \mathrm{Mon}(\mathcal{C}) \cong \mathrm{Mon}_{cl}(\mathcal{C})$ is fully faithful with image those monoids where the multiplication is commutative.*

PROOF. Note that the map $\mathrm{tw} : 2_+ \to 2_+$ sending 1 to 2 and 2 to 1 have the property $\chi_i \circ \mathrm{tw} = \chi_{1-i}$. Therefore the diagram

$$\begin{tikzcd} M(1_+)^2 \arrow[r, "\mathrm{tw}"] & M(1_+)^2 \\ M(2_+) \arrow[u, "(\chi_1,\chi_2)"] \arrow[r, "\mathrm{tw}"] \arrow[d, "M(\mu)"] & M(2_+) \arrow[u, "(\chi_1,\chi_2)"'] \arrow[d, "M(\mu)"] \\ M(1_+) \arrow[r, "\mathrm{id}"] & M(1_+) \end{tikzcd} .$$

Therefore the multiplication on $M(1_+)$ is commutative.

Let us now construct an inverse. If M is a classical commutative monoid, note that the multiplication maps $\mu_T : M^T \to M$ are independent of the order in the set T. Therefore we can define a functor

$$I_+ \mapsto M^I \qquad (f : I_+ \to J_+) \mapsto \prod_{j\in J} M^{f^{-1}j} \xrightarrow{\prod_{j\in J}\mu_{f^{-1}j}} M^J .$$

This produces an inverse to the previous functor □

Exercise 24. *Let* Cat *be the* ∞*-category of categories (i.e. the subcategory of* Cat_∞ *spanned by the nerves of ordinary categories). Then a commutative monoid in* Cat *is the same thing as a symmetric monoidal category, and the space of maps of commutative monoids is the nerve of the groupoid of symmetric monoidal functors.*

Lemma 3.23. *Let* $\mathcal{C}$ *be an* ∞*-category with finite products. Then the* ∞*-category* $\mathrm{CMon}(\mathcal{C})$ *has a zero object and direct sums.*

Proof. First we claim that the constant functor at the terminal object is the zero object. It is clear that it is the terminal object (since it is in functors). But

$$\mathrm{Map}_{\mathrm{CMon}(\mathcal{C})}(*, M) \cong \mathrm{Map}_{\mathcal{C}}(*, \lim_{\mathrm{Fin}_*} M) \cong \mathrm{Map}_{\mathcal{C}}(*, M(0_+)) \cong * .$$

Now to prove that it has direct sums, we need to prove that the two maps $M \to M \times N$ and $N \to M \times N$ exhibit $M \times N$ as the coproduct. First we claim that for every commutative monoid P there's a natural transformation

$$\mu : P \times P \to P$$

such that $P \times * \to P \times P \to P$ and $* \times P \to P \times P \to P$ are homotopic to the identity and such that $\mu_{P\times Q}$ is homotopic to $\mu_P \times \mu_Q$. Then we can construct an inverse to the map

$$\mathrm{Map}(M \times N, P) \to \mathrm{Map}(M, P) \times \mathrm{Map}(N, P)$$

by sending $(f : M \to P, g : N \to P)$ to the composition $M \times N \to P \times P \to P$. The map μ_P is constructed by taking the natural transformation

$$\mu_P : (P \times P)(I_+) \cong P(I_+) \times P(I) \cong P(I_+ \vee I_+) \to P(I_+)$$

induced by precomposing with the fold map $I_+ \vee I_+ \to I_+$. □

Lemma 3.24. *Let* M *be an* E_1*-space. Then the two* E_1*-spaces* $\Omega M([1])$ *(with the* E_1*-space structure of example 3.6) and* $\Omega \circ M$ *coincide.*

Proof. We need to prove that the two functors

$$\Omega(M([n])) = \lim \left(\begin{array}{ccc} & & M([0]) \\ & & \downarrow \\ M([0]) & \longrightarrow & M([1]) \end{array} \right) \qquad \text{and} \qquad (\Omega M)([n]) = \lim_{[n]^{\triangleright}} \Omega(M_{[n]})$$

are naturally equivalent. We will do it by constructing a third functor that is equivalent to either of them. Let $Y : \Delta^{\mathrm{op}} \to \mathscr{S}$ be the functor sending $[n]$ to the limit of the "staircase diagram"

$$\begin{array}{ccccccc} & & & & & & X([0]) \\ & & & & & & \downarrow \\ & & & & X([0]) & \longrightarrow & X([1]) \\ & & & & \downarrow & & \\ & & X([0]) & \longrightarrow & \iddots & & \\ & & \downarrow & & & & \\ X([0]) & \longrightarrow & X([1]) & & & & \end{array} .$$

This receive a map from the functor sending $[n]$ to the limit of the diagrams

$$\begin{tikzcd}
 & & & X([0]) \arrow[d] \\
 & & X([n]) \arrow[r] \arrow[d] & X([n]) \\
 & X([n]) \arrow[r] \arrow[d] & \iddots & \\
X([0]) \arrow[r] & X([n]) & &
\end{tikzcd}$$

where we send the i-th corner copy of $X([n])$ to $X([1])$ via g_i. By the Segal condition this map is an equivalence, and the latter diagram is simply $\Omega(X([n])$.

On the other direction $Y([n])$ maps to the limit of $X([1])_{[n]^{\triangleright}}$ and this map is an equivalence. □

There is a functor $\Omega^\infty : \mathrm{Sp} \to \mathrm{CMon}(\mathscr{S})$ sending E to $\Omega^\infty(\Sigma^\infty(-) \otimes E)$.

Construction 3.25. *Let M be a commutative monoid. Then the classifying space BM has a canonical structure of commutative monoid as well. Indeed we can take*

$$BM(I_+) := \operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} M(I_+ \wedge \mathrm{Gap}([n])_+)$$

Therefore we can iterate this construction obtaining $B^2M := B(BM)$, $B^3M := B(B^2M)$ etc. Note that B^nM is always group-like, since it is connected.

Lemma 3.26. *The functor $B : \mathrm{CMon}(\mathscr{S}) \to \mathrm{CMon}(\mathscr{S})$ gives the suspension in* $\mathrm{CMon}(\mathscr{S})$

Proof. This follows from the Bousfield-Kan formula (theorem 1.78): we can compute a pushout $B \amalg_A C$ as the geometric realization of

$$B \amalg C \leftleftarrows B \amalg A \amalg C \leftleftarrows B \amalg A \amalg A \amalg C \cdots$$

and plugging the characterization of coproduct in $\mathrm{CMon}(\mathscr{S})$ of lemma 3.23 we obtain the thesis. □

Denis: expand!

Definition 3.27. *Let M be an E_∞-space. Then we define its classifying spectrum $\mathbf{B}^\infty M$ as the spectrum*

$$\mathbf{B}^\infty M := \left(\{B^nM\}_{n\geq 1}, \sigma_n : B^nM \xrightarrow{\sim} \Omega B^{n+1}M \right).$$

This indeed defines a spectrum since B^nM is a group-like E_1-space for $n \geq 1$ and so σ_n is an equivalence by theorem 3.9. Note that $\Omega^\infty \mathbf{B}^\infty M \cong \Omega BM$.

Proposition 3.28. *There are natural transformations $\mathbf{B}^\infty\Omega^\infty \to \mathrm{id}_{\mathrm{Sp}}$ and $\mathrm{id}_{E_\infty - \mathscr{S}} \to \Omega^\infty\mathbf{B}^\infty$ exhibiting an adjunction $\mathbf{B}^\infty \dashv \Omega^\infty : \mathrm{Sp} \leftrightarrows E_\infty - \mathscr{S}$.*

Proof. The natural transformation

$$M \to \Omega^\infty \mathbf{B}^\infty M \cong \Omega BM$$

is just the same as the natural transformation defined levelwise by

$$\begin{tikzcd}
M(I_+) \arrow[r] \arrow[d] & M(*) \arrow[d] \\
M(*) \arrow[r] & \mathrm{colim}_{[n]} M(I_+ \wedge \mathrm{Gap}([n])_+)
\end{tikzcd}.$$

that is the unit of the adjunction $B \dashv \Omega$. In particular as a map of pointed spaces it coincides with the map $M \to \Omega BM$ of lemma 3.8.

The natural transformation

$$\mathbf{B}^\infty\Omega^\infty E \to E$$

is the one whose n-th component is the map

$$B^n\Omega^\infty E \to \Omega^\infty\Sigma^n E$$

induced by iterating the transformation

$$B\Omega^\infty E \to \Omega^\infty\Sigma E$$

given by the fact that B is the suspension in $\mathrm{CMon}(\mathscr{S})$ and Ω^∞ preserves the zero object. The triangular identities now are easy. □

THEOREM 3.29 (Recognition principle for connective spectra).

- *For any E_∞-space M the unit map $M \to \Omega^\infty\mathbf{B}^\infty M$ is an equivalence if and only if M is group-like.*
- *For every E_∞-space M the spectrum $\mathbf{B}^\infty M$ is connective and for every spectrum E the map $\mathbf{B}^\infty\Omega^\infty E \to E$ is an equivalence if and only if E is connective.*

Therefore the adjunction $\mathbf{B}^\infty \dashv \Omega^\infty$ restricts to an equivalence between the ∞-category of E_∞-groups and the ∞-category of connective spectra.

PROOF. The first statement follows from the analogous statement for E_1-spaces.

Now let M be an E_∞-space. Then we need to show that the spectrum $\mathbf{B}^\infty M$ is connective. But we have

$$\pi_{-n}\mathbf{B}^\infty M = \pi_0 B^n M = 0\,,$$

since $B^n M$ is connected.

Finally we need to show that the map $\mathbf{B}^\infty\Omega^\infty E \to E$ is an equivalence if and only if E is connective. One direction is obvious, now suppose that E is connective. Then $\Omega^\infty E$ is a group-like E_∞-space, therefore the map

$$\Omega^\infty E \to \Omega^\infty\mathbf{B}^\infty\Omega^\infty E$$

is an equivalence. But then by the triangular identities the map

$$\Omega^\infty\mathbf{B}^\infty\Omega^\infty E \to \Omega^\infty E$$

is an equivalence and so the map

$$\mathbf{B}^\infty\Omega^\infty E \to E$$

is an isomorphism on all non-negative homotopy groups. Since all the other homotopy groups are 0, it is an equivalence by lemma 2.22. □

EXERCISE 25. *Let $\mathrm{Sp}(\mathrm{CMon}(\mathscr{S}))$ be the ∞-category*

$$\mathrm{Sp}(\mathrm{CMon}(\mathscr{S})) \cong \lim\left(\mathrm{CMon}(\mathscr{S}) \xleftarrow{\Omega} \mathrm{CMon}(\mathscr{S}) \xleftarrow{\Omega} \cdots\right)$$

defined as in the definition of spectra but using E_∞-spaces instead of spaces. Prove that $\mathrm{Sp}(\mathrm{CMon}(\mathscr{S})) \cong \mathrm{Sp}$ and that the left adjoint of

$$\Omega^\infty : \mathrm{Sp} \cong \mathrm{Sp}(\mathrm{CMon}(\mathscr{S})) \to \mathrm{CMon}(\mathscr{S})$$

is exactly $\mathbf{B}^\infty$.

3. Group completion of commutative monoids

Most of this section is derived from [**Hoy19**] and section 3 of [**BEH$^+$20**]. The explicit construction of the plus construction follows [**Hat02**, Proposition 4.40]. An extremely good reference for the group completion theorem is [**Nik17**]

DEFINITION 3.30. *Let R be a ring. Then the groupoid of projective R-modules with the direct sum forms an E_∞-space. Its group completion is called the **algebraic K-theory spectrum** $K(R)$ of R. It contains important arithmetic information about R.*

In this section we fix M be an E_∞-space and $x \in \pi_0 M$ such that for every $y \in \pi_0 M$ there's $z \in \pi_0 M$ and $n \geq 0$ such that $y + z = nx$.

EXAMPLE 3.31. *In many cases we will have $\pi_0 M = \mathbb{N}$ and $x = 1$. More generally, if $\pi_0 M$ is finitely generated as a commutative monoid, with generators $x_1, \dots, x_n$, we can take $x = x_1 + \cdots + x_n$.*

In the above situation we can define a map of spaces

$$x+ : M \to M$$

given as the composition

$$M \cong * \times M \xrightarrow{(x, \mathrm{id}_M)} M \times M \xrightarrow{\mu} M$$

and a space (called the x-telescope of M)

$$\mathrm{tel}_x M := \mathrm{colim}\left(M \xrightarrow{x+} M \xrightarrow{x+} \cdots\right).$$

If we denote with x also the image in $\pi_0 M^{\mathrm{gp}}$, the map of E_∞-spaces $M \to M^{\mathrm{gp}}$ induces a map

$$t : \mathrm{tel}_x M \to \mathrm{tel}_x M^{\mathrm{gp}} \cong M^{\mathrm{gp}}$$

where the last equivalence follows from the fact that $x+ : M^{\mathrm{gp}} \to M^{\mathrm{gp}}$ is an equivalence. One could naively expect the map t to be an equivalence, but unfortunately this is not always the case, due to a fundamental group obstruction.

EXAMPLE 3.32. *Let R be a ring and let Mod_R be the E_∞-space associated to the symmetric monoidal groupoid of free finitely generated R-modules with monoidal operation given by the direct sum. Then we have*

$$\mathrm{Mod}_R \cong \amalg_{n \geq 0} BGL_n(R).$$

Then, if we let $x = 1 \in \mathbb{N} = \pi_0 \mathrm{Mod}_R$, we have

$$\mathrm{tel}_1 \mathrm{Mod}_R \cong \mathbb{Z} \times BGL_\infty(R)$$

where $GL_\infty(R)$ is the group of infinite invertible matrices that coincide with the identity outside of a block in the top left corner. Then

$$\pi_1(\mathrm{tel}_1 \mathrm{Mod}_R) \cong GL_\infty(R)$$

is not an abelian group (e.g. it contains $GL_2(R)$ as a subgroup). However $\pi_1 \mathrm{Mod}_R^{\mathrm{gp}}$ must be abelian, since it is the fundamental group of an H-space.

Luckily this is all that can go wrong. The map t is "an equivalence up to fundamental group issues" in some sense.

DEFINITION 3.33. *Let $f : X \to Y$ be a map of spaces. We will say that it is acyclic if for every $y \in Y$ the homotopy fiber $X_y := X \times_Y \{y\}$ is non-empty and has trivial reduced homology.*

The main theorem in this section (to be proven in appendix **??** is the following

THEOREM 3.34 (Group completion theorem). *Let M be an E_∞-space*[1]*. Let $x \in \pi_0 M$ be such that for every $y \in \pi_0 M$ there's z such that $y + z = nx$. Then let*

$$\operatorname{tel}_x M := \operatorname{colim}(M \xrightarrow{x\cdot} M \xrightarrow{x\cdot} M \xrightarrow{x\cdot} \cdots)$$

Then the canonical map

$$\operatorname{tel}_x M \to M^{\mathrm{gp}}$$

is an acyclic map.

To make yse of it we will need some results on acyclic maps.

LEMMA 3.35. *A map $f : X \to Y$ is acyclic if and only if the square*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Y \arrow[r] & Y \end{tikzcd}$$

is a pushout square.

PROOF. Suppose the square is a pushout square. Since in spaces pushout squares are stable under pullback by descent the square

$$\begin{tikzcd} X_y \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & * \end{tikzcd}$$

is a pushout square, i.e. $\Sigma X_y \cong *$. But then its reduced homology is trivial.

Viceversa, suppose all those squares are pushout. Then again by descent $X \cong \operatorname{colim}_y X_y$ and we are done since pushout squares are closed under colimits. $\square$

EXERCISE 26. *The pushout condition of lemma 3.35 is equivalent to say that f is a categorical epimorphism, i.e. that for every Z the map*

$$\operatorname{Map}(Y, Z) \to \operatorname{Map}(Y, X)$$

is an inclusion of connected components.

LEMMA 3.36. *Let $f : X \to Y$ be a map. If f is acyclic, then f induces an isomorphism $f_* : H_*(X) \to H_*(Y)$. The viceverse is true if Y is simply connected.*

PROOF. If f is acyclic, then by taking cofibers of the horizontal arrows in the square of lemma 3.35 we see that the cofiber of the map f is contractible, therefore f is an isomorphism in homology by the long exact sequence in homology.

Let us now assume that Y is simply connected and f is an isomorphism in homology. We need to prove that the square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Y \arrow[r] & Y \end{tikzcd}$$

is a pushout square, that is that the map $Y \amalg_X Y \to Y$ is an equivalence. A simple computation with the long exact sequence shows that it is a homology equivalence and then van Kampen shows that both spaces are simply connected, so we can conclude by the relative Hurewicz theorem. $\square$

LEMMA 3.37. *Acyclic maps are closed under pushouts.*

[1]In fact we only need that it is an E_1-space such that $\pi_0 M$ satisfies the Ore conditions

Proof. Let us consider a pushout square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

Then we can consider the cube

$$\begin{tikzcd}[row sep=small, column sep=small] & X \arrow[rr, "f"] \arrow[dl] \arrow[dd] & & Y \arrow[dl] \arrow[dd] \\ Y \arrow[rr, crossing over] \arrow[dd] & & Y & \\ & X' \arrow[rr, "f'"] \arrow[dl] & & Y' \arrow[dl] \\ Y' \arrow[rr] & & Y' \arrow[from=uu, crossing over] & \end{tikzcd}$$

since the front and bottom arrows are pushout squares, the whole cube is a pushout cube. Since f is acyclic, the top square is also a pushout square and so the bottom square is a pushout square as we wanted to prove. $\square$

The next few results will be to show that acyclic maps behave as equivalences for spaces with a mild condition on the fundamental groups. Remember that a group is perfect if it has trivial abelianization (i.e. it is generated by commutators). Then every group has a maximal perfect subgroup, which is therefore normal (just take the subgroup generated by all perfect subgroups). We say that a group G is hypoabelian if the maximal perfect subgroup is trivial. We say that a space X is hypoabelian if for every $x \in X$ the group $\pi_1(X, x)$ is hypoabelian.

Lemma 3.38. *Let $f : X \to Y$ be an acyclic map. Suppose $\pi_1(X, x)$ is hypoabelian for each $x \in X$. Then f is an equivalence.*

Proof. First of all, notice that $\pi_0 f$ is surjective (since the fibers are all nonempty). Let $y \in Y$ and let us consider the fiber X_y. Pick $x \in X_y$ and let us consider the long exact sequence in homotopy groups

$$\pi_2(Y, y) \to \pi_1(X_y, x) \to \pi_1(X, x) \to \pi_1(Y, x)$$

We know that $\pi_1(X_y, x)$ is perfect, hence its image in $\pi_1(X, x)$ is trivial. Therefore the map $\pi_2(Y, y) \to \pi_1(X_y, x)$ is surjective. But then $\pi_1(X_y, x)$ is abelian and hence trivial. Therefore X_y is contractible. $\square$

Lemma 3.39. *Let $X \to X'$ be an acyclic map and Y such that $\pi_1(Y, y)$ is hypoabelian for every y. Then*

$$\mathrm{Map}(X', Y) \to \mathrm{Map}(X, Y)$$

is an equivalence.

Proof. Let us fix $f : X \to Y$. We want to show that the fiber $\mathrm{Map}(X', Y) \times_{\mathrm{Map}(X,Y)} \{f\}$ is contractible. Consider the pushout diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] & Y' \end{tikzcd}$$

Then the map $Y \to Y'$ is an acyclic map with hypoabelian source, and therefore it is an equivalence. But then the following square is cartesian

$$\begin{tikzcd} \mathrm{Map}(Y, Y) \arrow[r] \arrow[d] & \mathrm{Map}(X', Y) \arrow[d] \\ \mathrm{Map}(Y, Y) \arrow[r] & \mathrm{Map}(X, Y) \end{tikzcd}$$

In particular the homotopy fiber over $f \in \mathrm{Map}(X, Y)$ is contractible. □

Our goal now is to construct for every X a space X^+ with hypoabelian π_1 and an acyclic map $X \to X^+$. The above result will force it to be the left adjoint of the inclusion of hypoabelian spaces into spaces.

THEOREM 3.40. *Let X be any space. Then there exist a space X^+ and an acyclic map $X \to X^+$ such that $\pi_1(X^+)$ is hypoabelian for every basepoint.*

PROOF. First by working on each connected component we can assume X is connected. Let us first assume $H_1(X) = 0$, i.e. $\pi_1(X)$ is perfect. Then let $\{\alpha_i\}$ be a set of generators for $\pi_1(X)$ and let us consider the cofiber sequence

$$\bigvee_i S^1 \to X \to X'$$

Then if we look at the long exact sequence in homology we deduce

$$H_* X' \cong \begin{cases} H_* X & \text{if } * \neq 2 \\ H_2(X) \oplus \bigoplus_I \mathbb{Z} & \text{otherwise} \end{cases}.$$

Moreover by van Kampen we have $\pi_1 X' = 0$, so $\pi_2 X' \cong H_2 X'$. Therefore we can choose $\beta_i : S^2 \to X'$ generators corresponding to the summand and taking the cofiber sequence

$$\bigvee_i S^2 \to X' \to X^+ .$$

Then $X \to X^+$ is a homology isomorphism and X^+ is simply connected, so it is acyclic.

Now let us do the case of a general X. Let $P < \pi_1(X)$ be the maximal perfect subgroup. Let $\tilde{X} \to X$ be the covering space corresponding to P, so that $\pi_1 \tilde{X} = P$. Then we can apply the previous construction to $\tilde{X}$ and take the pushout

$$\begin{tikzcd} \tilde{X} \arrow[r] \arrow[d] & \tilde{X}^+ \arrow[d] \\ X \arrow[r] & X^+ \end{tikzcd}$$

Then $X \to X^+$ is acyclic and $\pi_1(X^+) = \pi_1(X)/P$ is hypoabelian, as required. □

COROLLARY 3.41. *The inclusion of hypoabelian spaces into spaces has a left adjoint which sends X to the space X^+ of theorem 3.40. Moreover the unit of the adjunction is the acyclic map $X \to X^+$.*

REMARK 3.42. *The functor $X \mapsto X^+$ commutes with products, since both hypoabelian spaces and acyclic maps are closed under products.*

COROLLARY 3.43. *Let M be an E_∞-space and x as in the hypothesis of theorem 3.34. Then the map*

$$(\mathrm{tel}_x M)^+ \to M^{\mathrm{gp}}$$

is an equivalence. In particular if $\mathrm{tel}_x M$ is already hypoabelian, the map $\mathrm{tel}_x M \to M^{\mathrm{gp}}$ is an equivalence.

A1. The proof of the group completion theorem

LEMMA 3.44. *Let M be an E_1-space such that $\pi_0 M^{\mathrm{gp}}$ is hypoabelian (for example if M is an E_∞-space). Then the map $M^{\mathrm{gp}} \to (M^+)^{\mathrm{gp}}$ is an equivalence.*

PROOF. It is enough to show that $BM \to B(M^+)$ is an equivalence of pointed spaces. But since it is a colimit of acyclic maps, it is an acyclic map. Moreover $\pi_1 BM = \pi_0 M^{\mathrm{gp}}$, and so it is hypoabelian. Then the thesis follows from ... □

In this section we will prove the group completion theorem, following [**MS76**].

DEFINITION 3.45. *Let $\mathcal{C}$ be an ∞-category with finite products and let $M : \Delta^{\mathrm{op}} \to \mathcal{C}$ be an associative monoid. Then a left M-action is a simplicial object $X : \Delta^{\mathrm{op}} \to \mathcal{C}$ together with a map $X \to M$ of simplicial objects such that for every $T \in \Delta$ the map*

$$X(T) \to M(T) \times X([0])$$

induced by the arrow $[0] \to T$ picking the maximum of T is an equivalence. We will often say that M acts on $X([0])$.

EXAMPLE 3.46. *For any monoid M and object $X \in \mathcal{C}$ we can define $X(T) := M(T) \times X$. We call this the trivial action of M on X.*

EXAMPLE 3.47. *For any monoid M, there's an action of M on its underlying space $M([1])$ given by the simplicial object $\Delta^{\mathrm{op}} \to \mathcal{C}$ sending T to $M(T \amalg \{\infty\})$ (this is the **left multiplication action**).*

EXAMPLE 3.48. *Suppose that $\mathcal{C} = \mathscr{S}$ is the ∞-category of spaces and let M be an E_1-space acting on a space X. Then for any point $m \in M([1])$ of the underlying space of M we can find a map $m\cdot : X([0]) \to X([0])$ as follows:*

$$X([0]) = \{m\} \times X([0]) \to M([1]) \times X([0]) \cong X([1]) \to X([0])$$

where the last map is induced by $[0] \to [1]$ picking 0.

DEFINITION 3.49. *Let M be a monoid acting on X. Then the **homotopy orbits** of the action X_{hM} are just the colimit of the simplicial object $X : \Delta^{\mathrm{op}} \to \mathcal{C}$ (if it exists).*

EXAMPLE 3.50. *If M acts trivially on X we have $X_{hM} \cong X \times BM$.*

EXAMPLE 3.51. *If M acts on itself by left multiplication M_{hM} is contractible by lemma 3.13.*

To continue we will need the following generalization of ..

LEMMA 3.52. *Let X be a space with an M-action such that every $m \in M$ acts invertibly on X. Then the sequence*

$$X \to X_{hM} \to BM$$

is a fiber sequence.

PROOF. This is a generalization of the proof of ... Indeed by ... it is enough to prove that the natural transformation $X \to M$ is cartesian. Proceeding as in the proof of .. it suffices to see that

$$\begin{array}{ccc} X([1]) & \longrightarrow & X([0]) \\ \downarrow & & \downarrow \\ M([1]) & \longrightarrow & M([0]) \end{array}$$

is cartesian. But if we take $m \in M([1])$ the induced map on the fibers over m is exactly $m : X([0]) \to X([0])$, which is an equivalence by hypothesis. $\square$

Note that $x+ : M \to M$ is a map of spaces with an M-action by ...

LEMMA 3.53. *The map*

$$\mathrm{tel}_x M \to M^{\mathrm{gp}}$$

is an equivalence if and only if x acts invertibly on $\mathrm{tel}_x M$.

PROOF. One direction is obvious. Let us prove the other. Suppose x acts invertibly on $\operatorname{tel}_x M$. Then every element of M acts invertibly on $\operatorname{tel}_x M$, by the defining property of y. Therefore by lemma 3.52 we have a fiber sequence

$$\operatorname{tel}_x M \to (\operatorname{tel}_x M)_{hM} \to BM$$

But $(\operatorname{tel}_x M)_{hM} \cong \operatorname{tel}_x(M_{hM}) \cong \operatorname{tel}_x(*) = *$ by example **??**, so $\operatorname{tel}_x M \cong \Omega BM = M^{\mathrm{gp}}$. $\square$

Now let M be an E_∞-space and $x \in M$ be a point. Then we will construct for every $n \geq 1$ a group homomorphism

$$\Sigma_n \to \pi_1(M, nx)$$

$\boxed{\textbf{UNFINISHED}}$

CHAPTER 4

Vector bundles and K-theory

1. Vector bundles

Definition 4.1. *Let X be a topological space. A **complex vector bundle** over X is a map of topological spaces $p : E \to X$ equipped with maps $s : E \times_X E \to X$ and $\mu : \mathbb{C} \times E \to E$ over X such that for every point $x \in X$ there is a neighborhood U of x, a finite dimensional complex vector space V and an isomorphism $p^{-1}U \cong U \times V$ over U such that under this isomorphism*

$$s(y, v, w) = (y, v + w) \qquad \mu(\lambda, y, v) = (y, \lambda v)\,.$$

*Note that this implies that for every $x \in X$ the fiber $E_x := p^{-1}x$ is endowed with a canonical structure of finite dimensional complex vector space. The **rank** of a vector bundle is the locally constant function $X \to \mathbb{N}$ sending x to $\dim_{\mathbb{C}} E_x$.*

A map of vector bundles is just a continuous map $f : E \to E'$ over X that respects the sum and the scalar multiplication.

We will write the isomorphism classes of vector bundles over X as $\pi_0 \operatorname{Vect}(X)$.

Lemma 4.2. *Let X be a paracompact Hausdorff space and $\{U_i\}$ be an open cover. Then we can find a locally finite countable cover $\{V_n\}_{n\geq 0}$ such that every V_n is the disjoint union of open subsets contained in one of the U_i's.*

Proof. Let $\{\psi_i\}$ be a partition of unity subordinated to our cover. For any finite subset $S \subseteq I$ let

$$V_S = \{x \in X \mid \forall i \in S,\ j \notin S\ \psi_i(x) > \psi_j(x)\}$$

Since in the neighborhood of any point only finitely many of the ψ_i are non-zero, the subset V_S is open. Moreover for every $x \in X$ we have $x \in V_S$ where $S = \{i \in I \mid \psi_i(x) > 0\}$, so the V_S are an open cover. Furthermore if $i \in I$ we have $V_S \subseteq U_i$ (because for all $x \notin U_i$ $\psi_i(x) = 0$). Finally let

$$V_n := \bigcup_{\#S=n} V_S$$

and to prove the thesis we only need to check that $V_S \cap V_T = \varnothing$ if S, T are distinct sets with the same cardinality. But this is obvious from the definition, since we can find $i \in S \smallsetminus T$ and $j \in T \smallsetminus S$ and so for every $x \in V_S \cap V_T$ we would need to have $\psi_i(x) > \psi_j(x)$ and $\psi_j(x) > \psi_i(x)$ simultaneously. $\square$

In particular notice that every vector bundle over a paracompact Hausdorff space can be trivialized over an open cover.

Lemma 4.3. *Let X be a paracompact Hausdorff space and $p : E \to X \times |\Delta^1|$ be a vector bundle. Let E' be the restriction of E to $X \times \{0\} \cong X$. Then there is an isomorphism of vector bundles $E \cong E' \times |\Delta^1|$.*

Proof. First we will show that there is an open cover $\{U_i\}_{i\in I}$ such that E is trivial over $U_i \times |\Delta^1|$. Indeed by the compactness of $|\Delta^1|$, for every $x \in X$ we can find open neighborhoods $U_{x,1}, \dots, U_{x,n}$ and $t_1, \dots, t_{n-1} \in [0, 1]$ such that E is trivial over $U_{x,i} \times [t_{i-1}, t_i]$. But then we can glue the trivializations on $U_x := \bigcap_i U_{x,i}$ and find a trivialization over $U_x \times |\Delta^1|$. Moreover by lemma 4.2 we can find such a countable cover $\{U_n\}_{n\geq 0}$. Then pick a partition of unity $\{\psi_n\}$ subordinate to the cover and let $\Psi_n := \sum_{i\leq n} \psi_n$. Then if we let

$$X_n := \{(x,t) \in X \times |\Delta^1| \mid t \leq \Psi_n(x)\}$$

we will construct a map $p_n : E|_{X_n} \to E'$ over X that is an iso on every fiber inductively. Since $\Psi_{-1} = 0$ we have $X_{-1} = X \times \{0\}$ and we can take p_{-1} to be the identity. Now let us construct p_n from p_{n-1}. Now there is a retraction $X_n \to X_{n-1}$ sending (x,t) to $(x, \max(t, \Psi_{n-1}x))$. We want to construct a map $E_n \to E_{n-1}$ over this projection that is an isomorphism on every fiber. By construction we have $X_n \setminus X_{n-1} \subseteq U_n \times |\Delta^1|$, and therefore we can construct such a projection over $X_n \cap U_n \times |\Delta^1|$, since E_n is trivial there. But then we can just take the identity over X_{n-1}. Therefore we get p_n as the composition $E_n \to E_{n-1} \to E'$. Finally we get a continuous map $E \to E'$ by sending e to $p_n e$ if $x \in E_n$ (this is continuous because $\{E_n\}$ is a locally finite closed cover). $\square$

LEMMA 4.4. *Let $p : E \to X$ be a vector bundle over a paracompact Hausdorff space X. Then there exists a map $f : E \to \mathbb{C}^\infty$ which is a $\mathbb{C}$-linear embedding on each fiber. Moreover if X is compact we can find such a map $E \to \mathbb{C}^N$ for N finite.*

PROOF. Let us find a countable open cover $\{U_i\}_{i\geq 0}$ such that E is trivialized on each U_i, that is there is $f_i : E|_{U_i} \to \mathbb{C}^{d_i}$ inducing $E|_{U_i} \cong U_i \times \mathbb{C}^{d_i}$. Moreover let $\{\psi_i\}_{i\geq 0}$ be a partition of unit subordinate to this cover. Then we can let

$$f(e) = (\psi_0 f_0, \psi_1 f_1, \dots) \in \mathbb{C}^{d_0} \times \mathbb{C}^{d_1} \times \cdots \cong \mathbb{C}^\infty .$$

Then f is the required map. Moreover it is clear that if X is compact we can choose a finite subcover. $\square$

EXERCISE 27. *In the situation of lemma 4.4 suppose there is a closed cofibration $Z \to X$ and a map $f_0 : E|_Z \to \mathbb{C}^\infty$ which is an injection on every fiber. Then we can choose f that coincides with f_0 over Z.*

Let $M_\infty(\mathbb{C}) \subseteq \prod_{\mathbb{N}\times\mathbb{N}} \mathbb{C}$ be the space of $\mathbb{C}$-linear maps $\mathbb{C}^\infty \to \mathbb{C}^\infty$.

DEFINITION 4.5. *Let $d \geq 0$ be a non-negative integer. Then the Grassmannian of d-planes Gr_d is the space*

$$\mathrm{Gr}_d := \{P \in M_\infty(\mathbb{C}) \mid P^2 = P,\ P = P^H,\ \mathrm{rk}\, P = d\}$$

of orthogonal projection matrices of rank d. Note that such matrices are in bijection with the subspaces of $\mathbb{C}^\infty$ they are projecting to, so we are going to think about it as the space of subspaces of $\mathbb{C}^\infty$ of rank d.

EXERCISE 28. *Let $V \subseteq \mathbb{C}^\infty$ be a d-dimensional subspace and P_V. Then, if $V^\perp \subseteq \mathbb{C}^\infty$ is the orthogonal complement of V we have a map*

$$\mathrm{Hom}_{\mathbb{C}}(V, V^\perp) \to \mathrm{Gr}_d$$

sending f to the graph subspace of f, i.e. the subspace of all elements $\{(v, fv) \in V \oplus V^\perp \cong \mathbb{C}^\infty \mid v \in V\}$. Show that this is an open embedding, identifying $\mathrm{Hom}_{\mathbb{C}}(V, V^\perp)$ with the subspace of those $W \subseteq \mathbb{C}^\infty$ such that the orthogonal projection $W \to V$ is an isomorphism.

There is a vector bundle $\xi_d \to \mathrm{Gr}_d$ where $\xi_d = \{(v, P) \mid Pv = v\}$, called the tautological vector bundle over Gr_d. This is a vector bundle because it can be trivialized by the charts of exercise 28.

THEOREM 4.6. *Let X be a paracompact Hausdorff space. Then the map*

$$[X, \mathrm{Gr}_d] \to \pi_0 \operatorname{Vect}_d(X) \qquad f \mapsto f^*\xi_d$$

is a bijection.

PROOF. First of all let us notice that this is well defined. Indeed if $H : X \times |\Delta^1| \to \mathrm{Gr}_d$ is a homotopy, lemma 4.3 we have that the restrictions of $H^*\xi_d$ to $X \times \{0\}$ and $X \times \{1\}$ are isomorphic, so the map is in fact well defined.

Now notice that there is a bijection between pairs of a map $f : X \to \mathrm{Gr}_d$ and an isomorphism $\varphi : E \cong f^*\xi_d$, and maps $E \to \mathbb{C}^\infty$ that are linear embeddings on each fiber. In fact in one direction, if we have a fiberwise embedding $g : E \to \mathbb{C}^\infty$ we can send it to the pairs $f(x) = P_{gE_x}$ and the induced map $E \to \xi_d \subseteq \mathbb{C}^\infty \times \mathrm{Gr}_d$. Viceversa if we have such a pair (f, φ) we can send it to the composition $E \to \xi_d \to \mathbb{C}^\infty$.

Therefore by lemma 4.4 the map in the statement is surjective. Let us prove that it is injective.

Suppose that $f, g : E \to \mathbb{C}^\infty$ are two different fiberwise embeddings. We need to construct a homotopy of them through fiberwise embeddings. First of all by postcomposing through the homotopy

$$(t, (x_0, x_1, \dots)) \mapsto (1 - t)(x_0, x_1, \dots) + t(x_0, 0, x_1, \dots)$$

we can move f and g to maps that land in the even (resp. odd) numbered coordinates. But then we can just do a straight-line homotopy between them. □

EXERCISE 29. *Let $p : E \to X$ be a vector bundle. Then $\Omega_p \operatorname{Map}(X, \mathrm{Gr}_d)$ is equivalent to the space of automorphisms of the vector bundle p (hint: use exercise 27.*

Our next goal is to show that the space Gr_d is a model for BU_d. We will do so by using a criterion that will be useful also in the following.

LEMMA 4.7. *Let $p : E \to B$ be a fibration of topological spaces and G be a topological group acting on E over B such that the map $G \times E \to E \times_B E$ sending (g, e) to (ge, e) is a homeomorphism. Then if E is contractible there's an equivalence of E_1-spaces $G \cong \Omega B$.*

PROOF. Note that the E_1-group ΩB is the simplicial space sending

$$[n] \mapsto * \times_B * \times_B \cdots \times_B * \cong E \times_B E \times_B \cdots \times_B E$$

since E is contractible. But since $E \to B$ is a fibration we can compute the homotopy pullbacks as pullbacks of topological spaces. Note that there is a homeomorphism

$$G \times E \cong E \times_B E \qquad (g, e) \mapsto (ge, e)$$

over E, and so the group ΩB can be realized as the simplicial space

$$[n] \mapsto G \times \cdots G \times E$$

But since E is contractible this is just the simplicial space representing the group G. □

Let the infinite Stiefel manifold be $V_d := \operatorname{Emb}(\mathbb{C}^d, \mathbb{C}^\infty)$ be the space of isometric embeddings of $\mathbb{C}^d$ into $\mathbb{C}^\infty$. Equivalently, this is the space of d-uples $(v_1, \dots, v_d)$ of pairwise orthogonal elements in $\mathbb{C}^\infty$ such that $|v_i|^2 = 1$. For example we can identify V_1 with the infinite sphere S^∞. There is an obvious map $V_d \to \mathrm{Gr}_d$ sending every embedding to its image.

PROPOSITION 4.8. *The space V_d is contractible.*

PROOF. Let us consider $\tilde{V}_d$ the space of all embeddings $\mathbb{C}^d \to \mathbb{C}^\infty$, not necessarily isometric. There's an obvious inclusion $V_d \subseteq \tilde{V}_d$. We claim that it is a homotopy equivalence. In fact we can write a sequence $V_d := \tilde{V}_d^{(d)} \subseteq \tilde{V}_d^{(d-1)} \subseteq \cdots \tilde{V}_d^{(0)}$ where $\tilde{V}_d^{(i)}$ is the subspace of those embeddings $f : \mathbb{C}^d \to \mathbb{C}^\infty$ such that $f|_{\mathbb{C}^i}$ is isometric. Then we claim that the inclusion $V_d^{(i)} \subseteq V_d^{(i-1)}$ is a homotopy equivalence. Indeed we can retract first $\tilde{V}_d^{(i-1)}$ onto $\tilde{V}_d^{(i-1)'}$, spanned by the subspace of those embeddings such that fe_i is orhtogonal to $f(\mathbb{C}^{i-1})$ via the homotopy

$$(t, f) \mapsto f(P_i -) + (1 - tP_i)f \circ (1 - P_i)(-)$$

where P_i is the projection onto $f(\mathbb{C}^i)$. Finally we can retract $\tilde{V}_d^{(i-1)'}$ onto $\tilde{V}_d^{(i)}$ by rescaling.

So it suffices to show that $\tilde{V}_d$ is contractible. To do so let us consider $W_d \subseteq \tilde{V}_d$ be the subspace of those embeddings f such that $f(\mathbb{C}^d) \perp \mathbb{C}^d$. We claim that this inclusion has a deformation retraction given by

$$(t, f) \mapsto (1 - t + tT^d) \circ f$$

where $T^d : \mathbb{C}^\infty \to \mathbb{C}^\infty$ is the linear map sending x to $(T^d x)_i = x_{i-d}$ (where we assume $x_i = 0$ for $i < 1$). It is easy to see that $1 - t + tT^d : \mathbb{C}^\infty \to \mathbb{C}^\infty$ is injective for every t, and so the map is indeed a deformation retraction. On the other hand we want to show that the inclusion $W_d \to \tilde{V}_d$ is nullhomotopic. This is because if $f_0 : \mathbb{C}^d \to \mathbb{C}^\infty$ is the standard embedding we can do a straight-line homotopy for every map $f \in W_d$

$$(t, f) \mapsto tf_0 + (1 - t)f$$

which is obviously injective for every t. $\square$

Let U_d be the group of $d \times d$ unitary matrices, that is the group of isometries from $\mathbb{C}^d$ to $\mathbb{C}^d$.

COROLLARY 4.9. *The space* Gr_d *is equivalent to* BU_d.

PROOF. The space V_d has an action of the group U_d by precomposition and the projection $p : V_d \to \mathrm{Gr}_d$ is equivariant, when we give Gr_d the trivial action. We claim that this projection satisfies the hypotheses of lemma 4.7. Indeed for $V \in \mathrm{Gr}_d$ let $H_V := \mathrm{Hom}(V, V^\perp) \subseteq \mathrm{Gr}_d$ be the open neighborhood of exercise 28. Then for every isometry $f : \mathbb{C}^d \cong V$, the map $U_d \times H_V \to p^{-1}H_V$ sending (A, W) to is a U_d-equivariant homeomorphism. $\square$

Let now $\mathrm{Gr} := \coprod_d \mathrm{Gr}_d \cong \coprod_d BU_d$

DEFINITION 4.10. *The* E_∞*-space of vector spaces is the functor* $\mathrm{Fin}_* \to \mathrm{Top}$

$$\mathrm{Vect} : I_+ \mapsto \{(V_i)_{i \in I} \mid V_i \perp V_j \ \forall i \neq j\} \subseteq \mathrm{Gr}^I$$

where a map $f : I_+ \to J_+$ *sends* $(V_i)_{i \in I}$ *to* $\left(\bigoplus_{fi=j} V_j\right)_{j \in J}$.

LEMMA 4.11. Vect *is an* E_∞*-space. Moreover for any space* X *we have that* $\pi_0 \mathrm{Map}(X, \mathrm{Vect})$ *is the monoid of isomorphism classes of vector bundles on* X *under direct sum.*

PROOF. Let $I_+ \in \mathrm{Fin}_*$ and $d = (d_1, \ldots, d_n)$. Then we can write $\mathrm{Vect}_d(n_+)$ as the subspace of $\mathrm{Vect}(n_+)$ given by those V_i such that $\dim_{\mathcal{C}} V_i = d_i$. Then we have a decomposition

$$\mathrm{Vect}(n_+) = \coprod_{d_1, \ldots, d_n} \mathrm{Vect}_d(n_+)$$

Therefore to prove that Vect is an E_∞-space it suffices to prove that the map

$$\mathrm{Vect}_d(n_+) \to \prod_{i=1}^{n} \mathrm{Vect}_{d_i}$$

is an equivalence. But not that there is a fiber bundle

$$V_{d_1+\cdots+d_n} \to \mathrm{Vect}_d(n_+)$$

sending $f : \mathbb{C}^{d_1+\cdots d_n} \to \mathbb{C}^\infty$ to $(f(\mathbb{C}^{d_i}))_{i=1,\dots,n}$. Working as in the proof of corollary 4.9 we see that $\mathrm{Vect}_d(n_+) \cong B(\prod_{i=1}^n U_i) \cong \prod_{i=1}^n BU_{d_i}$. Moreover the projection $\mathrm{Vect}_d(n_+) \to \mathrm{Vect}_{d_i}$ corresponds to the projection to the i-th factor under this isomorphism.

To prove the second statement, it is enough to reduce to the case where X is connected (since both functors send coproducts to products). But then

$$\pi_0 \operatorname{Map}(X, \mathrm{Vect}) \cong \pi_0 \operatorname{Map}(X, \coprod_{n\geq 0} BU_n) \cong \coprod_{n\geq 0} \operatorname{Map}(X, BU_n) \cong \coprod_{n\geq 0} \mathrm{Vect}_n(X)\,.$$

□

Recall that a space X is called finite if it can be obtained by the contractible space $*$ and the empty space $\varnothing$ via pushouts. In particular the homotopy type of every finite CW-complex if finite.

Exercise 30. *Prove that if X is a finite space, it is the homotopy type of a finite CW complex (hint: first prove that all its homotopy groups are finitely generated, then construct a cell structure inductively by attaching cells).*

Lemma 4.12. *For any finite space X the underlying space of* $\operatorname{Map}(X, \mathrm{Vect})^{\mathrm{gp}}$ *is* $\operatorname{Map}(X, BU \times \mathbb{Z})$. *In particular we have an equivalence* $\mathrm{Vect}^{\mathrm{gp}} \cong BU \times \mathbb{Z}$.

Proof. First of all we let $1 \in \pi_0 \mathrm{Vect} = \mathbb{N}$, so we know that the underlying space of $\mathrm{Vect}^{\mathrm{gp}}$ is $\mathrm{tel}_1 \mathrm{Vect}^+$. Note that

$$\mathrm{tel}_1 \mathrm{Vect} = \operatorname{colim}\left(\coprod_{n\geq 0} BU_n \xrightarrow{+1} \coprod_{n\geq 0} BU_n \xrightarrow{+1} \cdots \right)$$

and so we have

$$\mathrm{tel}_1 \mathrm{Vect} \cong \coprod_{n\in\mathbb{Z}} \operatorname*{colim}_{m\to\infty} BU_m \cong BU \times \mathbb{Z}\,.$$

Since the right hand side is simply connected, the plus construction does not do anything.

Now when X is finite, the functor $\operatorname{Map}(X, -)$ commutes with filtered colimits, therefore we have an equivalence

$$\mathrm{tel}_1 \operatorname{Map}(X, \mathrm{Vect}) \cong \operatorname{Map}(X, \mathrm{tel}_1 \mathrm{Vect}) \cong \operatorname{Map}(X, \mathrm{Vect}^{\mathrm{gp}})$$

Now the π_0 of the right hand side is a group, so 1 satisfies the hypothesis of the group completion theorem. Therefore

$$\operatorname{Map}(X, \mathrm{Vect})^{\mathrm{gp}} \cong \mathrm{tel}_1 \operatorname{Map}(X, \mathrm{Vect}) \cong \operatorname{Map}(X, \mathrm{Vect}^{\mathrm{gp}})^+ \cong \operatorname{Map}(X, \mathrm{Vect}^{\mathrm{gp}})\,.$$

□

Remark 4.13. *The finiteness of X is necessary (it can be slightly weakened to finitely dominated spaces but no more), one can construct counterexamples otherwise. However one can give an interpretation of* $\operatorname{Map}(X, BU \times \mathbb{Z})$ *in terms of vector bundles by using the theory of Fredholm complexes.*

Definition 4.14. *The connective (complex) topological K-theory spectrum* ku *is the spectrum* $\mathbf{B}^\infty \mathrm{Vect}$. *It has the property that* $\Omega^\infty \mathrm{ku} \cong BU \times \mathbb{Z}$.

PROPOSITION 4.15. $B\,\mathrm{Vect} \cong U$

PROOF. Our goal is then to show that

$$\operatorname*{colim}_n \mathrm{Vect}([n]) \cong U$$

We will show that it is actually *homeomorphic* to the model of example 1.75. In fact there is a map

$$\coprod_n \mathrm{Vect}([n]) \times |\Delta^n| \to U \qquad (V_1, \dots, V_n, (t_0, \dots, t_n)) \mapsto \sum_{i=1}^n e^{2\pi i \sum_{j \leq i} t_j} P_{V_i} + P_{(\bigoplus V_i)^\perp}$$

where $P_V : \mathbb{C}^\infty \to \mathbb{C}^\infty$ is the orthogonal projection onto the subspace $V \subseteq \mathbb{C}^\infty$. Said it differently, this sends $(V_1, \dots, V_n, (t_0, \dots, t_n))$ to the matrix $A \in U$ such that V_i is an eigenspace of A with eigenvalue $e^{2\pi t_i}$ and A restricts to the identity on the orthogonal complement of $\bigoplus_i V_i$. It is easy to see that this is compatible with the simplicial relations.

The spectral theorem then says that this map is a homeomorphism. □

This proof of Bott periodicity is due to Bruno Harris [**Har80**]

COROLLARY 4.16 (Bott periodicity). *There is a homotopy equivalence $\Omega U \cong BU \times \mathbb{Z}$. In particular we have $\Omega^2 U \cong U$.*

PROOF. From the proposition 4.15 it follows that

$$\Omega U \cong \Omega B\,\mathrm{Vect} \cong \mathrm{Vect}^{\mathrm{gp}}$$

Note that $\pi_0 \mathrm{Vect} \cong \mathbb{N}_{\geq 0}$, since complex vector spaces are determined up to isomorphism by their dimension. □

DEFINITION 4.17. *The (complex) topological K-theory spectrum* KU *is the spectrum with spaces $(BU \times \mathbb{Z}, U, BU \times \mathbb{Z}, \dots)$ with the connecting maps given by the Bott periodicity theorem.*

REMARK 4.18. *The Bott periodicity theorem implies that $U \cong \Omega^\infty \Sigma\,\mathrm{KU}$ is an E_∞-space, even if its underlying E_1-space comes from a non-commutative topological group.*

REMARK 4.19. *Using the same ideas one can prove half of the real Bott periodicity theorem: $\Omega U/O \cong BO \times \mathbb{Z}$ and $\Omega U/Sp \cong BSp \times \mathbb{Z}$. There are four more equivalences that need to be proven to deduce the complete theorem. To do so a new idea is required (the Wood fiber sequence).*

2. Computations in topological K-theory

Let us choose an isometry $\varphi : \mathbb{C}^\infty \otimes \mathbb{C}^\infty \cong \mathbb{C}^\infty$ (for example by choosing a bijection $\varphi : \mathbb{N} \times \mathbb{N} \cong \mathbb{N}$ and sending $e_i \otimes e_j$ to $e_{\varphi(i,j)}$). Then we can define a natural transformation

$$\mathscr{V}(I_+) \wedge \mathscr{V}(J_+) \to \mathscr{V}(I_+ \wedge J_+)$$

sending $(V_i), (W_j)$ to $\varphi(V_i \otimes W_j)$. Therefore we can get by induction maps

$$B^n\mathscr{V}(I_+) \wedge B^m\mathscr{V}(J_+) \to B^{n+m}\mathscr{V}(I_+ \wedge J_+)$$

and thus a map

$$\mu : \mathrm{ku} \otimes \mathrm{ku} \to \mathrm{ku}$$

where $\mathrm{ku} = B^\infty \mathscr{V}$ is the classifying spectrum. Moreover we have a map $\eta : \mathbb{S} \to \mathrm{ku}$ corresponding to the point $1 \in \mathbb{Z} = \pi_0 \mathrm{ku}$ (i.e. to the selection of $\mathbb{C} \subseteq \mathbb{C}^\infty$. If we choose the isometry so that it sends $\mathbb{C} \otimes \mathbb{C}$ to $\mathbb{C}$ then we see that $\mu(\eta \circ 1) = \mu(1 \circ \eta) = \mathrm{id}_{\mathrm{ku}}$.

LEMMA 4.20. *The pair η and μ turn* ku *into a homotopy ring spectrum, i.e. an algebra object in* hSp.

PROOF. It boils down to proving that any two isometries $\mathbb{C}^\infty \otimes \mathbb{C}^\infty \otimes \mathbb{C}^\infty \cong \mathbb{C}^\infty$ are homotopic, i.e. that any two isometries $\mathbb{C}^\infty \cong \mathbb{C}^\infty$ are homotopic. But the contractibility of the space of isometries is proven in a similar way to proposition 4.8. $\square$

CONSTRUCTION 4.21. *Let E be a homotopy ring spectrum. Then the cohomology theory E^*X is multiplicative, i.e. there's a natural multiplication map*

$$E^*X \otimes E^*Y \to E^*(X \wedge Y)$$

turning $E^*(X_+)$ into a ring. Moreover the ring is graded commutative if E is homotopy commutative.

Moreover we have a class $\beta \in \pi_2 \mathrm{ku} \cong \pi_0 B^2\mathscr{V} \cong \pi_0 BU$, such that the corresponding map $\Omega^\infty\Sigma^{-1}\mathrm{ku} \to \Omega^\infty\Sigma\,\mathrm{ku}$ is exactly the identification of proposition 4.15. Therefore we can write

$$\mathrm{KU} = \mathrm{colim}(\mathrm{ku} \xrightarrow{\beta} \Sigma^{-2}\mathrm{ku} \xrightarrow{\beta} \Sigma^{-4}\mathrm{ku} \to \cdots)$$

If $p : E \to X$ is a vector bundle, we let $\mathbb{P}E := (E \smallsetminus 0)/\mathbb{C}^\times$ with the obvious projection to X inherited by E. This corresponds on a local trivialization to replace $U \times V$ by $U \times \mathbb{P}V$. It is called the projective bundle associated to E. There is a canonical line bundle η_E on $\mathbb{P}E$ given by

$$\eta_E := \{(x, L) \in E \times_X \mathbb{P}_E \mid x \in L\}$$

PROPOSITION 4.22 (Projective line bundle formula). *Let $L \to X$ be a line bundle (i.e. a vector bundle of rank 1). Then there is a canonical decomposition*

$$\mathrm{KU}^*(\mathbb{P}(1 \oplus L)) \cong \mathrm{KU}^* X \oplus \mathrm{KU}^* X \cdot [\eta_L]\,.$$

of $\mathrm{KU}^* X$*-modules.*

PROOF. There's a map $p^* : \mathrm{KU}^* X \to \mathrm{KU}^*(\mathbb{P}(1 \oplus L))$ induced by the projection $\mathbb{P}(1 \oplus L) \to X$. Therefore we get a natural transformation

$$\mathrm{KU}^* X \oplus \mathrm{KU}^* X \to \mathrm{KU}^*(\mathbb{P}(1 \oplus L)) \qquad (a, b) \mapsto a + b[\eta_L]$$

corresponding to a map of spectra

$$\mathrm{map}(\Sigma^\infty_+ X, \mathrm{KU}) \oplus \mathrm{map}(\Sigma^\infty_+ X, \mathrm{KU}) \to \mathrm{map}(\mathbb{P}(1 \oplus L), \mathrm{KU})\,.$$

We want to claim it is an equivalence. We can consider the poset $\mathcal{P}$ of open subsets $U \subseteq X$ where this map is an isomorphism. Note that if $U, V, U \cap V \in \mathcal{P}$ then $U \cup V \in \mathcal{P}$ by the Mayer-Vietoris sequence. Moreover if we have a chain of open subsets $\{U_i\}_{i \in I}$ in $\mathcal{P}$, then their union is in $\mathcal{P}$. Therefore by Zorn's lemma we can conclude if we can show that any trivialinzing open subset is in $\mathcal{P}$, that is if we can show the case when L is trivial. But then $\square$

UNFINISHED

CHAPTER 5

Localizations and completions

In this section we will study the concept of *Bousfield localization* of spectra. Pursuing the analogy between spectra and abelian groups, we would like to study a spectrum by concentrating in its simpler parts, e.g. its "p-torsion" for a prime p, or its "torsion-free" part. It turns out that the ∞-category of spectra is much more complicated than the category of abelian groups, and it contains many interesting phenomena of arithmetic nature. In this section we will follow Bousfield's original articles [**Bou79a**] and [**Bou79b**]. For a modern introduction to the notion of Bousfield localization in a more general context, see [**Law20**].

1. Smallness conditions in spectra

Let κ be a regular cardinal. We will say that a simplicial set T is κ-small if it has at most κ non-degenerate simplices. Equivalently it can be constructed from simplices by sequential colimits, pushouts and coproducts of cardinality at most κ.

Definition 5.1. *A simplicial set S is κ-filtered if for every κ-small simplicial set K and every map $f : K \to S$ there exists an extension $\bar{f} : K^{\triangleright} \to S$.*

Example 5.2. *If $\mathcal{C}$ is an ∞-category with all κ-small colimits, $\mathcal{C}$ is κ-filtered.*

Exercise 31. *Let κ be a cardinal which is not regular. Then a simplicial set S is κ-filtered if and only if S is κ_+-filtered, where κ_+ is the successor cardinal.*

Proposition 5.3. *Let S be a κ-filtered simplicial set. Then the functor*

$$\operatorname{colim} : \operatorname{Fun}(S, \mathscr{S}) \to \mathscr{S}$$

commutes with κ-small limits.

Proof. [**Lur09**, Proposition 5.3.3.3] ■ □

Denis: There should be a more direct proof using the fact that $S \to S^K$ is cofinal and descent

Definition 5.4. *Let Sp^κ be the full subcategory of Sp generated by desuspensions $\Sigma^n\mathbb{S}$ of the sphere under κ-small colimits.*

Remark 5.5. *Let $E \in \mathrm{Sp}^\kappa$. Then the functor $\mathrm{Map}_{\mathrm{Sp}}(E, -) : \mathrm{Sp} \to \mathscr{S}$ commutes with κ-filtered colimits. Indeed by proposition 5.3 the collection of spectra E such that $\mathrm{Map}_{\mathrm{Sp}}(E, -)$ commutes with κ-filtered colimits is closed under κ-small limits, and it contains all spheres.*

Lemma 5.6. *There are only a set of equivalence classes of elements in Sp^κ.*

Proof. We will construct Sp^κ as the union of κ subcategories $\{\mathcal{C}_\alpha\}_{\alpha<\kappa}$ of Sp, each of which is small, defined by transfinite recursion. Let $\mathcal{C}_0$ be the full subcategory spanned by $\Sigma^n\mathbb{S}$ for $n \in \mathbb{Z}$. Then for $\alpha < \kappa$ ordinal number suppose we have defined $\mathcal{C}_\alpha$. Then we let $\mathcal{C}_{\alpha+1}$ the full subcategory spanned by all colimits of all diagrams $K \to \mathcal{C}_\alpha$ for K κ-small simplicial set. Since the set of all possible K is small, then the set of all possible diagrams $K \to \mathcal{C}_\alpha$ is small as well, and therefore $\mathcal{C}_{\alpha+1}$ is also small. Finally for $\lambda < \kappa$ limit ordinal we define

$$\mathcal{C}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{C}_\alpha\,.$$

Clearly $\mathcal{C}_\alpha \subseteq \mathrm{Sp}^\kappa$ for all $\alpha \leq \kappa$. It is enough to show that $\mathcal{C}_\kappa$ is closed under κ-small colimits, so that $\mathrm{Sp}^\kappa = \mathcal{C}_\kappa$. Let now K be a κ-small simplicial set and $f : K \to \mathcal{C}_\kappa$ be a diagram. Then for every $k \in K$, there is $\alpha_k < \kappa$ such that $fk \in \mathcal{C}_\alpha$. But then we can let $\beta = \sup_{k\in K} \alpha_k$, which is still less than κ. So f factors through $\mathcal{C}_\beta$, and therefore $\operatorname{colim}_K f$ lies in $\mathcal{C}_{\beta+1} \subseteq \mathcal{C}_\kappa$. □

PROPOSITION 5.7. *Let $E \in \mathrm{Sp}$ be a spectrum. Then one can write E as a κ-filtered colimit of elements of Sp^κ.*

PROOF. Let $J = \mathrm{Sp}^\kappa \times_{\mathrm{Sp}} \mathrm{Sp}_{/E}$ be the ∞-category of spectra with a map to E. Then J has κ-small colimits, and therefore it is κ-filtered. Then let us consider the map

$$\operatorname*{colim}_{X\in J} X \to E\,.$$

We want to prove that this map is an equivalence. Since J is filtered, it commutes with taking homotopy groups, so it is enough to prove that the map

$$\operatorname*{colim}_{X\in J} \pi_* X \to \pi_* E$$

is an isomorphism. It is surjective because if we have a map $\alpha : \Sigma^n\mathbb{S} \to E$, then we can interpret it as an element of J and then α Is the image of $1 \in \pi_n\Sigma^n\mathbb{S} \to \operatorname{colim}_{X\in J} \pi_* X$.

Now let us show it is injective. Suppose we have $[f : X \to E] \in J$ and $\alpha \in \pi_n X$ such that $f\alpha = 0$. Then if we interpret α as a map $\Sigma^n\mathbb{S} \to X$, then f factors through the cofiber Y of X, Therefore we have

$$\alpha \in \pi_n X \to \pi_n Y \to \operatorname*{colim}_{Z\in J} Z$$

is zero. □

EXERCISE 32. *Let κ be a regular cardinal. Then a spectrum E has the property that $\mathrm{Map}_{\mathrm{Sp}}(E, -)$ commutes with κ-filtered colimits if and only if E is a retract of an element of Sp^κ.*

EXERCISE 33. *Let κ be a regular cardinal. Then a spectrum E has the property that $\mathrm{Map}_{\mathrm{Sp}}(E, -)$ commutes with κ-filtered colimits if and only if E is in Sp^κ. (Hint: if κ is uncountable, retract can be computed via a sequential colimit. If κ is countable work by induction on the degree of the lowest homology group of E).*

To prove the following theorem we need a result by Serre.

THEOREM 5.8 (Serre). *For $i > n$ the homotopy group $\pi_i S^n$ is finite, unless $i = 2n - 1$ and n is even. In particular $\pi_i\mathbb{S} = \pi_{2i+2}S^{i+2}$ is a finite group for every positive i.*

LEMMA 5.9. *Let κ be an uncountable regular cardinal. Then a spectrum E is in Sp^κ if and only if $\pi_* E < \kappa$. In particular every spectrum lies in Sp^κ for some κ.*

PROOF. First let us show that the collection of spectra with $\pi_* E < \kappa$ is closed under cofibers and κ-small coproducts. For the coproducts it follows immediately from the fact that κ-small groups are closed under κ-small direct sums. For the cofibers, if $f : E \to E'$ is a map of spectra and $\#\pi_* E, \#\pi_* E' < \kappa$, we have that if E'' is the cofiber of f we can write the short exact sequence

$$0 \to \operatorname{coker} f_* \to \pi_* E'' \to \ker f_{*-1} \to 0$$

and $\#\operatorname{coker} f_*, \#\ker f_* < \kappa$, therefore $\#\pi_* E'' < \kappa$. Since all κ-small colimits can be built out of cofibers and κ-small coproducts, and Serre's theorem proves that $\#\pi_*\mathbb{S} < \kappa$, we have that all κ-small spectra have $\#\pi_* E < \kappa$.

Let us suppose $\#\pi_*E < \kappa$. We will construct a sequence

$$E_0 \to E_1 \to E_2 \to \cdots \to E$$

such that for every $n \geq 0$ we have $\#\pi_*E_n < \kappa$, E_n is obtained by E_{n-1} by κ-small coproducts and cofibers, and the map $F_n \to F_{n+1}$ is zero on homotopy, where F_n is the fiber of $E_n \to E$. The second condition implies that $\pi_*(F_n) = \mathrm{colim}_n\, \pi_*(F_n) = 0$, so that $\mathrm{colim}_n E_n \cong E$. Since sequential colimits are κ-small this implies the thesis.

We start by letting $E_0 := \bigoplus_{\alpha\in\pi_*E} \Sigma^{|\alpha|}\mathbb{S}$ with the tautological map $E_0 \to E$. Suppose now we have constructed E_n, we want to construct E_{n+1}. By the previous argument we have $\#\pi_*F_n < \kappa$, so we can write a map

$$g_n : \bigoplus_{\alpha\in\pi_*F_n} \Sigma^{|\alpha|}\mathbb{S} \to F_n \to E_n$$

such that the first map in the composition is tautologically surjective on homotopy. Then note that the composition $f_n g_n$ is canonically nullhomotopic (since it factors through $F_n \to E_n \to E$, and so f_n factors through the cofiber E_{n+1} of g_n. Moreover, if we let F_{n+1} the fiber of $f_{n+1} : E_{n+1} \to E$ we have that the induced map $F_n \to F_{n+1}$ is zero in homotopy by the long exact sequence of the fiber sequence

$$\bigoplus_{\alpha\in\pi_*F_n} \Sigma^{|\alpha|}\mathbb{S} \to F_n \to F_{n+1}$$

This proves all the properties we wanted, therefore proving the thesis. □

2. Bousfield localizations

Throughout this section, we will assume that κ is an uncountable regular cardinal.

Recall that for a spectrum E, the E-homology of a spectrum X is the graded abelian group $E_*X := \pi_*(E \otimes X)$.

Definition 5.10. *Let E be a spectrum. We say that a map $f : X \to Y$ of spectra is an E-**equivalence** if the induced map $E \otimes f : E \otimes X \to E \otimes Y$ is an equivalence. Equivalently, a map if an E-equivalence if and only if it induces an equivalence in E-homology. We say that a spectrum X is E-**acyclic** if the map $0 \to X$ is an E-equivalence, that is if $E \otimes X = 0$.*

*A spectrum Z is E-**local** if for every E-equivalence $f : X \to Y$ the map $f^* : \mathrm{map}(Y, E) \to \mathrm{map}(X, E)$ is an equivalence. We will denote the full subcategory of E-local spectra with Sp_E.*

Example 5.11. *Let $n \in \mathbb{Z}$. We will write $\mathbb{S}[1/n]$ for the spectrum obtained by the colimit*

$$\mathrm{colim}\left(\mathbb{S} \xrightarrow{n} \mathbb{S} \xrightarrow{n} \mathbb{S} \xrightarrow{n} \cdots\right).$$

Then we will see that a spectrum is $\mathbb{S}[1/n]$-local if and only if n acts invertibly on it or, equivalently, if the homotopy groups are modules over $\mathbb{Z}[1/n]$.

Example 5.12. *Let $\mathbb{S}/n$ be the cofiber of the multiplication by n map $\mathbb{S} \to \mathbb{S}$. Then a spectrum X is $\mathbb{S}/n$-acyclic if and only if the map $n : X \to X$ multiplication by n acts invertibly. If $n = p$ is a prime an $\S/p$-local spectrum is called p-**complete**.*

The following lemma is the crucial smallness condition that will be necessary to make the theory of localizations work.

Lemma 5.13. *Let E be a spectrum and κ be a regular cardinal such that $\#\pi_*E < \kappa$. Then the subcategory of Sp generated under colimit by κ-small E-acyclic spectra is the full subcategory of E-acyclic spectra.*

To prove the lemma we will need to auxiliary results first.

LEMMA 5.14. *If $\#\pi_*E < \kappa$, then for every κ-small spectrum W, we have $\#E_*W < \kappa$.*

PROOF. It's enough to prove that the collection of spectra W with $\#E_*W < \kappa$ is closed under κ-small colimits, that is cofibers and κ-small coproducts. For coproducts it's obvious, for cofibers if we have $X \to Y \to Z$ cofiber sequence with X and Y satisfying the required condition, we have an exact sequence

$$E_*Y \to E_*Z \to E_{*-1}X$$

and so $\#E_*Z \leq \#E_*Y + \#E_*X < \kappa$. □

LEMMA 5.15. *Let E be a spectrum with $\#\pi_*E < \kappa$. Then for every E-acyclic spectrum X and every $\alpha \in \pi_*E$ there is a κ-small E-acyclic spectrum W and a map $f : W \to X$ such that α is in the image of $\pi_*f : \pi_*W \to \pi_*X$.*

PROOF. Let $\alpha : \Sigma^n\mathbb{S} \to X$ We will construct a sequence

$$\Sigma^n\mathbb{S} = W_0 \to W_1 \to W_2 \to \cdots \to X$$

of κ-small spectra with maps $f_i : W_i \to X$ such that $f_0 = \alpha$, W_n is κ-small and $\#E_*W_n < \kappa$ and the map $W_i \to W_{i-1}$ is trivial in E-homology. Therefore, if we let $W = \operatorname{colim}_n W_n$ we have that

$$E_*W = \operatorname*{colim}_n E_*W_n = 0$$

and so W is E-acyclic, as required. Moreover it is κ-small because κ is uncountable.

The pair $(W_0, f_0) = (\Sigma^n\S, \alpha)$ is determined by our conditions. Suppose we have W_n, let us construct W_{n+1}. We can write the fiber sequence

$$W_n \xrightarrow{f_n} X \to \bar{W}_n$$

Using the long exact sequence in E-homology and the fact that $E_*X = 0$ we have that the map $E_{*+1}\bar{W}_n \to E_*W_n$ is an isomorphism. In particular $\#E_*\bar{W}_n < \kappa$. Now we can write

$$E_*\bar{W}_n \cong \operatorname*{colim}_{F \in J} E_*F$$

where J is the category of finite spectra mapping to $\bar{W}_n$. Therefore for every $\beta \in E_*\bar{W}_n$ we can find F_β finite spectrum with a map to $\bar{W}_n$ and $\tilde{\beta}$ lift of β to E_*F_β. That is the map

$$K := \bigoplus_{\beta \in E_*\bar{W}_n} F_\beta \to \bar{W}_n$$

is surjective in E-homology. Hence the composite $\Omega K \to \Omega\bar{W}_n \to W_n$ is surjective on E-homology as well. Moreover the composte $\Omega K \to W_n \to X$ is the zero map, so if we let W_{n+1} be the cofiber of $\Omega K \to W_n$ we have that $W_n \to X$ factors (perhaps not uniquely) to W_{n+1}. It is easy now to verify that W_{n+1} has all the properties we required of it. □

COROLLARY 5.16. *There exists an E-acyclic spectrum A such that X is E-local if and only if* $\operatorname{map}(A, X) = 0$.

PROOF. Let κ be a sufficiently big regular cardinal. Then let A be the direct sum of all κ-small E-acyclic spectra. Then $\operatorname{map}(A, X) = 0$ if and only if $\operatorname{map}(W, X) = 0$ for every κ-small E-acyclic spectrum W. But since by lemma 5.13 every E-acyclic spectrum is a colimit of κ-small E-acyclic spectra, this implies that $\operatorname{map}(A, X) = 0$ if and only if $\operatorname{map}(W, X) = 0$ for every E-acyclic spectrum W, which was what we needed to prove. □

We can finally conclude the proof of our crucial lemma.

PROOF OF LEMMA 5.13. Let X be an E-acyclic spectrum and J be the subcategory of $\mathrm{Sp}_{/X}$ spanned by κ-small E-acyclic spectra. Since it has κ-small colimits it is κ-filtered (therefore filtered). Then we claim that

$$\operatorname*{colim}_{W \in J} W \to X$$

is an equivalence. It is enough to verify that

$$\operatorname*{colim}_{W \in J} \pi_* W \to \pi_* X$$

is an isomorphism. The lemma 5.15 tells us that it is surjective. Let us prove now that it is injective. Let $f : W \to X$ be an element of J and $\alpha \in \pi_n W$ such that $f\alpha = 0$. Then, if $g : Y \to W$ is the fiber of f, we can find a lift $\beta \in \pi_n Y$ of α. But Y is E-acyclic, and so by lemma ... we can find V κ-small acyclic spectrum and $h : V \to Y$ such that β is in the image of h, hence α is in the image of gh. In particular, if we let $p : W \to W'$ be the cofiber of $gh : V \to W$ we have that $p\beta = 0$. Moreover, since $fgh = 0$, the map f factors through $f' : W' \to X$ and W' is κ-small and E-acyclic. Therefore the class of α dies in the colimit. □

PROPOSITION 5.17. *Let X be a spectrum. Then there exists an E-local spectrum $L_E X$ and an E-equivalence $l : X \to L_E X$. In particular the inclusion $\mathrm{Sp}_E \subseteq \mathrm{Sp}$ has a left adjoint $L_E : \mathrm{Sp} \to \mathrm{Sp}_E$, which we call the E-localization functor.*

PROOF. Let A be the spectrum of corollary 5.16. Then by lemma 5.9 we know that A is κ-small for some regular cardinal κ (which could be bigger than the one we chose earlier). Then we construct a sequence $\{X_\alpha\}_{\alpha \leq \kappa}$ as follows: we let $X_0 = X$. Given X_α we define $X_{\alpha+1}$ to be the cofiber of

$$\bigoplus_{n \in \mathbb{Z}, f : \Sigma^n A \to X_\alpha} \Sigma^n A \to X_\alpha$$

and if λ is a limit ordinal we let $X_\lambda = \operatorname{colim}_{\alpha < \lambda} X_\alpha$. Then we claim that X_κ is the spectrum we want. First notice that the map $X_\alpha \to X_{\alpha+1}$ is an E-equivalence for every α (since its fiber is E-acyclic) and so the map $X \to X_\kappa$ is an E-equivalence. Then it remains to prove that X_κ is E-local or, equivalently, that

$$\mathrm{map}(A, X_\kappa) = 0$$

It is enough to prove that for every $n \in \mathbb{Z}$

$$\pi_0 \operatorname{Map}(\Sigma^n A, X_\kappa) = 0$$

that is that every map $\Sigma^n A \to X_\kappa$ is nullhomotopic. So suppose we have a map $\Sigma^n A \to X_\kappa$. Since $\Sigma^n A$ is κ-small, this factors through some X_α for $\alpha < \kappa$. But then it factors through the composite

$$\Sigma^n A \to X_\alpha \to X_{\alpha+1}$$

which is zero, by the definition of $X_{\alpha+1}$. □

3. Inverting primes

Let p be a prime number. Then we can let

$$\mathbb{S}[1/p] = \operatorname{colim}\left(\mathbb{S} \xrightarrow{p} \mathbb{S} \xrightarrow{p} \cdots\right)$$

Note that for every spectrum X we have

$$\pi_*(\mathbb{S}[1/p] \otimes X) = \operatorname{colim}\left(\pi_* X \xrightarrow{p} \pi_* X \xrightarrow{p} \cdots\right) \cong \pi_* X[1/p]$$

In particular a spectrum X is $\mathbb{S}[1/p]$-acyclic iff $\pi_* X$ is locally p-power torsion, that is if every element of $\pi_* X$ is p-power torsion.

LEMMA 5.18. *A spectrum X is $\mathbb{S}[1/p]$-local if and only if p acts invertibly on $\pi_* X$.*

PROOF. If $p : X \to X$ is an equivalence and Y is a $\mathbb{S}[1/p]$-acyclic spectrum we have

$$0 \cong \mathrm{map}(Y[1/p], X) \cong \lim\left(\mathrm{map}(Y, X) \xrightarrow{p} \mathrm{map}(Y, X) \xrightarrow{p} \cdots\right) \cong \mathrm{map}(Y, X)\,.$$

Where we have used that $p : \mathrm{map}(Y, X) \to \mathrm{map}(Y, X)$ acts invertibly (since it does so on X). □

THEOREM 5.19. *Let X be a spectrum. The Bousfield localization of X at $\mathbb{S}[1/p]$ is $L_{\mathbb{S}[1/p]} X \cong \mathbb{S}[1/p] \otimes X =: X[1/p]$.*

PROOF. Note that the cofiber of $X \to \mathbb{S}[1/p] \otimes X$ is the colimit of the cofibers of the maps $X \xrightarrow{p^n} X$, so it is X/p^n. Since X/p^n is $\mathbb{S}[1/p]$-acyclic for every n we have that the cofiber of $X \to \mathbb{S}[p^{-1}]$ is □

More generally, if $S = \{p_1, p_2, \dots\}$ is a set of primes, we can let $\mathbb{S}[S^{-1}]$ to be the spectrum

$$\mathbb{S}[S^{-1}] \cong \mathrm{colim}\left(\mathbb{S} \xrightarrow{p_1} \mathbb{S} \xrightarrow{p_1 p_2} \mathbb{S} \xrightarrow{p_1 p_2 p_3} \cdots\right)$$

where the products gain more factors as long as there are more primes in S (so that if S is infinite, they gain more and more factors). As before we have

$$\pi_*(\mathbb{S}[S^{-1}] \otimes X) \cong \pi_* X[S^{-1}]\,.$$

LEMMA 5.20. *A spectrum X is $\mathbb{S}[S^{-1}]$-local if and only if every prime $p \in S$ acts invertibly on X, that is if X is $\mathbb{S}[1/p]$-local for every $p \in S$.*

PROOF. Note that for every $p \in S$ we have $\mathbb{S}/p \otimes \mathbb{S}[S^{-1}] = 0$, since p acts invertibly on $\mathbb{S}[S^{-1}]$. Therefore for any spectrum X $p : X \to X$ is an $\mathbb{S}[S^{-1}]$-equivalence. In particular if X is $\mathbb{S}[S^{-1}]$-local, it is an equivalence. Let us prove the viceversa. Suppose that p acts invertibly on X. Then we have

$$X \cong \mathrm{map}(\mathbb{S}[S^{-1}], X) \cong \lim(X \xleftarrow{p_1} X \xleftarrow{p_1 p_2} \cdots)\,.$$

So if A is $\mathbb{S}[S^{-1}]$-acyclic we have

$$\mathrm{map}(A, X) \cong \mathrm{map}(A, \mathrm{map}(\mathbb{S}[S^{-1}], X)) \cong \mathrm{map}(A \otimes \mathbb{S}[S^{-1}], X) \cong 0\,.$$

□

THEOREM 5.21. *Let X be a spectrum. The Bousfield localization of X at $\mathbb{S}[S^{-1}]$ is $L_{\mathbb{S}[S^{-1}]} X \cong \mathbb{S}[S^{-1}] \otimes X =: X[S^{-1}]$.*

PROOF. Note that the map $X \to \mathbb{S}[S^{-1}] \otimes X$ is an $\mathbb{S}[S^{-1}]$-equivalence, for example by checking on homotopy groups. Therefore it suffices to show that $\mathbb{S}[S^{-1}] \otimes X$ is $\mathbb{S}[S^{-1}]$-local. But it is clear that every p acts invertibly on it. □

REMARK 5.22. *These kind of Bousfield localizations, given by $L_E X \cong L_E \mathbb{S} \otimes X$ are called* ***smashing localizations****.*

When S is the set of all primes, we call this localization **rationalization** and we write it as $L_{\mathbb{S}[S^{-1}]} X =: X_{\mathbb{Q}}$. Note that

$$\pi_* \mathbb{S}_{\mathbb{Q}} = \begin{cases} \mathbb{Q} & \text{if } * = 0 \\ 0 & \text{otherwise.} \end{cases}$$

So $\mathbb{S}_{\mathbb{Q}} \cong H\mathbb{Q}$. It turns out that rational spectra are very easy.

Proposition 5.23. *Let X be a rational spectrum. Then there is an equivalence*

$$X \cong \bigoplus_{n\in\mathbb{Z}} \Sigma^n H\pi_n X\,.$$

Proof. Since X is rational, for every n the group $\pi_n X$ is a rational vector space. Let $(e_i^{(n)})$ be a basis. Then we have a map

$$\bigoplus_{i\in I} \Sigma^n \mathbb{S} \xrightarrow{e_i^{(n)}} X$$

which factors through the rationalization of the left hand side

$$i_n : H\pi_n X \cong \bigoplus_{i\in I} \Sigma^n H\mathbb{Q} \xrightarrow{e_i^{(n)}} X\,.$$

This map is an isomorphism on π_n and zero everywhere else. But then $\bigoplus_n i_n$ is an isomorphism on homotopy and therefore an equivalence. $\square$

Corollary 5.24 (Chern character). *Let X be a finite space. Then there's an equivalence*

$$\mathrm{KU}^0(X)_{\mathbb{Q}} \cong \bigoplus_{n\in\mathbb{Z}} H^{2n}(X;\mathbb{Q})\,.$$

Proof. Since $\mathrm{map}(\Sigma^\infty X_+, -)$ commutes with filtered colimits we have

$$\begin{aligned}\mathrm{KU}^0(X)_{\mathbb{Q}} &= \pi_0\,\mathrm{map}(\Sigma^\infty X_+, \mathrm{KU})_{\mathbb{Q}} \cong \pi_0\,\mathrm{map}(\Sigma^\infty X_+, \mathrm{KU}_{\mathbb{Q}}) \cong \\ &\cong \pi_0\,\mathrm{map}\left(\Sigma^\infty X_+, \bigoplus_{n\in\mathbb{Z}} \Sigma^{2n} H\mathbb{Q}\right) \cong \bigoplus_{n\in\mathbb{Z}} \pi_{-2n}\,\mathrm{map}(\Sigma^\infty X_+, H\mathbb{Q})\,.\end{aligned}$$

$\square$

In fact we have the following theorem (which follows more generally from the Schwede-Shipley Morita theory):

Theorem 5.25. *The ∞-category $\mathrm{Sp}_{\mathbb{Q}}$ of rational spectra is equivalent to the derived ∞-category $\mathscr{D}(\mathbb{Q})$ of the rational numbers.*

Exercise 34. *Let E be a spectrum. Show that there is a natural equivalence $\pi_* E_{\mathbb{Q}} \cong H_*(E;\mathbb{Q})$.*

4. Completion at a prime

Let p be a prime number. Then we say that a spectrum is p-complete if it is local for the Moore spectrum $\mathbb{S}/p$. We will write $E_p^\wedge := L_{\mathbb{S}/p}E$ for the Bousfield localization at $\mathbb{S}/p$, which we call p-completion.

Our first job is to justify this name. Let $\mathbb{S}/p^n$ be the cofiber of the map $\mathbb{S} \to \mathbb{S}[1/p]$. This is a connective spectrum whose integral homology is given by

$$H_*\mathbb{S}/p^\infty \cong \begin{cases} \mathbb{Z}/p^\infty & \text{if } * = 0 \\ 0 & \text{otherwise}\end{cases}$$

where $\mathbb{Z}/p^\infty \cong \mathbb{Z}[1/p]/\mathbb{Z} \cong \mathrm{colim}_n(\mathbb{Z}/p \xrightarrow{p} \mathbb{Z}/p^2 \xrightarrow{p} \mathbb{Z}/p^3 \xrightarrow{p} \cdots)$. Note that we can write, with the same proof

$$\mathbb{S}/p^\infty \cong \mathrm{colim}\left(\mathbb{S} \to \mathbb{S}/p \to \mathbb{S}/p^2 \to \mathbb{S}/p^3 \to \cdots\right)$$

Proposition 5.26. *There is a natural equivalence*

$$\mathrm{map}(\Sigma^{-1}\mathbb{S}[1/p]/\mathbb{S}, E) \cong E_p^\wedge$$

PROOF. It suffices to show that the map $E \to \mathrm{map}(\Sigma^{-1}\mathbb{S}[1/p]/\mathbb{S}, E)$ induced by $\Sigma^{-1}\mathbb{S}/\mathbb{S}[1/p] \to \mathbb{S}$ is an $\mathbb{S}/p$-equivalence with p-complete target. Its fiber is $\mathrm{map}(\mathbb{S}[1/p], E)$, so and so it is $\mathbb{S}/p$-acyclic. $\square$

COROLLARY 5.27. *A spectrum E is p-complete if and only if the spectrum*

$$\mathrm{map}(\mathbb{S}[1/p], E) \cong \lim(E \xleftarrow{p} E \xleftarrow{p} \cdots)$$

is contractible.

COROLLARY 5.28. *There is a natural equivalence*

$$E_p^\wedge \cong \lim_n E/p^n .$$

PROOF. We can write

$$\mathbb{S}/p^\infty \cong \operatorname*{colim}_n \mathbb{S}/p^n$$

so that

$$E_p^\wedge \cong \mathrm{map}(\Sigma^{-1}\mathbb{S}/p^\infty, E) \cong \lim_n \mathrm{map}(\Sigma^{-1}\mathbb{S}/p^n, E) \cong E/p^n .$$

$\square$

LEMMA 5.29. *Let E be a spectrum. Then there is a natural short exact sequence of abelian groups*

$$0 \to \mathrm{Ext}(\mathbb{Z}/p^\infty, \pi_* E) \to \pi_*(E_p^\wedge) \to \mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_{*-1}E) \to 0$$

PROOF. We will need a presentation of $\mathbb{S}/p^\infty$. Let us consider the diagram of fiber sequences

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathbb{S} \arrow[r] \arrow[d, "\begin{pmatrix} -1 \\ p \end{pmatrix}"] & \mathbb{S}\oplus\mathbb{S} \arrow[r] \arrow[d, "\begin{pmatrix} -1 & 0 \\ p & -1 \\ 0 & p \end{pmatrix}"] & \cdots \\
\mathbb{S} \arrow[r] \arrow[d, "1"] & \mathbb{S}\oplus\mathbb{S} \arrow[r] \arrow[d, "{(p,1)}"] & \mathbb{S}\oplus\mathbb{S}\oplus\mathbb{S} \arrow[r] \arrow[d, "{(p^2,p,1)}"] & \cdots \\
\mathbb{S} \arrow[r, "p"] & \mathbb{S} \arrow[r, "p"] & \mathbb{S} \arrow[r, "p"] & \cdots
\end{tikzcd}$$

Taking the colimit this gives a fiber sequence

$$\bigoplus_{n\geq 1} \mathbb{S} \to \bigoplus_{n\geq 0} \mathbb{S} \to \mathbb{S}[1/p]$$

and taking the cofiber from the first row we get a fiber sequence

$$\bigoplus_{n\geq 1} \mathbb{S} \to \bigoplus_{n\geq 1} \mathbb{S} \to \mathbb{S}/p^\infty .$$

where the first map has n-th component the map $\mathbb{S} \to \bigoplus_{n\geq 1}\mathbb{S}$ given by -1 on the n-th coordinate and p on the $(n+1)$-th coordinate. Plugging this fiber sequence in $\mathrm{map}(-, X)$ we obtain a fiber sequence

$$\prod_{n\geq 1} X \to \prod_{n\geq 1} X \to X_p^\wedge .$$

where the first map has n-th coordinate the map $\prod_{n\geq 1} X \to X$ sending the n-th factor to X via the opposite of the identity, the second factor to X via the multiplication by p map and all the other factors to 0. Therefore the long exact sequence in homotopy yields

$$\cdots \to \pi_{*+1}(X_p^\wedge) \to \prod_n \pi_* X \to \prod_n \pi_* X \to \pi_*(X_p^\wedge) \to \cdots$$

so our thesis follows from the identification of the kernel of the cokernel of this map. But if we consider the corresponding short exact sequence of abelian groups

$$0 \to \bigoplus_{n\geq 1} \mathbb{Z} \to \bigoplus_{n\geq 1} \mathbb{Z} \to \mathbb{Z}/p^\infty \to 0$$

constructed in a similar fashion, and we consider the long exact sequence induced by $\mathrm{Hom}(-, \pi_* X)$, we obtain a long exact sequence

$$0 \to \mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_* X) \to \prod_{n\geq 1} \pi_* X \to \prod_{n\geq 1} \pi_* X \to \mathrm{Ext}(\mathbb{Z}/p^\infty, \pi_* X) \to 0$$

so the kernel and cokernel are identified as promised. □

If A is an abelian group, the group $\mathrm{Ext}(\mathbb{Z}/p^\infty, A)$ is called the **derived p-completion** of A (in fact it is the 0-th derived functor of the p-completion functor $\mathrm{Ab} \to \mathrm{Ab}$). An abelian group A is called derived p-complete if the map $A \to \mathrm{Ext}(\mathbb{Z}/p^\infty, A)$ induced by the long exact sequence

$$0 \to \mathrm{Hom}(\mathbb{Z}/p^\infty, A) \to \mathrm{Hom}(\mathbb{Z}[1/p], A) \to \mathrm{Hom}(\mathbb{Z}, A) = A \to \mathrm{Ext}(\mathbb{Z}/p^\infty, A) \to \mathrm{Ext}(\mathbb{Z}[1/p], A) \to \mathrm{Ext}(\mathbb{Z}, A) =$$

is an isomorphism. Note that by the proof of the previous lemma the map $\pi_n A \to \pi_n(A_p^\wedge)$ factors through the inclusion of $\mathrm{Ext}(\mathbb{Z}/p^\infty, A)$. This is equivalent to $\mathrm{Hom}(\mathbb{Z}[1/p], A) = \mathrm{Ext}(\mathbb{Z}[1/p], A) = 0$, therefore it follows immediately that derived p-complete groups are closed under kernels, cokernel, extensions and direct sums. Moreover the inclusion of derived p-complete groups into all abelian groups has a left adjoint sending A to $\mathrm{Ext}(\mathbb{Z}/p^\infty, A)$.

Corollary 5.30. *If a group A is derived p-complete, then* $\mathrm{Hom}(\mathbb{Z}/p^\infty, A) = 0$. *Therefore a spectrum E is p-complete if and only if all its homotopy groups are derived p-complete.*

Proof. If A is derived p-complete, then the map

$$\mathrm{Hom}(\mathbb{Z}/p^\infty, A) \to \mathrm{Hom}(\mathbb{Z}[1/p], A)$$

is an isomorphism. Therefore p acts invertibly on the group $\mathrm{Hom}(\mathbb{Z}/p^\infty, A)$, and so

$$\mathrm{Hom}(\mathbb{Z}/p^\infty, A) \cong \mathrm{Hom}(\mathbb{Z}[1/p], \mathrm{Hom}(\mathbb{Z}/p^\infty, A)) \cong \mathrm{Hom}(\mathbb{Z}[1/p] \otimes \mathbb{Z}/p^\infty, A) = 0\,.$$

Now let a spectrum E. If all its homotopy groups are derived p-complete, then we have $\pi_* E \to \pi_*(E_p^\wedge)$ is an isomorphism for all $*$, and so E is p-complete. Viceversa we need to prove that for a spectrum E the homotopy groups of $E_p^\wedge$ are derived p-complete. By the previous lemma it suffices to show that $\mathrm{Hom}(\mathbb{Z}/p^\infty, A)$ is derived p-complete, which is left as an exercise for the reader. □

Exercise 35. *An abelian group A is derived p-complete if and only if the spectrum HA is p-complete.*

Recall (or take it as a definition) that if $A_0 \xleftarrow{p_0} A_1 \xleftarrow{p_1} A_2 \leftarrow \cdots$ is a tower of groups, then $\lim^1 A_n$ is defined as the cokernel of the map

$$\prod_n A_n \to \prod_n A_n \qquad (a_n)_n \mapsto (a_n - p_n a_{n+1})\,.$$

This is indeed the derived functor of the limit. Note that if all the maps in the tower are surjective, $\lim^1 A_n = 0$.

Lemma 5.31. *Let A an abelian group. Then there is a short exact sequence*

$$0 \to \lim^1 \mathrm{Hom}(\mathbb{Z}/p^n, A) \to \mathrm{Ext}(\mathbb{Z}/p^\infty, A) \to \lim_n A/p^n \to 0\,.$$

In particular for an abelian group with bounded p-torsion the derived p-completion and the classical p-completion coincide.

PROOF. Let us consider the short exact sequence

$$0 \to \bigoplus_n \mathbb{Z}/p^n \to \bigoplus_n \mathbb{Z}/p^n \to \mathbb{Z}/p^\infty \to 0$$

where the first map sends (a_n) to $(a_n - pa_{n-1})$ and the second map is (a_n) to $\sum_n a_n/p^n$. This induces a long exact sequence

$$0 \to \operatorname{Hom}(\mathbb{Z}/p^\infty, A) \to \prod_n \operatorname{Hom}(\mathbb{Z}/p^n, A) \to \prod_n \operatorname{Hom}(\mathbb{Z}/p^n, A) \to$$
$$\to \operatorname{Ext}(\mathbb{Z}/p^\infty, A) \to \prod_n \operatorname{Ext}(\mathbb{Z}/p^n, A) \to \prod_n \operatorname{Ext}(\mathbb{Z}/p^n, A)$$

and therefore a short exact sequence

$$0 \to \lim^1 \operatorname{Hom}(\mathbb{Z}/p^n, A) \to \operatorname{Ext}(\mathbb{Z}/p^\infty, A) \to \lim_n \operatorname{Ext}(\mathbb{Z}/p^n, A)$$

Now it is enough to check that $\operatorname{Ext}(\mathbb{Z}/p^n, A) \cong A/p^n$ canonically and that the maps induced by the inclusion $\mathbb{Z}/p^n \to \mathbb{Z}/p^{n+1}$ are exactly the projections $A/p^{n+1} \to A/p^n$. $\square$

The following theorem proves that any spectrum can be reconstructed from $E[1/p]$, $E_p^\wedge$ plus some gluing data

THEOREM 5.32 (Arithmetic fracture square). *There is a natural cartesian square*

$$\begin{tikzcd} E \arrow[r] \arrow[d] & E_p^\wedge \arrow[d] \\ E[1/p] \arrow[r] & (E_p^\wedge)[1/p] \end{tikzcd} .$$

PROOF. Let us consider F to be the fiber of $E \to E_p^\wedge \times_{(E_p^\wedge)[1/p]} E_p^\wedge$ (the so-called "total fiber" of the square. First we observe that its p-completion $F_p^\wedge$ is the total fiber of the square

$$\begin{tikzcd} E_p^\wedge \arrow[r] \arrow[d] & E_p^\wedge \arrow[d] \\ E[1/p]_p^\wedge = 0 \arrow[r] & (E_p^\wedge[1/p])_p^\wedge = 0 \end{tikzcd}$$

and so it is trivial. Therefore F is $\mathbb{S}/p$-acylic, so p acts invertibly on F. Thus $F = F[1/p]$. But $F[1/p]$ can be computed as the total fiber of the square

$$\begin{tikzcd} E[1/p] \arrow[r] \arrow[d] & E_p^\wedge[1/p] \arrow[d] \\ E[1/p] \arrow[r] & E_p^\wedge[1/p] \end{tikzcd}$$

and so it is zero. Therefore $F = 0$, which was the thesis. $\square$

The spectrum $E_p^\wedge$ is sometimes denoted by $E_{\mathbb{Z}_p}$ and $E_p^\wedge[1/p]$ by $E_{\mathbb{Q}_p}$.

EXERCISE 36. *For any spectrum E the homotopy groups of $E_{\mathbb{Q}_p}$ have a canonical $\mathbb{Q}_p$-module structure.*

We can also consider a kind of completion "at all primes at once". We say that a spectrum is **profinitely complete** if it is local for $\bigoplus_p \mathbb{S}/p$ where p ranges through all primes. In particular a spectrum A is profinitely acyclic if and only if $A/p = 0$ for every p, that is if p is rational.

PROPOSITION 5.33. *A spectrum X is profinitely complete if and only if $X_{\mathbb{Q}} = 0$. Moreover for every spectrum X the map*

$$X \to \prod_p X_p^\wedge$$

exhibits the right hand side as the profinite completion of X

PROOF. First let us prove that the target is profinitely complete. In fact if A is a rational spectrum, it is $\mathbb{S}[1/p]$-local for every p. Therefore

$$\mathrm{map}(A, \prod_p X_p^\wedge) \cong \prod_p \mathrm{map}(A, X_p^\wedge) = 0\,.$$

Now we need to show that the map is a profinite equivalence. This is equivalent to say that the map is an equivalence after tensoring by $\mathbb{S}/p$ for every p. But if $q \neq p$ prime we have that $p : X/q \to X/q$ is an equivalence (as $\pi_* X/q$ is q^2-torsion), and so $(X_q^\wedge)/p = (X/p)_q^\wedge = 0$. But then

$$X/p \to \left(\prod_q X_q^\wedge\right)/p \cong \prod_q X_q^\wedge/p = X_p^\wedge/p$$

is an equivalence are required. ☐

THEOREM 5.34 (Arithmetic fracture square). *Let X be a spectrum. Then the following square is cartesian*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \prod_p X_p^\wedge \arrow[d] \\ X_{\mathbb{Q}} \arrow[r] & \left(\prod_p X_p^\wedge\right)_{\mathbb{Q}} \end{tikzcd}.$$

PROOF. As before it is enough to show that the map is an equivalence rationally and after tensoring with $\mathbb{S}/p$ for every prime p, and in both cases it is straightforward. ☐

The bottom left corner of the square is the rational part, the top right corner is the profinite part and the bottom right corner is the "adelic" part (so-called by analogy with the ring of finite adeles $\left(\prod_p \mathbb{Z}_p^\wedge\right) \otimes \mathbb{Q}$).

5. Localization at topological K-theory

Our next goal is to study KU-localization, where KU is topological K-theory. We will use the arithmetic fracture square to reduce it a study of $(L_{\mathrm{KU}}X)_{\mathbb{Q}}$ and $(L_{\mathrm{KU}}X)_p^\wedge$ for every p.

LEMMA 5.35. *Let X be a spectrum and E be a spectrum such that $E_{\mathbb{Q}} \neq 0$. Then the map*

$$X \to L_E X$$

is a rational equivalence. In particular we have $(L_E X)_{\mathbb{Q}} \cong X_{\mathbb{Q}}$ and every rational spectrum is KU*-local.*

Proof. It suffices to show that every E-acyclic spectrum is rationally trivial (since the fiber of $X \to L_E X$ is E-acyclic). But if A is a spectrum such that $A \otimes E = 0$ then we have by ..

$$0 = (A \otimes E)_{\mathbb{Q}} = A \otimes E_{\mathbb{Q}} \cong \bigoplus_n H\pi_n(E_{\mathbb{Q}}) \otimes A$$

and the right hand side is a direct sum of shifted copies of $H\mathbb{Q} \otimes A \cong A_{\mathbb{Q}}$. Therefore $A_{\mathbb{Q}} = 0$, which is what we wanted to prove. $\square$

Lemma 5.36. *Let E be a spectrum and X be a spectrum. Then $X_p^\wedge$ is E-local if and only if X/p is E-local*

Proof. Since $X_p^\wedge/p \cong X/p$, if $X_p^\wedge$ is E-local, so is X/p. Viceversa note that if X/p is E-local we can use the fiber sequence

$$X/p \to X/p^n \to X/p^{n-1}$$

to prove that X/p^n is E-local for every n by induction on n. Therefore

$$X_p^\wedge \cong \lim_n X/p^n$$

is E-local. $\square$

Therefore we have a good understanding of $(L_{\mathrm{KU}} X)_{\mathbb{Q}} = X_{\mathbb{Q}}$. To understand the p-complete parts we will need the following statement as input, that in fact comes from combining several non-trivial theorems

Proposition 5.37. *Let p be an odd prime. There exists a map of spectra $v_1 : \Sigma^{2(p-1)}\mathbb{S}/p \to \mathbb{S}/p$ and a map of rings $\psi^r : \mathrm{KU}_p^\wedge \to \mathrm{KU}_p^\wedge$ such that*

- *v_1 is a* KU*-equivalence.*
- *The mapping telescope*

  $$\mathbb{S}/p[v_1^{-1}] := \operatorname{colim}(\mathbb{S}/p \xrightarrow{v_1} \Sigma^{-2(p-1)}\mathbb{S}/p \xrightarrow{v_1} \Sigma^{-4(p-1)}\mathbb{S}/p \xrightarrow{v_1} \cdots)$$

  is equivalent to the fiber of the map

  $$\psi^r - 1 : \mathrm{KU}/p \to \mathrm{KU}/p\,.$$

For $p = 2$ a similar statement holds, where we replace v_1 with a map $v_1^4 : \Sigma^8\mathbb{S}/2 \to \mathbb{S}/2$ such that $\mathbb{S}/2[v_1^{-4}]$ is the fiber of a map $\mathrm{KO}/2 \to \mathrm{KO}/2$.

Corollary 5.38. *Let X be a spectrum. Then X/p is* KU*-local if and only if the map $v_1 : \Sigma^{2(p-1)}X/p \to X/p$ is an equivalence. Moreover for any spectrum X we have*

$$L_{\mathrm{KU}} X/p \cong X/p[v_1^{-1}]$$

Proof. For simplicity let us work only in the case p odd.

Since the map v_1 is a KU-equivalence, if X/p is KU-local then multiplication by v_1 is necessarily an equivalence. Let us prove the other direction. If v_1 is an equivalence we have an equivalence

$$X/p \cong X/p[v_1^{-1}] = X \otimes \mathbb{S}/p[v_1^{-1}]$$

but by the proposition we can identify the right hand side with the fiber of a map

$$X \otimes \mathrm{KU}/p \to X \otimes \mathrm{KU}/p$$

So it is enough to prove that $X \otimes \mathrm{KU}/p$ is KU-local. But that is a KU-module, and so it is necessarily KU-local.

Finally the map $X/p \to X/p[v_1^{-1}]$ is a KU-equivalence since it is a composition of KU-equivalences and therefore it is a KU-localization map since the target is KU-local. $\square$

THEOREM 5.39. *The* KU*-localization is smashing, that is the map*

$$L_{\mathrm{KU}}X \to L_{\mathrm{KU}}\mathbb{S} \otimes X$$

is an equivalence. Moreover we can write a pullback square

$$\begin{tikzcd} L_{\mathrm{KU}}\mathbb{S} \arrow[r] \arrow[d] & \prod_p J_p \arrow[d] \\ H\mathbb{Q} \arrow[r] & \left(\prod_p J_p\right)_{\mathbb{Q}} \end{tikzcd}$$

where J_p is the fiber of the map $\psi^r - 1 : \mathrm{KU}_p^\wedge \to \mathrm{KU}_p^\wedge$ for p odd, and the same with KU *replaced by* KO *for $p = 2$.*

PROOF. Remember that the map $X \to L_{\mathrm{KU}}\mathbb{S}\otimes X$ is always a KU-equivalence, so it's enough to show that the target is KU-local. By the arithmetic square and since every rational spectrum is KU-local, it is enough to show that $(L_{\mathrm{KU}}\mathbb{S}\otimes X)_p^\wedge$ is KU-local for every prime, that is that $L_{\mathrm{KU}}\mathbb{S}/p \otimes X$ is KU-local. But by corollary.. we have

Denis: Insert ref!

$$L_{\mathrm{KU}}\mathbb{S}/p \otimes X \cong \mathbb{S}/p[v_1^{-1}] \otimes X \cong X/p[v_1^{-1}] \cong L_{\mathrm{KU}}(X/p)$$

and therefore the left hand side is KU-local.

Finally we just need to prove that $(L_{\mathrm{KU}}\mathbb{S})_p^\wedge \cong J_p$. Let $\eta : \mathbb{S} \to \mathrm{KU}$ be the unit. Since $\psi^r : \mathrm{KU}_p^\wedge \to \mathrm{KU}_p^\wedge$ is a map of rings, we have $\psi^r\eta \cong \eta$, therefore we can choose a nullhomotopy of $(\psi^r - 1) \circ \eta$. So we have a map $\mathbb{S} \to J_p$. Moreover J_p is obviously KU-local and p-complete so we get a map

$$(L_{\mathrm{KU}}\mathbb{S})_p^\wedge \to J_p\,.$$

Then proposition .. and corollary ... tells us this is an equivalence after tensoring by $\mathbb{S}/p$, and so it is an equivalence. □

The spectrum J_p is called the p-complete image of J spectrum, and the spectrum $L_{\mathrm{KU}}\mathbb{S}$ is also sometimes called the image of J spectrum. That's because the map $\pi_*\mathbb{S} \to \pi_*L_{\mathrm{KU}}\mathbb{S}$ is a split surjection for $* > 0$ and it identifies $\pi_*L_{\mathrm{KU}}\mathbb{S}$ with the image of the J-homomorphism $J : \pi_*O \to \pi_*\mathbb{S}$.

CHAPTER 6

Manifolds and duality

In this chapter we will cover some applications of stable homotopy theory to the study of smooth manifolds, culminating on the Pontryagin-Thom theorem (Theorem 6.39). In the course of this chapter we will need a few basic facts of differential topology. We will use [**Kos07**] as our main source for those.

1. Thom spectra and the Thom isomorphism

Definition 6.1. *Let X be a space and $p : V \to X$ be a vector bundle over X. Then the **Thom space** of V $Th(V)$ is the pointed space given by the cofiber of $V \setminus \{0\} \to V$.*

The following lemma will be useful during the construction of the Pontryagin-Thom map.

Lemma 6.2. *Let X be a compact topological space and $p : V \to X$ be a vector bundle over X. Then the Thom space $\mathrm{Th}(V)$ is also equivalent to the one-point compactification of V.*

Proof. Put a metric on V. Then we can replace the map $V \setminus \{0\} \to V$ with the homotopy equivalent map $S(V) \to X$ where $S(V)\{v \in V \mid |v| = 1\}$ is the unit sphere. The the cofiber is given by

$$\mathrm{Th}(V) \cong * \cup_{S(V)\times\{0\}} S(V) \times [0, 1] \cup_{S(V)\times\{1\}} X\,.$$

We get then a map from the right hand side of the above equivalence to S^V, the one-point compactification of V by sending

$$S(V) \times (0, 1] \to V \qquad (v, t) \mapsto tv$$

and notice that this induces a homeomorphism of $S(V) \times (0, 1] \cup_{S(V)\times\{1\}} X \cong V$. Since both are locally compact Hausdorff spaces, they have equivalent one point compactifications. □

If $p : V \to X$ is a vector bundle classified by a map $f : X \to BO_n$, we notice that $V \setminus \{0\}$ is the colimit of the composition $X \to BO_n \to \mathscr{S}$ where the second functor is the functor induced by O_n acting on $\mathbb{R}^n \setminus \{0\}$.

Lemma 6.3. *Let $p : V \to X$ be a vector bundle classified by $f : X \to BO_n$. Then $\mathrm{Th}(V)$ is equivalent to the colimit of the composite*

$$X \to BO_n \to \mathscr{S}_*$$

where the second map is the map $BO_n \to \mathscr{S}_*$ inducing the action of O_n on the pointed space S^n, seen as the one point compactification of $\mathbb{R}^n$.

Denis: I forgot to explain at some point why BG is the groupoid with one object and G automorphisms – add it somewhere!

Proof. We note that $V \setminus \{0\} \to X$ is a fiber bundle (and therefore a Serre fibration), and so $V \setminus \{0\}$ is the colimit of the functor $X \to \mathscr{S}$ sending x to $V_x \setminus \{0\}$. But then, using that $\mathrm{cof} : \mathrm{Fun}([1], \mathscr{S}) \to \mathscr{S}_*$ commutes with colimits, we have

$$\mathrm{Th}(V) \cong \mathrm{cof}(V \setminus \{0\} \to V) \cong \operatorname*{colim}_{x\in X} \mathrm{cof}(V_x \setminus \{0\} \to V_x) \cong \operatorname*{colim}_{x\in X} S^{V_x}\,.$$

□

DEFINITION 6.4. *The Thom spectrum X^V of $p : V \to X$ is the suspension spectrum of* $\mathrm{Th}(V)$*. It is equivalent to the colimit of the composition*

$$X \to BO_n \to BO \to \mathrm{Sp}\,.$$

More generally, if $f : X \to BO \times \mathbb{Z}$ is a virtual vector bundle, we can define the Thom spectrum of f X^f as the colimit of the composition

$$X \to BO \times \mathbb{Z} \to \mathrm{Sp}$$

where the second functor is induced by the map of E_1-groups

$$J : O = \operatorname{colim} O_n \to \operatorname{colim} \mathrm{Aut}_*(S^n) \cong \mathrm{Aut}_{\mathrm{Sp}}(\mathbb{S})\,.$$

Finally sometimes we will write X^V for the colimit of a functor $S : X \to \mathrm{Sp}$ such that $S(x)$ has value $\Sigma^{n_x}\mathbb{S}$ for every $x \in X$. These are called stable spherical fibrations*.*

LEMMA 6.5. *Let $V : X \to \mathrm{Sp}$ and $W : Y \to \mathrm{Sp}$ be two stable spherical fibrations. Then if we write $V \otimes W$ for the composite*

$$V \otimes W : X \times W \to \mathrm{Sp} \times \mathrm{Sp} \xrightarrow{\otimes} \mathrm{Sp}$$

we have $(X \times W)^{V \otimes W} \cong X^V \otimes Y^W$. In particular we have $X^{\Sigma V} \cong \Sigma X^V$.

PROOF. Follows from the fact that the tensor product commutes with colimits in each variable. $\square$

In some sense Thom spectra are the easiest type of spectra that there is: homology and cohomology of Thom spectra are easy to compute in terms of the homology and cohomology of the base.

DEFINITION 6.6. *If V is a virtual vector bundle over X of rank n (or more generally a stable spherical fibration), and E is a homotopy ring spectrum a* ***Thom class*** *is a map $\theta : X^V \to \Sigma^n E$ such that for every $x \in X$ the map $\Sigma^n\mathbb{S} \cong \{x\}^{V_x} \to X \to \Sigma^n E$ is an invertible element of the ring $\pi_0 E$. A Thom class is sometimes also called an E-orientation of V, and a vector bundle that admits a Thom class is called E-orientable.*

REMARK 6.7. *A Thom class is the same datum as a natural transformation $S^{V_x} \to \Sigma^n E$ such that for every $x \in X$ the composite*

$$E \otimes S^{V_x} \to E \otimes \Sigma^n E \to \Sigma^n E$$

is an equivalence. That is, it is equivalent to an equivalence of the local system $E \otimes S^{V_x}$ with the constant local system at E.

EXAMPLE 6.8. *If $E = H\mathbb{F}_2$ every stable spherical fibration is E-orientable. Indeed the functor $X \to \mathrm{Sp}$ sending x to $H\mathbb{F}_2 \otimes S^{V_x}$ takes values in the full subgroupoid of spectra spanned by $\Sigma^n H\mathbb{F}_2$. But this is contractible, and so it is a constant functor.*

EXAMPLE 6.9. *If $E = H\mathbb{Z}$ an orientation of a vector bundle $V : X \to BO_n$ is equivalent to a nullhomotopy of the composite*

$$X \to BO_n \to BC_2\,.$$

Indeed $BC_2 \times \mathbb{Z}$ is the subgroupoid of Sp *spanned by spectra of the form $\Sigma^n H\mathbb{Z}$. But this is the same thing as a lift of V to the fiber BSO_n of $BO_n \to BC_2$. It is left as an exercise to the reader to see that this is the same thing as a trivialization of the determinant line bundle (that is that homotopy classes of maps $X \to BSO_n$ correspond to pairs of a vector bundle of rank n together with a trivialization of the determinant line bundle).*

THEOREM 6.10 (Thom isomorphism). *Let X be a space, V be a stable spherical fibration of rank n over X and $\theta : X^V \to \Sigma^n E$ a Thom class. Then there are equivalences*

$$E \otimes X^V \cong E \otimes \Sigma^n \Sigma^\infty_+ X \quad \textit{and} \quad \mathrm{map}(X^V, E) \cong \mathrm{map}(\Sigma^n \Sigma^\infty_+ X, E)\,.$$

PROOF. We have that

$$E \otimes X^V \cong \operatorname*{colim}_X E \otimes S^{V_x} \cong \operatorname*{colim}_X \Sigma^n E \cong E \otimes \Sigma^n \Sigma^\infty_+ X\,.$$

Moreover

$$\mathrm{map}(X^V, E) \cong \lim_X \mathrm{map}(S^{V_x}, E)\,.$$

Now we need to prove that an orientation induces a natural equivalence

$$\mathrm{map}(S^{V_x}, E) \cong \Sigma^{-n} E\,.$$

But note that we have a map

$$\Sigma^{-n} E \to \Sigma^{-n} \mathrm{map}(E, E) \cong \mathrm{map}(\Sigma^n E, E) \to \mathrm{map}(S^{V_x}, E)$$

that is an equivalence for every $x \in X$, and therefore it is an equivalence after taking the limit. □

2. Spanier-Whitehead duality

DEFINITION 6.11. *Let $X \in \mathrm{Sp}$. A spectrum Y is called the* ***dual*** *of X if there are maps*

$$\mathrm{ev} : X \otimes Y \to \mathbb{S} \qquad \mathrm{coev} : \mathbb{S} \to Y \otimes X$$

and commutative diagrams

$$\begin{tikzcd} X \arrow[r, "\mathrm{ev} \otimes 1_X"] \arrow[dr, "1_X"'] & X \otimes Y \otimes X \arrow[d, "1_X \otimes \mathrm{coev}"] \\ & X \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} Y \arrow[r, "1_Y \otimes \mathrm{ev}"] \arrow[dr, "1_Y"'] & Y \otimes X \otimes Y \arrow[d, "\mathrm{coev} \otimes 1_Y"] \\ & Y \end{tikzcd}$$

If X has a dual we write it with $\mathbb{D}X$ and call it the ***Spanier-Whitehead dual*** *of X. In this case we say that X is dualizable.*

Denis: strongly?

LEMMA 6.12. *Let X be a spectrum. Then X is dualizable if and only if there is a spectrum $\mathbb{D}X$ and a natural equivalence*

$$\mathbb{D}X \otimes - \cong \mathrm{map}(X, -)\,.$$

In particular if X is dualizable, its dual is necessarily of the form $\mathbb{D}X \cong \mathrm{map}(X, \mathbb{S})$.

PROOF. If X is dualizable, the functor $\mathbb{D}X \otimes -$ is the right adjoint of the functor $X \otimes -$ because we can use ev and coev to build the triangular identities. Therefore we have an equivalence

$$\mathrm{map}(X, Z) \cong \mathbb{D}X \otimes Z$$

for every $Z \in \mathrm{Sp}$.

Viceversa, suppose that there is a spectrum $\mathbb{D}X$ with the above property. Then it is necessarily of the form $\mathbb{D}X \cong \mathrm{map}(X, \mathbb{S})$. In particular there is a map

$$\mathrm{ev} : X \otimes \mathbb{D}X \cong X \otimes \mathrm{map}(X, \mathbb{S}) \otimes X \to \mathbb{S}\,.$$

But there is also a map

$$\mathbb{S} \to \mathrm{map}(X, X) \cong \mathbb{D}X \otimes X\,,$$

adjoint to the identity map $X \to X$. It is then an exercise to see that these two maps satisfy the defining identities for a dual object. □

Exercise 37. *Show that every functor* $F : \mathrm{Sp} \to \mathrm{Sp}$ *commuting with all colimits is of the form* $F(X) \cong F(\mathbb{S}) \otimes X$*. Therefore* X *is dualizable if and only if* $\mathrm{map}(X, -)$ *commutes with all colimits.*

Remark 6.13. *Suppose* X *is a dualizable spectrum with dual* $\mathbb{D}X$*. Then for any spectra* Z, T *there is a natural equivalence*

$$\mathrm{map}(X \otimes Z, T) \cong \mathrm{map}(Z, \mathbb{D}X \otimes T)$$

In particular, by choosing $Z = T = \mathbb{S}$ *we have*

$$\mathbb{D}X \cong \mathrm{map}(\mathbb{S}, \mathbb{D}X \otimes \mathbb{S}) \cong \mathrm{map}(X, \mathbb{S}) .$$

Therefore $X \mapsto \mathbb{D}X$ *is a functor from dualizable spectra to spectra (given by the restriction of* $\mathrm{map}(-, \mathbb{S})$*).*

Lemma 6.14. *Let* $f : X \to Y$ *be a map of dualizable spectra, then the cofiber is still dualizable with dual given by the fiber of* $\mathbb{D}f$*.*

Proof. We have for any spectrum Z an equivalence

$$\begin{aligned}\mathrm{map}(\mathrm{cof}\, f, Z) &\cong \mathrm{fib}(\mathrm{map}(Y, Z) \to \mathrm{map}(X, Z)) \cong \\ &\cong \mathrm{fib}(\mathbb{D}Y \otimes Z \to \mathbb{D}X \otimes Z) \cong \mathrm{fib}(\mathbb{D}Y \to \mathbb{D}X) \otimes Z .\end{aligned}$$

□

Proposition 6.15. *A spectrum is dualizable if and only if it is finite.*

Proof. Since $\mathbb{S}$ is obviously dualizable, all finite spectra are dualizable (add details!). Now let X be a dualizable spectrum. Note that $\mathrm{map}(X, -) := \mathbb{D}X \otimes -$ commutes with all colimits, therefore X is compact. We conclude that X is finite by exercise ... □

Remark 6.16. *Let* X *be a dualizable spectrum and* E *a spectrum. Then we have a natural equivalence*

$$E_*\mathbb{D}X \cong \pi_*(\mathbb{D}X \otimes E) \cong \pi_* \mathrm{map}(X, E) \cong E^{-*}X .$$

Theorem 6.17 (Adams' homological Brown representability). *Let* $E_* : h\mathscr{S}_* \to \mathrm{grAb}$ *be a homology theory. Then there is a spectrum* E *and a natural equivalence*

$$E_*(X) \cong \pi_*(E \otimes X) .$$

3. Atiyah duality

In this section we will identify the dual of a closed manifold (or more generally a compact manifold with boundary) as a certain Thom spectrum. This combined with the Thom isomorphism (Theorem 6.10) will allow us to deduce Poincaré duality as a corollary. The theorem is usually attributed to Atiyah, which proved it in the present version as [**Ati61**, Proposition 3.2], although the first version of the theorem appears as [**MS60**, Lemma 2]. In this section we will broadly follow the exposition in [**Rez13**], although without using the language of Frobenius algebras.

Lemma 6.18. *Let* M *be a closed manifold and* N *be an embedded closed submanifold with normal bundle* ν*. Then there is a well-defined homotopy equivalence*

$$M/M \smallsetminus N \cong \mathrm{Th}(\nu)$$

depending only on the isotopy class of the embedding. In particular we have a map $M \to \mathrm{Th}(\nu)$ *which is called the* ***Pontryagin-Thom collapse map****.*

PROOF. By the tubular neighborhood theorem, we can find an embedding $\nu \subseteq E$ that is unique up to isotopy. Then the open cover $\{M \setminus N, \nu\}$ of M yields a homotopy cocartesian square

$$\begin{tikzcd} \nu \setminus 0 \arrow[r] \arrow[d] & \nu \arrow[d] \\ M \setminus N \arrow[r] & M \end{tikzcd}$$

and this induces an equivalence of horizontal cofibers, as required. □

REMARK 6.19. *More generally, in the situation of lemma* **??** *if V is a vector bundle on M we obtain a map*

$$\mathrm{Th}(V) \to \mathrm{Th}(V \oplus \nu)$$

by considering the embedding of M into the disc bundle of V. Finally, if V is a virtual vector bundle on M we obtain a map

$$M^V \to N^{V \oplus \nu}$$

whose homotopy class depends only on the isotopy class of the embedding. We will write the map as $PT(j, V)$ or simply $PT(j)$ where $j : N \to M$ is the embedding.

EXAMPLE 6.20. *Let M, N be two closed manifolds. Then for every $m \in M$ the Pontryagin-Thom map induced by the embedding $N \to M \times N$ sending x to (m, x) is the map*

$$\Sigma^\infty (M \times N)_+ \cong \Sigma^\infty M_+ \otimes \Sigma^\infty N_+ \to \Sigma\infty_+ S^{T_x M} \Sigma^\infty N_+$$

where $M \to S^{T_x M}$ is the Pontyagin-Thom collapse map for M. In particular if $M \cong S^n$ this is homotopic to the identity on the nontrivial summand.

THEOREM 6.21 (Whitney's embedding theorem). *Let M be a closed smooth manifold. Then the space*

$$\operatorname*{colim}_n \mathrm{Emb}(M, \mathbb{R}^n)$$

is contractible. In particular M can always be embedded in some $\mathbb{R}^n$ and for n sufficiently big any two embeddings are isotopic.

PROOF. It suffices to show that every continuous map $\partial\Delta^m \times M \to \mathbb{R}^n$ which is an embedding on $\{t\} \times M$ can be extended to a continuous map $\Delta^m \times M \to \mathbb{R}^n$ if n is sufficiently big. □

DEFINITION 6.22. *Let us choose an embedding of M into $\mathbb{R}^n$. Since M is compact, we can extend it to an embedding of M into S^n. Then we have a Pontryagin-Thom collapse map*

$$\mathbb{S} \oplus \Sigma^{-n}\mathbb{S} = \Sigma^{-n}\Sigma^\infty S^n_+ \to M^{n-\nu} \cong M^{-TM}.$$

Note that the component on $\Sigma^{-n}\mathbb{S}$ is nullhomotopic, and so we can safely discard it. Since all embeddings of M into $\mathbb{R}^n$ are isotopic for n sufficiently large, this map is well defined up to homotopy. We call the resulting element of $\pi_0 M^{-TM}$ the ***fundamental class*** *of M.*

Equipped with this structure we define the evaluation and coevaluation maps

$$\mathrm{ev} : \Sigma^\infty_+ M \otimes M^{-TM} \to \Sigma^\infty_+ M \to \mathbb{S}$$

where the first map is the Pontryagin-Thom collapse map for the diagonal $M \to M \times M$ and the second induced by the canonical map $M \to *$. The coevaluation map instead is the composite

$$\mathrm{coev} : \mathbb{S} \to M^{-TM} \to \Sigma^\infty_+ M \otimes M^{-TM}$$

where the first map is the fundamental map and the second is induced by the diagonal map $M \to M \times M$, twisted by the virtual vector bundle $(-TM, 0)$.

Our goal is to prove the following result

Theorem 6.23 (Atiyah duality). *The evaluation and coevaluation maps described above exhibit M^{-TM} as the Spanier-Whitehead dual of $\Sigma^\infty_+ M$.*

Corollary 6.24 (Poincaré duality). *Let E be a homotopy ring spectrum and M a manifold equipped with an E-orientation of the tangent bundle. Then there is a natural isomorphism*

$$E_{\dim M - *} M \cong E^* M\,.$$

Proof. Combining Atiyah duality and the Thom isomorphism we have

$$E^*M \cong \pi_{-*}\operatorname{map}(\Sigma^\infty_+ M, E) \cong \pi_{-*}(M^{-TM} \otimes E) \cong \pi_{-*}(\Sigma^{-\dim M}\Sigma^\infty_+ M \otimes E) \cong E_{\dim M - *} M\,.$$

□

To prove this result we will need to introduce for the first time the notion of transversality.

Definition 6.25. *Let $f : M \to M'$ be a map of smooth manifolds and $N' \subseteq M'$ be a submanifolds. We say that f is transversal to N' if for every point $x \in f^{-1}N'$ we have that $T_{fx}N'$ and $df(T_xM)$ generate $T_{fx}N'$.*

Lemma 6.26. *Let $f : M \to M'$ be a map of smooth manifolds transverse to some submanifold $N' \subseteq M'$. Then $f^{-1}N'$ is a submanifold of M and df restricts to an isomorphism of the normal bundle of N in M with the pullback of the normal bundle of N' in M'.*

Proof. This is basically the implicit function theorem after choosing a neighborhood of fx such that the pair (M', N') is isomorphic to $(\mathbb{R}^m, \mathbb{R}^n)$. □

Lemma 6.27. *Let $f : M_1 \to M_2$ be a map of smooth manifolds transverse to some submanifold $N_2 \subseteq M_2$ and let $N_1 = f^{-1}N_2$. Moreover let V be a vector bundle over M_2. Then there is a commutative diagram*

$$\begin{tikzcd}
M_1^{f^*V} \arrow[r] \arrow[d] & M_2^V \arrow[d] \\
N_1^{f^*(V\oplus\nu)} \arrow[r] & N_2^{V\oplus\nu}
\end{tikzcd}$$

Proof. We can find a tubular neighborhood $j : \nu \subseteq M_2$ for N_2 such that $f^{-1}\nu$ is a tubular neighborhood for N_1 (e.g. by picking any tubular neighborhood for N_2 and isotopying it so that it lands in trivializing charts for N_2). Then the commutativity of the diagram follows immediately from the construction of the Pontryagin-Thom map. □

Proof of Atiyah duality. We need to prove that ev and coev satisfy the relations in definition ... Let us do the first. Consider the diagram

$$\begin{tikzcd}
(M\times M)^{(-TM,-TM)} \cong M^{-TM}\otimes M^{-TM} \arrow[r, "f_*"] & M^{-TM}\otimes\Sigma^\infty M_+\otimes M^{-TM} \arrow[r, "PT(g)"] \arrow[dr, "1\otimes\mathrm{ev}"] & M^{-TM}\otimes\Sigma^\infty M_+ \arrow[d, "(pr_1)_*"] \\
M^{-TM} \arrow[u] \arrow[ur, "\mathrm{coev}\otimes 1"] & & M^{-TM}
\end{tikzcd}$$

where $f : M \times M \to M \times M \times M$ is the map sending (x, y) to (x, x, y) and $g : M \times M \to M \times M \times M$ is the closed embedding sending (x, y) to (x, y, y). Note that f and g are transverse and fit into a pullback diagram

$$\begin{tikzcd} M \arrow[r, "\Delta"] \arrow[d, "\Delta"] & M \times M \arrow[d, "g"] \\ M \times M \arrow[r, "f"] & M \times M \times M \end{tikzcd}$$

and so by lemma 6.27 we have that the top row can be also written as the composition

$$(M \times M)^{-TM} \xrightarrow{PT(\Delta)} M \xrightarrow{\Delta_*} M^{-TM} \otimes \Sigma^\infty M_+$$

and therefore the whole composition is homotopic to

$$M^{-TM} \xrightarrow{PT(\Delta(j\times 1))} M^{-TM} \xrightarrow{(pr_1\Delta)_*} M^{-TM}$$

and we claim that both arrows in this composition are identities. Indeed $pr_1\Delta = \mathrm{id}_M$ and the embedding $(j, \mathrm{id}_M) : M \to S^N \times M$ is isotopic to $(0, \mathrm{id}_M)$ whose Pontryagin-Thom collapse map is the identity.

Similarly for the other composition we have a diagram

$$\begin{tikzcd} \Sigma^\infty M_+ \otimes M^{-TM} \arrow[r, "g_*"] & \Sigma^\infty M_+ \otimes M^{-TM} \otimes \Sigma^\infty M_+ \arrow[r, "PT(f)"] \arrow[dr, "\mathrm{ev}\otimes\mathrm{id}"'] & \Sigma^\infty M_+ \otimes \Sigma^\infty M_+ \arrow[d, "(pr_2)_*"] \\ \Sigma^\infty M_+ \arrow[u, "1\otimes PT(j)"] \arrow[ur, "\mathrm{id}\otimes\mathrm{coev}"] & & \Sigma^\infty M_+ \end{tikzcd}$$

and we can use the transversality of f and g again to replace the composition with the composition

$$\Sigma^\infty M_+ \xrightarrow{PT(\mathrm{id},j)} \Sigma^\infty M_+ \xrightarrow{(pr_2\Delta)_*} \Sigma^\infty_+ M$$

and finally use the fact that the embedding (id, j) is isotopic to $(\mathrm{id}, 0)$ to conclude. □

In fact one can do prove an analogous theorem for manifolds with boundary, although this requires some more care with the embeddings. Let say that an embedding of manifolds with boundary $N \subseteq M$ is calle **neat** if $\partial M \cap N = \partial N$ and locally near the points of ∂N it is diffeomorphic to $[0,\infty)\times\mathbb{R}^k \subseteq [0,\infty)\times\mathbb{R}^n$. Then one can, similarly as before, construct Pontryagin-Thom collapse maps $M^V \to \mathrm{Th}(V \oplus \nu)$ where V is a vector bundle over M and ν is the normal bundle of N in M.

Let M be a manifold with boundary and embed it neatly into D^n (such an embedding always exists for n sufficiently large by ??). Then we can find a tubular neighborhood $\nu \subseteq D^n$ of M such that $\nu \cap \partial D^n$ is a tubular neighborhood of ∂M in ∂D^n ([**Kos07**, Theorem III.4.2]). Therefore we can construct a commutative square

$$\begin{tikzcd} S^{n-1} \cong \partial D^n \arrow[r] \arrow[d] & \mathrm{Th}(\nu|_{\partial M}) \arrow[d] \\ D^n \arrow[r] & \mathrm{Th}(\nu) \end{tikzcd}$$

and therefore after stabilizing a commutative diagram

$$\begin{tikzcd} \Sigma^{-1}\mathbb{S} \arrow[r] \arrow[d] & (\partial M)^{-1-T\partial M} \arrow[r] \arrow[d] & \Sigma^\infty \partial M_+ \otimes M^{-TM} \arrow[d] \\ 0 \arrow[r] & M^{-TM} \arrow[r] & \Sigma^\infty M_+ \otimes M^{-TM} \end{tikzcd}$$

where the top right arrow is the Pontryagin-Thom collapse map induced by the diagonal embedding $\partial M \subseteq \partial M \times M$. Therefore we get a map between vertical cofibers

$$\mathrm{coev} : \mathbb{S} \to \Sigma^\infty M/\partial M \otimes M^{-TM}\,.$$

Viceversa, to construct the evaluation map we need to consider map on vertical cofibers in the diagram

$$\begin{tikzcd}
M^{-TM} \otimes \Sigma^\infty \partial M_+ \arrow[r] \arrow[d] & \Sigma^{-1}\Sigma^\infty \partial M_+ \arrow[r] \arrow[d] & \Sigma^{-1}\mathbb{S} \arrow[d] \\
M^{-TM} \otimes \Sigma^\infty M_+ \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

where the top horizontal map is the Pontryagin-Thom collapse map. Once we have this construction the proof proceeds in a similar fashion as before.

Theorem 6.28 (Atiyah duality for manifolds with boundary). *Let $(M, \partial M)$ be a compact manifold with boundary. Then there is an equivalence of spectra*

$$\mathbb{D}\Sigma^\infty M/\partial M \cong M^{-TM}\,.$$

Corollary 6.29. *Let M be a closed manifold and V be a virtual vector bundle. Then there is an equivalence of spectra*

$$\mathbb{D}M^V \cong M^{-V-TM}\,.$$

Proof. By adding enough copies of the trivial bundle to both sides we can assume that V is a vector bundle. Then put a metric on V and let $D(V)$ be the unit disc bundle. This is a manifold with boundary such that $M^V \cong D(V)/\partial D(V)$ and so we have

$$\mathbb{D}M^V \cong \mathbb{D}D(V)/\partial D(V) \cong D(V)^{-T(D(V))} \cong M^{-V-TM}\,,$$

where in the last passage we have used that the projection $D(V) \to M$ is an equivalence that identifies $T(D(V))$ with $V + TM$. $\square$

Exercise 38. *If M is a manifold with boundary, the Spanier-Whitehead dual of the boundary map $M/\partial M \to \Sigma M_+$ is the map $M^{-TM} \to (\partial M)^{-1-T\partial M}$ (hint: $\Sigma M_+ = W/\partial W$ where $W = M\times[0,1]$. In particular the composite $\mathbb{S} \to (\partial M)^{-T\partial M} \to M^{1-TM}$ is nullhomotopic (since it is a shift of the dual to the composite $M/\partial M \to \Sigma\partial M_+ \to S^0$).*

4. Thom spectra and bordism

In this section we will use Thom spectra to study the bordism groups of manifolds. This section is essentially a modern retelling of Chapter II of [**Sto68**].

For the rest of the section we fix a map $\xi : B \to BO$. We will denote with $M\xi$ the Thom spectrum of the corresponding rank 0 stable vector bundle.

Definition 6.30. *Let $V : X \to BO$ a virtual vector bundle of rank 0. Then a ξ-structure on V is the datum of a factorization V_ξ of V through ξ. When a virtual vector bundle has rank different than 0, a ξ-structure on V is the same as a ξ-structure on $V - \mathrm{rk}\,V$.*

Example 6.31.
(1) *When $B = BSO = \mathrm{fib}(BO \to BC_2)$ a ξ-structure is just an orientation.*
(2) *When $B = BU$ a ξ-structure is called a stable complex structure.*
(3) *When $B = *$ a ξ-structure is called a stable framing.*

Exercise 39. *Show that the datum of a stable framing on a vector bundle V is equivalent to an equivalence class of pairs (n, φ) where $\varphi : V \oplus \mathbb{R}^n \cong \mathbb{R}^{\mathrm{rk}\,V+n}$ is a trivialization of $V \oplus \mathbb{R}^n$ and two such pairs are equivalent when they are isomorphic after adding a trivial vector bundle.*

DEFINITION 6.32. *Let M be an n-dimensional manifold (possibly with boundary). Then a ξ-structure on M is a ξ-structure on its stable normal bundle $-TM$. That is, it is a lift of $n - TM : M \to BO$ along ξ.*

CONSTRUCTION 6.33. *Let W be a manifold with boundary M. Then notice that there are essentially two isomorphisms $TW|_M \cong \mathbb{R} \oplus TM$, given by taking the "inward" or the "outward" normal. We choose always the inward pointing normal, so that we have a consistent isomorphism of stable vector bundles* $(\dim W - TW)|_M \cong \dim M - TM$. *In particular any ξ-structure on W induces a ξ-structure on M.*

DEFINITION 6.34. *Let us fix a non-negative integer $n \geq 0$. Then the n-**dimensional cobordism group** Ω_n^ξ is the quotient of the commutative monoid of n-manifolds with ξ-structure under disjoint union by the submonoid of those that are boundaries of $(n+1)$-manifolds with ξ-structure.*

LEMMA 6.35. *The commutative monoid Ω_n^ξ is a group.*

PROOF. Let (M, φ) be a closed n-manifold with ξ-structure. Then let us consider the manifold with boundary $M \times [0,1]$. Its tangent bundle is just equivalent to $TM \times T[0,1] \cong TM \times \mathbb{R}$, where we identify it with the "left-to-right" orientation (i.e. the one induced by the canonical embedding $[0,1] \subseteq \mathbb{R}$). Then the inclusion $M \subseteq M \times [0,1]$ is a homotopy equivalence, and so the ξ-structure on M induces a ξ-structure $\tilde{\varphi}$ on $M \times [0,1]$ such that its restriction to $M \times \{0\}$ is φ. Therefore we have that

$$\partial(M \times [0,1], \tilde{\varphi}) \cong (M, \varphi) \amalg (M, -\varphi)$$

for some other ξ-structure $-\varphi$. But then $[M, \varphi] + [M, -\varphi] = 0$ in Ω_n^ξ. □

REMARK 6.36. *The ξ-structure $-\varphi$ is given by the same lift $\varphi : M \to B$ but the homotopy of the composite $M \to B \to BO$ with* $\dim M - TM$ *is twisted by the generator of $\pi_1 BO = \mathbb{Z}/2$.*

CONSTRUCTION 6.37. *Let $M\xi$ be the Thom spectrum of the virtual vector bundle $\xi : B \to BO$. Let (M, φ) be a closed n-manifold with ξ-structure. Then the class $[M, \varphi] \in \pi_n M\xi$ is given by the composite*

$$\Sigma^n \mathbb{S} \to M^{n-TM} \xrightarrow{\varphi_*} M\xi$$

where the first map is the n-th suspension of the fundamental class of M and φ_* is the map induced on Thom spectra.

LEMMA 6.38. *Let (M, φ) and (M', φ') be two closed n-manifolds with ξ-structure. Then $[M \amalg M', \varphi \amalg \varphi'] = [M, \varphi] + [M', \varphi']$.*

Moreover if (M, φ) is the boundary of a compact manifold with ξ-structure (W, ψ), we have $[M, \varphi] = 0$. Therefore the assignment $(M, \varphi) \mapsto [M, \varphi]$ induces a map of groups

$$\Omega_n^\xi \to \pi_n M\xi\,.$$

PROOF. The first statement follows since the fundamental class of a disjoint union is the direct sum of the fundamental classes, because it is the dual of the map $(M \amalg M')_+ \to S^0$. To prove the second statement we use exercise 38. Suppose (M, φ) is the boundary of (W, ψ). Then we can factor $[M]$ as

$$\Sigma^n \mathbb{S} \to M^{n-TM} \to W^{n+1-TW} \to M\xi$$

due to the similar factorization of virtual vector bundles. But the composition of the first two maps is nullhomotopic. □

THEOREM 6.39 (Pontryagin-Thom). *The map*

$$\Omega_n^\xi \to \pi_n M\xi$$

is an isomorphism of groups.

EXAMPLE 6.40. *This means for example that framed cobordism groups are equivalent to the stable homotopy groups of spheres. You can use this fact to compute low degree homotopy groups, in particular $\pi_0\mathbb{S}$ and $\pi_1\mathbb{S}$ are not hard to do by hand.*

COROLLARY 6.41. *Let X be a space. Then $\pi_n(M\xi \otimes \Sigma^\infty_+ X)$ is equivalent to the group of cobordism classes of triples (M, φ, f) where M is a closed n-manifold, φ is a ξ-structure on M and $f : M \to X$ is a continuous map.*

PROOF. Note that $M\xi \otimes \Sigma^\infty_+ X \cong M\xi_X$ where $\xi_X : B \times X \to B \to BO$. Therefore $\pi_n(M\xi \otimes \Sigma^\infty_+ X)$ is equivalent to $\Omega_n^{\xi_X}$. But it is immediate to verify that a ξ_X class on a manifold is equivalent to the pair of a ξ-structure and a map $f : M \to X$. □

Denis: Elaborate on how we can reduce to taking homotopy classes of maps using cylinders.

In order to prove the Pontryagin-Thom theorem, we need a couple of results from differential topology.

LEMMA 6.42. *Let $f : M \to N$ be a proper continuous map of smooth manifolds. Then f is proper homotopic two a smooth map, that is there exists a proper continuous map $H : M \times [0, 1] \to N$ such that $H_0 = f$ and H_1 is proper and smooth.*

PROOF. This is an immediate consequence of [**Kos07**, Theorem III.2.5.(2)]. □

THEOREM 6.43 (Transversality theorem). *Let $f : M \to M'$ be smooth map of smooth manifolds and N be a compact submanifold of M'. Then there is a smooth homotopy $H : M \times [0, 1] \to M'$ such that $H_0 = f$, H is the constant homotopy outside of a relatively compact neighborhood of N and H_1 is transverse to N. Moreover if f is already transverse to N when restricted to a closed subset A, we can choose the homotopy to be relative to A.*

PROOF. This is an immediate consequence of [**Kos07**, Corollary IV.2.4]. □

Then we can prove the Pontryagin-Thom theorem. The proof will be divided in two parts: surjectivity and injectivity, which will be similar. Let us first prove surjectivity. So suppose we are given a map $f : \Sigma^n\mathbb{S} \to M\xi$. We need to show that it comes from a manifold. First let $B_n = B \times_{BO} BO_n$ and ξ_n be the rank n vector bundle on B_n coming from $B_n \to BO_n$. Then we have

$$M\xi \cong \operatorname*{colim}_k \Sigma^{-k}\Sigma^\infty \operatorname{Th}(\xi_k)\,.$$

Therefore we can find some $k \geq 0$ such that f factors through a map $\Sigma^{n+k}\mathbb{S} \to \Sigma^\infty \operatorname{Th}(\xi_k)$. Moreover note that $\operatorname{Th}(\xi_k)$ is k-connected (since it is a colimit of S^k, and k-connected spaces are closed under colimits). Therefore up to increasing k we can assume that $n + k$ is in the stable range for $\Sigma^\infty \operatorname{Th}(\xi_k)$, and so f comes from a map $f : S^{n+k} \to \operatorname{Th}(\xi_k)$. Now let us recall that $BO_k = \operatorname{Gr}_k$, the Grassmannian of k-planes in $\mathbb{R}^\infty$. We can write

$$\operatorname{Gr}_k = \bigcup_m \operatorname{Gr}_k(\mathbb{R}^m)$$

where $\operatorname{Gr}_k(\mathbb{R}^m)$ is the subspace of those k-dimensional subspaces lying in $\mathbb{R}^m$. Equivalently, it is the space of $\{P \in M_m(\mathbb{R}) \mid P = {}^TP,\ P^TP = I, \operatorname{rk} P = m\}$, and therefore $\operatorname{Gr}_k(\mathbb{R}^m)$ is a compact Hausdorff space. In fact using the charts of ... it is a compact smooth manifold. Therefore there exist some m such that

f comes from a map $S^{n+k} \to \mathrm{Th}(\xi_{k,m})$ where $\xi_{k,m}$ is the restriction of ξ_k to $B_{k,m} = B \times_{BO} \mathrm{Gr}_k(\mathbb{R}^m)$. Therefore we can assume that f is a continuous pointed map $S^{n+k} \to \mathrm{Th}(\xi_{k,m})$.

Let $\eta_{k,m}$ be the universal bundle over $\mathrm{Gr}_k(\mathbb{R}^m)$, so that $\xi_{k,m} = \xi^*\eta_{k,m}$. Now recall that $\mathrm{Th}(\eta_{k,m})$ is the one point compactification of $\eta_{k,m}$ by lemma 6.2. Let $U = f^{-1}\xi_{k,m}$. Then U is a open subset of S^{k+m}, in particular it is a smooth manifold. Moreover the restriction to U of the map

$$S^{n+k} \to \mathrm{Th}(\xi_{k,m}) \to \mathrm{Th}(\eta_{k,m})$$

is a proper continuous map $U \to \eta_{k,m}$ and both objects are smooth manifolds. Therefore by lemma 6.42 we can assume up to homotopy that $f|_U$ is smooth. Moreover by the transversality theorem 6.43 we can also assume that it is transverse to the zero section $\mathrm{Gr}_k(\mathbb{R}^m) \subseteq \eta_{k,m}$, which is a compact submanifold. Then we let $M = f^{-1}\,\mathrm{Gr}_k(\mathbb{R}^m)$, which therefore is a compact submanifold of S^{n+k}. Moreover its normal bundle is identified with the pullback of $\eta_{k,m}$ and so it has a canonical ξ-structure φ given by the restriction of f to M. Unwrapping the definitions we see that (M, φ) indeed is a closed manifold with ξ-structure such that $[M, \varphi] = [f]$.

Now to prove that it is injective. Let (M, φ) be an n-manifold such that its class in $\pi_n M\xi$ is trivial. This means that the composite

$$\Sigma^n \mathbb{S} \to M^{n-TM} \to M\xi$$

is nullhomotopic. Now we claim that we can find $k \gg 0$, an embedding $M \subseteq \mathbb{R}^{n+k}$ with normal bundle ν and a smooth map $g : M \to \mathrm{Gr}_{n,k}$ classifying ν such that the map $M^{n-TM} \to M\xi$ is homotopic to the composite

$$M^{n-TM} \cong \Sigma^{-k}\Sigma^\infty \mathrm{Th}(\nu) \xrightarrow{\mathrm{Th}(g)} \Sigma^{-k}\Sigma^\infty \mathrm{Th}(\xi_{k,m}) \to M\xi$$

and the composite

$$f : S^{n+k} \to \mathrm{Th}(\nu) \xrightarrow{\mathrm{Th}(g)} \mathrm{Th}(\xi_{k,m})$$

is nullhomotopic. Indeed, M is a finite CW complex, in particular a finite space, and so the map

$$g : M \to BO = \operatorname*{colim}_{k,m} \mathrm{Gr}_{k,m}$$

classifying $n - TM$ factors through some finite $\mathrm{Gr}_{k,m}$. Write ν for the pullback of $\xi_{k,m}$ under g (this is a vector bundle whose K-theory class is $n+k-TM$). Moreover we can, after possibly enlarging $k$choose an embedding $M \subseteq \mathbb{R}^{n+k}$ such that the normal bundle is isomorphic to ν. This is because if two vector bundles V, V' have the same K-theory class, there's some $m \gg 0$ such that $V \oplus 1^m \cong V' \oplus 1^{m'}$.

Finally we need to show that, up to possibly enlarging k, the composite

$$S^{n+k} \to \mathrm{Th}(\nu) \to \mathrm{Th}(\xi_{k,m})$$

But this follows as before from the Freudenthal suspension theorem and the compactness of $\mathbb{S}$.

Note that the preimage of $\xi_{k,m}$ under f is exactly $\nu \subseteq S^{n+k}$, in particular $f|_\nu$ is smooth. Therefore there is some continuous map $H : D^{n+k+1} \to \mathrm{Th}(\xi_{k,m})$ restricting to f on the boundary. Without loss of generality we can take some $\epsilon > 0$ such that $H(tx) = f(x)$ for all $t \in [1-\epsilon, 1]$. By applying again the transversality theorem 6.43 we can change H up to homotopy relative to $\{x \in D^{n+k} \mid |x| \geq 1-\epsilon/2\}$ so that pH is smooth in a neighborhood of $\mathrm{Gr}_{k,m}$ and transverse to it (where $p : \mathrm{Th}(\xi_{k,m}) \to \mathrm{Th}(\eta_{k,m})$ is the canonical projection. Finally this shows that $H^{-1}\,\mathrm{Gr}_{k,m}$ is a manifold with boundary exactly $M = H^{-1}\,\mathrm{Gr}_{k,m} \cap \partial D^{n+k+1}$ and whose normal bundle can be given a ξ structure (since its normal bundle is some subspace of D^{n+k+1} and the whole map $D^{n+k+1} \to \mathrm{Gr}_{k,m} \to BO$ has a lift to B). This concludes our thesis.

REMARK 6.44. *The Pontryagin-Thom theorem 6.39 can in fact be strengthened. There exists an ∞-category Cob_d^ξ whose objects are $(d-1)$-dimensional manifolds with ξ-structure and morphisms are bordisms between them. Then the Galatius-Madsen-Tillman-Weiss theorem ([?,]) says that its realization (i.e. the ∞-groupoid obtained by inverting all arrows) has homotopy type*

$$|\mathrm{Cob}_d^\xi| \cong \Omega^\infty \Sigma MT\xi_d$$

where $T\xi_d$ is the composite

$$T\xi_d : B \times_{BO} BO_d \xrightarrow{\eta_d^*\xi} BO_d \xrightarrow{d-\eta_d} BO$$

and $BO_d \to BO$ is the map classifying the virtual vector bundle $-\eta_d$ (so the opposite *of the tautological bundle), while in the pullback we take the usual inclusion $BO_d \to BO$. This E_∞-space can be thought of as a categorification of the $(d-1)$-st bordism group as we have defined it. Using the Galatius-Madsen-Tillman-Weiss, the Pontryagin-Thom theorem can be deduced from the fact that the map*

$$\Sigma^d MT\xi_d \cong M(d + T\xi_d) \to M\xi$$

is d-connected, which follows from connectedness of the map $d - \eta_d : BO_d \to BO$.

5. The Steenrod problem

The Steenrod problem is a classical topological problem, which was the main motivation for Thom to develop this machinery. It comes in two flavours: oriented and unoriented. Recall that every closed smooth d-manifold M has a fundamental class $[M] \in H_d(M;\mathbb{Z}/2)$. Moreover, if M is oriented the class can be lifted to a class $[M] \in H_d(M;\mathbb{Z})$.

- **Unoriented:** Let X be a topological space. For which classes $\alpha \in H_d(X;\mathbb{Z}/2)$ there exists a closed d-manifold M and a continuous map $f : M \to X$ such that $\alpha = f_*[M]$?
- **Oriented:** Let X be a topological space. For which classes $\alpha \in H_d(X;\mathbb{Z})$ there exists a closed oriented d-manifold M and a continuous map $f : M \to X$ such that $\alpha = f_*[M]$?

Notice that there are Thom classes $MO \to H\mathbb{Z}/2$ and $MSO \to H\mathbb{Z}$ representing the fact that MO (resp. MSO) are the Thom spectra of the universal (oriented) vector bundle. It is easy to see that these classes are an isomorphism on π_0. Therefore we can rephrase the Steenrod problem as asking what are the images of the maps

$$\pi_*(MO \otimes \Sigma^\infty X_+) \to \pi_*(H\mathbb{Z}/2 \otimes \Sigma^\infty X_+) \text{ and } \pi_*(MSO \otimes \Sigma^\infty_+ X) \to \pi_*(H\mathbb{Z} \otimes \Sigma^\infty_+ X)\,.$$

It turns out that MO is equivalent to a sum of Eilenberg-MacLane spaces. In particular the projection $MO \to H\mathbb{Z}/2$ has a section. Therefore the map

$$\pi_*(MO \otimes \Sigma^\infty X_+) \to \pi_*(H\mathbb{Z}/2 \otimes \Sigma^\infty X_+)$$

is always surjective, and the answer to the unoriented Steenrod problems is "always". For the oriented case the situation is trickier, but we can already with what we have done so far give a partial answer. In fact note that the map

$$MSO \otimes H\mathbb{Q} \to H\mathbb{Q}$$

has always a section (since every rational spectrum is a sum of Eilenberg-MacLane spectra by lemma ..) and so it has a section. In particular the map

$$\pi_*(MSO \otimes \Sigma^\infty X_+)_\mathbb{Q} \to H_*(X,\mathbb{Q})$$

is always surjective, so for every $\alpha \in H_d(X,\mathbb{Z})$ there exists $n \in \mathbb{Z}$, and an oriented d-manifold $f : M \to X$ with a map to X such that $n\alpha = f_*[M]$.

To study the integral problem one can try something more subtle. Using the long exact sequence of the fiber sequence

$$t_{\geq 1}MSO \to MSO \to H\mathbb{Z}$$

we obtain that the subgroup we want to analyze is the kernel of the map

$$\pi_*(H\mathbb{Z} \otimes \Sigma^\infty X_+) \to \pi_{*+1}(t_{\geq 1}MSO \otimes \Sigma^\infty X_+)$$

In particular one can compute the low degree homotopy groups of MSO (for example, in dimension 1 and 2 noting that all oriented manifolds are boundaries, and doing something more refined in dimension 3) and see that there is a map $t_{\geq 1}MSO \to \Sigma^4 H\mathbb{Z}$. Therefore we obtain an obstruction

$$H_*(X;\mathbb{Z}) \to H_{*+5}(X;\mathbb{Z})$$

showing that all representable classes must be in the kernel. Thom computes explicitly this homology operation, and used it to prove the existence of a homology class that cannot be represented.

6. Computing bordism spectra

DEFINITION 6.45. *Let p be a prime. Then the mod p Steenrod algebra is graded abelian group*

$$\mathcal{A}^* := H^*(H\mathbb{F}_p;\mathbb{F}_p) = \pi_{-*}(H\mathbb{F}_p, H\mathbb{F}_p)\,.$$

It is given an algebra structure by composition.

Throghout the following, let $p = 2$.

THEOREM 6.46. *The Steenrod algebra is the graded associative algebra generated by elements $\{\mathrm{Sq}^i\}_{i\geq 0}$ in degree i under the relations*

$$\mathrm{Sq}^0 = 1,\ \mathrm{Sq}^i \mathrm{Sq}^j = \sum_{k=0}^{\lfloor i/2 \rfloor} \binom{j-k-1}{i-2k} \mathrm{Sq}^{i+j-k} \mathrm{Sq}^k \ \ \forall i < 2j\,.$$

Say that a sequence of positive integers $I = (i_0, i_1, i_2, \ldots, i_r)$ is admissible if $i_j \leq 2i_{j+1}$. If I is an admissible sequence write $\mathrm{Sq}^I = \mathrm{Sq}^{i_0} \cdots \mathrm{Sq}^{i_r}$. By convention we set $\mathrm{Sq}^\varnothing = 1$.

COROLLARY 6.47. *The elements Sq^I as I runs through the admissible sequences form an additive basis for $\mathcal{A}^*$.*

PROPOSITION 6.48. *Let X, Y be two spectra. Then the endomorphism Sq^n of $H^*(X \otimes Y) \cong H^*X \otimes H^*Y$ can be written as $\sum_{i+j=n} \mathrm{Sq}^i \otimes \mathrm{Sq}^j$.*

PROPOSITION 6.49. *Let X be a space. Then $\mathrm{Sq}^i x = 0$ if $|x| > i$ and $\mathrm{Sq}^i x = x^2$ if $i = |x|$.*

EXAMPLE 6.50. *Let $X = \mathbb{R}¶^\infty$. Then $H^*X \cong \mathbb{F}_2[x]$ for $|x| = 1$. Therefore we have $\mathrm{Sq}^0 x = x$, $\mathrm{Sq}^1 x = x^2$ and $\mathrm{Sq}^i x = 0$ for $i > 2$. Using this one can prove by induction*

$$\mathrm{Sq}^i x^n = \binom{n}{i} x^{n+1}\,.$$

UNFINISHED

Denis: We need to finish writing the proof of Thom's theorem!

A1. The transversality theorem

INCLUDE A PROOF OF THE TRANSVERSALITY THEOREM FROM SARD'S THEOREM FOLLOWING [**Kos07**, IV.2].

A2. Atiyah duality for topological manifolds

SKETCH OF HOW TO USE MICROBUNDLES TO WIN!

CHAPTER 7

The Adams spectral sequences

In this chapter we will construct the Adams spectral sequence and do some example computations, mainly following Adams' blue book [**Ada74**]. Mark Behrens' minicourse at MSRI in 2013 ([**Beh**]) is a good introduction for computations with the Adams spectral sequence. The canonical reference for all the most important computations is [**Rav03**].

Contrary to most expositions, we will use the action on the Steenrod algebra on cohomology instead of the coaction of the dual Steenrod algebra on homology. This will force us to impose some unnecessary finiteness hypothesis.■

Denis: After the course is done I will have to rewrite this section using homology

1. Exact couples and spectral sequences

Definition 7.1. *A* ***filtered spectrum*** *is a functor $X^\bullet : \mathbb{Z} \to \mathrm{Sp}$ where $\mathbb{Z}$ is the poset of integers with the decreasing order. That is a filtered spectrum is a diagram of the form*

$$\cdots \to X^{s+1} \to X^{s+1} \to X^s \to X^{s-1} \to \cdots .$$

We will write $X^\infty := \lim X^\bullet$ and $X^{-\infty} := \operatorname{colim} X^\bullet$.

The associated graded of a filtered spectrum $X^\bullet$ is the spectrum

$$\mathrm{gr}(X^\bullet) := \bigoplus_{s\in\mathbb{Z}} \mathrm{gr}^s(X^\bullet)$$

where $\mathrm{gr}^s(X^\bullet)$ is the cofiber of the map $X^{s+1} \to X^s$.

The goal in this chapter is to find a way to recover information about X^∞ and $X^{-\infty}$ from $\mathrm{gr}(X^\bullet)$.

Example 7.2. *Let X be any spectrum. Then the Postnikov tower of X is the filtered spectrum*

$$t_{\leq\bullet}X : \qquad \cdots t_{\leq s+1}X \to t_{\leq s}X \to t_{\leq s-1}X \to \cdots .$$

We have $\mathrm{gr}^s(t_{\leq\bullet}X) = \Sigma^s H\pi_s X$, $X^\infty = X$ and $X^{-\infty} = 0$.

Dually the Whitehead tower of X is the filtered spectrum

$$t_{\geq\bullet}X : \qquad \cdots t_{\geq -s-1}X \to t_{\geq -s}X \to t_{\geq -s+1}X \to \cdots .$$

We have $\mathrm{gr}^s(t_{\geq\bullet}X) = \Sigma^{-s} H\pi_{-s} X$, $X^\infty = 0$ and $X^{-\infty} = X$.

One can also tensor $t_{\leq\bullet}X$ and $t_{\geq\bullet}X$ by another spectrum Y. If Y is bounded below we have $(t_{\geq\bullet}X \otimes Y)^\infty = 0$ and $(t_{\geq\bullet}X \otimes Y)^{-\infty} = X \otimes Y$.

Example 7.3. *We can do a parametrized version of the previous example. Let $p : X \to B$ be a map of spaces and F be a spectrum. Then we can write $X = \operatorname{colim}_{b\in B} X_b$ where $X_b := X \times_B \{b\}$. Consider now the filtered spectrum $\operatorname{colim}_{b\in B} t_{\geq\bullet}(F \otimes \Sigma^\infty_+ X_b)$. It has associated graded $\operatorname{colim}_{b\in B} H\pi_s(F \otimes X_b)$ (what's called the cohomology with local coefficients of the local system $\pi_s(F \otimes X_b)$).*

Lemma 7.4 (Milnor exact sequence). *We have $\pi_* X^{-\infty} \cong \operatorname{colim}_s \pi_* X^s$ and there is a short exact sequence*

$$0 \to \lim_s{}^1 \pi_{*+1} X^s \to \pi_* X^\infty \to \lim_s \pi_* X^s \to 0 .$$

Proof. This follows from the fiber sequence

$$X^\infty \to \prod_s X^s \to \prod_s X^s$$

□

Definition 7.5. *An (unrolled) exact couple is a diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & A^{s+2} \arrow[r, "i"] \arrow[d, "j"] & A^{s+1} \arrow[r, "i"] \arrow[d, "j"] & A^s \arrow[r, "i"] \arrow[d, "j"] & A^{s-1} \arrow[r, "i"] \arrow[d, "j"] & \cdots \\
& E^{s+2} \arrow[ul, "k"] & E^{s+1} \arrow[ul, "k"] & E^s \arrow[ul, "k"] & E^{s-1} \arrow[ul, "k"] & \cdots \arrow[ul, "k"]
\end{tikzcd}$$

where the triangles are exact (that is $\ker i = \operatorname{im} k$, $\ker j = \operatorname{im} i$ *and* $\ker k = \operatorname{im} j$. *Here we let* A^s *be* E^s *are elements of some abelian category (typically graded* R*-modules for some ring* R*, but more general objects sometimes appear).*

Example 7.6. *If* $X^\bullet$ *is a filtered spectrum we have an exact couple of the form*

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_* X^{s+2} \arrow[r, "i"] \arrow[d, "j"] & \pi_* X^{s+1} \arrow[r, "i"] \arrow[d, "j"] & \pi_* X^s \arrow[r, "i"] \arrow[d, "j"] & \pi_* X^{s-1} \arrow[r, "i"] \arrow[d, "j"] & \cdots \\
& \pi_* \operatorname{gr}^{s+2}(X) \arrow[ul, "k"] & \pi_* \operatorname{gr}^{s+1}(X) \arrow[ul, "k"] & \pi_* \operatorname{gr}^s(X) \arrow[ul, "k"] & \pi_* \operatorname{gr}^{s-1}(X) \arrow[ul, "k"] & \cdots \arrow[ul, "k"]
\end{tikzcd}$$

where the maps i *and* j *have degree 0 and the map* k *has degree -1. All exact couples appearing in these notes will be of this form.*

Given an exact couple we will set $A^{-\infty} := \operatorname{colim}_s A^s$, $A^\infty := \lim_s A^s$ and $RA^\infty := {\lim_s}^1 A^s$. Our goal is often going to be to reconstruct A^∞ or $A^{-\infty}$ using information contained in E^s. We define increasing filtrations on A^∞ and $A^{-\infty}$ by

$$F^s A^{-\infty} := \Im(A^s \to A^{-\infty}) \text{ and } F^s A^\infty = \ker(A^\infty \to A^s)\,.$$

The first step

Lemma 7.7. *Let* $\widehat{A^{-\infty}} := \lim_s A^{-\infty}/F^s A^{-\infty}$ *and* $iA^\infty = \ker(A^\infty \to A^{-\infty})$. *Then we have*

$$\widehat{A^{-\infty}} \cong \lim_{m \geq n} \operatorname*{colim}_n F^n A^{-\infty}/F^m A^{-\infty} \text{ and } iA^\infty \cong \lim_{m \geq n} \operatorname*{colim}_n F^n A^\infty/F^m A^\infty\,.$$

In particular if $A^{-\infty} = 0$ *we have*

$$A^\infty \cong \lim_{m \geq n} \operatorname*{colim}_n F^n A^\infty/F^m A^\infty\,,$$

while if $A^s = 0$ *for* $s \gg 0$ *(or more generally if* $A^\infty = 0$ *and the maps we have*

$$A^{-\infty} \cong \lim_{m \geq n} \operatorname*{colim}_n F^n A^{-\infty}/F^m A^{-\infty}\,.$$

The goal of spectral sequences is to give a description of the groups $F^{n+1}A^{\pm\infty}/F^n A^{\pm\infty}$ in terms of the groups E^s. From that one can hope to reconstruct $A^{\pm\infty}$ using the above lemma and solving "extension problems"

$$0 \to F^n A^{\pm\infty}/F^m A^{\pm\infty} \to F^{n+1} A^{\pm\infty}/F^m A^{\pm\infty} \to F^{n+1} A^{\pm\infty}/F^n A^{\pm\infty} \to 0$$

Definition 7.8. *A spectral sequence is an object* E *(typically a bigraded abelian group) together with correspondences* $d^r : E \to E$ *such that* $\ker d^r = \operatorname{dom} d^{r+1}$ *and* $\operatorname{im} d^r = \operatorname{ind} d^{r+1}$. *We let* $Z^r = \ker d^r$ *and* $B_r = \operatorname{im} d^r$.

DEFINITION 7.9. *We let*

$$Z_r^s = \{x \in E^s \mid kx \in \operatorname{im} i^{r-1}\},\ Z_\infty^s = \bigcap_r Z_r^s\,.$$

$$B_r^s = \{x \in E^s \mid \exists y \in \ker i^{r-1}\ jy = x\},\ B_\infty^s = \bigcup B_r^s$$

$$E_r^s = Z_r^s / B_r^s$$

Note that there is a map $d^r : E_r^s \to E_r^{s+r}$ sending x to jy where $y \in A^{s+r}$ is such that $i^r y = kx$. This has the property that $(d^r)^2 = 0$, $\ker d^r = Z_{r+1}^s / B_r^s$ and $\operatorname{im} d^r = B_{r+1}^s / B_r^s$, so that the homology of d^r is E_{r+1}^s.

The collection $(\{E_r^s\}, \{d^r\})$ is called the ***spectral sequence*** *associated to the exact couple. The group E_r^s is called the* ***r-th page*** *of the spectral sequence.*

We can define $Z_\infty^s := \bigcap_r Z_r^s$, $B_\infty^s := \bigcup_r B_r^s$ and $E_\infty^s := Z_\infty^s / B_\infty^s$. The elements of Z_∞^s are called the ***permanent cycles*** *of the spectral sequence*

Let us define $RE^s := \lim_r{}^1 Z_r^s = \lim_r{}^1 Z_r^s / B_\infty^s$. This group is the main (often the only) difference between the E_∞ page of the spectral sequence and the associated graded.

LEMMA 7.10. *Suppose $RE^s = 0$ for every s. Then there is a short exact sequence*

$$0 \to F^s A^{-\infty} / F^{s+1} A^{-\infty} \to E_\infty^s \to F^s A^\infty / F^{s+1} A^\infty \to 0\,.$$

PROOF. If we have $x \in F^s A^{-\infty}$ we want to send it to the image $j\tilde{x}$ of a lift $\tilde{x} \in A^s$. This is well defined since any two lifts differ by an element of $\ker i^r$ for r sufficiently big. Moreover $\operatorname{im} j = \ker k \subseteq Z_\infty^s$ so this map lands in E_∞^s. Its kernel is exactly those elements such that $j\tilde{x} \in B_\infty^s$ or equivalently such that there is $y \in \ker j = \operatorname{im} i$ such that $\tilde{x} - y \in \ker i^\infty$. But these are exactly the elements coming from $F^{s+1} A^{-\infty}$. So we have obtained the first half of our exact sequence and we have shown that the cokernel is $Z_\infty^s / \ker k$.

$$0 \to F^s A^{-\infty} / F^{s+1} A^{-\infty} \to E_\infty^s \to Z_\infty^s / \ker k \to 0\,.$$

To prove the thesis we need to show $Z_\infty^s / \ker k \cong F^s A^\infty / F^{s+1} A^\infty$. We have a map $F^s A^\infty \to \ker i \subseteq A^{s+1}$ whose kernel is exactly $F^{s+1} A^\infty$. On the other hand $Z_\infty^s / \ker k \cong k(Z_\infty^s) \subseteq \ker i$ and the image of $F^s A^\infty$ lies in this subgroup by definition of Z_∞^s. Therefore it is enough to show that the map $F^s A^\infty \to k(Z_\infty^s)$ is surjective. Note that

$$k(Z_\infty^s) = \ker i \cap \bigcap_n \operatorname{im}(A^{s+n} \to A^{s+1})$$

Write

$$Q^s := \bigcap_n \operatorname{im}(A^{s+n} \to A^s)$$

We claim that $i(Q^{s+1}) = Q^s$. Indeed we have a short exact sequence

$$0 \to Z_r^s / \ker k \to \operatorname{im}(A^{s+n} \to A^{s+1}) \to \operatorname{im}(A^{s+n} \to A^s) \to 0$$

and so by passing at the limit and using $RE^s = 0$ we get a short exact sequence

$$0 \to Z_\infty^s / \ker k \to Q^{s+1} \to Q^s \to 0$$

Finally, since $Q^{s+1} \to Q^s$ is surjective, we obtain that $Q^s = \operatorname{im}(A^\infty \to A^s)$ (one inclusion is obvious, the other follows by constructing by induction a compatible system of lifts), thus proving the thesis. $\square$

EXAMPLE 7.11. *Let us consider the filtration of example 7.2. This produces the Atiyah-Hirzebruch spectral sequence*

$$H_p(X;\pi_q E) \Rightarrow E_{p+q}X\,.$$

Doing the parametrized version we obtain the Serre spectral sequence

$$H_p(B;E_qF) \Rightarrow E_{p+q}X$$

where the left hand side is the homology of the base with local coeffients given by the E-homology of the fiber with the obvious π_1-action.

REMARK 7.12. *When X is a ring spectrum one can ask the filtration to be a filtration of rings. In that case the spectral sequence becomes multiplicative: all pages E_r^s have a ring structure and the differentials d_r become derivations.*

Denis: Find the master thesis of the person working out this example explicitly and cite it...

2. The Adams spectral sequence

From now on let p be a prime number (2 in every example we will consider).

DEFINITION 7.13. *Let X be a spectrum. The* ***Adams filtration*** *on its homotopy groups π_*X is the decreasing filtration where $F^{\geq s}\pi_*X$ is the subgroup spanned by those maps that can be written as the composition of s-maps that are trivial on $H\mathbb{F}_p$-homology. One can define similarly a filtration on $\pi_*\operatorname{map}(Y,X)$ for any Y.*

LEMMA 7.14. *Let X be a spectrum. Then there is an equivalence of spectra*

$$H\mathbb{F}_p \otimes X \cong \bigoplus_{n\in\mathbb{Z}} \Sigma^n H(H_n(X;\mathbb{F}_p))\,.$$

PROOF. Let $\{\alpha_i : \Sigma^{n_i}\mathbb{S} \to H\mathbb{F}_p \otimes X\}$ be a basis of $H_*(X;\mathbb{F}_p)$ over $\mathbb{F}_p$. Then for every i we can construct a map

$$\Sigma^{n_i} H\mathbb{F}_p \to H\mathbb{F}_p \otimes H\mathbb{F}_p \otimes X \to H\mathbb{F}_p \otimes X$$

such that on the n_i-th homotopy group it is the inclusion of the summanda $\mathbb{F}_p\alpha_i \subseteq H_i(X;\mathbb{F}_p)$. Then summing all of them we obtain

$$\bigoplus_i \Sigma^{n_i} H\mathbb{F}_p \to H\mathbb{F}_p \otimes X\,,$$

which is an isomorphism on homotopy groups. □

LEMMA 7.15. *A map $f : X \to Y$ is nullhomotopic after being tensored by $H\mathbb{F}_p$ if and only if the composition $X \to Y \xrightarrow{\eta\otimes\mathrm{id}_Y} H\mathbb{F}_p \otimes Y$ is nullhomotopic.*

PROOF. The lemma follows by looking at the two commutative diagrams

$$\begin{tikzcd} X \arrow[r, "\eta\otimes\mathrm{id}_X"] \arrow[d, "f"] & H\mathbb{F}_p\otimes X \arrow[d, "H\mathbb{F}_p\otimes f"] \\ Y \arrow[r, "\eta\otimes\mathrm{id}_Y"] & H\mathbb{F}_p\otimes Y \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} H\mathbb{F}_p\otimes X \arrow[r, "\mathrm{id}\otimes\eta\otimes\mathrm{id}"] \arrow[drr, "H\mathbb{F}_p\otimes f"] & H\mathbb{F}_p\otimes H\mathbb{F}_p\otimes X \arrow[r, "\mathrm{id}\otimes f"] & H\mathbb{F}_p\otimes H\mathbb{F}_p\otimes Y \arrow[d, "\mu\otimes\mathrm{id}"] \\ & & H\mathbb{F}_p\otimes Y \end{tikzcd} .$$

□

PROPOSITION 7.16. *Let $\overline{H\mathbb{F}_p} \to \mathbb{S}$ be the fiber of $\eta : \mathbb{S} \to H\mathbb{F}_p$. The s-th layer of the Adams filtration $F^{\geq s}\pi_*X$ is the image of the map $\pi_*(\overline{H\mathbb{F}_p}^{\otimes s} \otimes X) \to \pi_*X$, and similarly for the filtration on $\pi_*\operatorname{map}(Y,X)$.*

PROOF. Induction using the previous lemma. □

DEFINITION 7.17. *Let X be any spectrum. We write $X^s = \overline{H\mathbb{F}_p}^{\otimes s} \otimes X$ and then $X^\infty := \lim_s X^s$. Then the* ***Adams spectral sequence*** *is the spectral sequence associated with the filtered spectrum $\operatorname{map}(Y,X^s)$. Its abutment is $\operatorname{map}(Y,X)$.*

PROPOSITION 7.18. *Suppose X is p-complete and connective. Then $X^\infty = 0$, that is the spectral Adams filtration is exhaustive.*

The proof of the proposition will need a lemma

LEMMA 7.19. *Let $\{Y_i\}$ be a collection of connective spectra. Then the natural map*

$$H\mathbb{F}_p \to \prod_i Y_i \to \prod_i (H\mathbb{F}_p \otimes Y_i)$$

is an equivalence.

PROOF. Let us first assume $Y_i = HA_i$ for A_i abelian groups. Then we need to prove that the map

$$H_*(H\mathbb{F}_p; \prod_i A_i) \to \prod_i H_*(H\mathbb{F}_p, A_i)$$

is an isomorphism. But we can use the universal coefficient theorem and write a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_n(H\mathbb{F}_p, \mathbb{Z}) \otimes \prod_i A_i & \longrightarrow & H_n(H\mathbb{F}_p, \prod_i A_i) & \longrightarrow & \mathrm{Tor}_1(H_{n-1}(H\mathbb{F}_p; \mathbb{Z}), \prod_i A_i) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \prod_i (H_n(H\mathbb{F}_p, \mathbb{Z}) \otimes A_i) & \longrightarrow & \prod_i H_n(H\mathbb{F}_p, A_i) & \longrightarrow & \prod_i \mathrm{Tor}_1(H_{n-1}(H\mathbb{F}_p; \mathbb{Z}), A_i) & \longrightarrow & 0 \end{array}$$

and the left and right vertical arrows are isomorphisms since $H_n(H\mathbb{F}_p; \mathbb{Z})$ is a finitely generated abelian group.

Now to prove it for general Y_i it's enough to prove it for $t_{\leq n} Y_i$ for every n and take the limit. But this follows by the case concentrated in one degree. □

PROOF OF PROPOSITION 7.18. First of all, note that $H\mathbb{F}_p \otimes X$ is p-complete for every spectrum X, since its homotopy groups are p-torsion. Therefore, by induction, we see that X^s is p-complete for every s if X is. Moreover it is connective, since the tensor product of connective spectra is connective. Therefore it follows that X^∞ is p-complete and bounded below (Milnor exact sequence). So it suffices to show that $H\mathbb{F}_p \otimes X^\infty = 0$. But we have a fiber sequence

$$X^\infty \to \prod_s X^s \to \prod_s X^s$$

and therefore, using lemma 7.19 we have $H\mathbb{F}_p \otimes X^\infty \cong \lim_s (H\mathbb{F}_p \otimes X^s)$. However the map $H\mathbb{F}_p \otimes X^{s+1} \to H\mathbb{F}_p \otimes X^s$ is nullhomotopic and so at the limit we have $H\mathbb{F}_p \otimes X^\infty = 0$. □

Now let us identify the E_2-term of the spectral sequence. Let us fix some notation for brevity: let $F^s = H\mathbb{F}_p \otimes \overline{H\mathbb{F}_p}^{\otimes s} \otimes X$. Then by lemma... F^s is a sum of copies of $H\mathbb{F}_p$. In particular its mod p cohomology is free over the Steenrod algebra. Note that we have a map

$$F^s \to \Sigma X^{s+1} \to \Sigma F^{s+1}\,.$$

LEMMA 7.20. *The complex*

$$0 \to H^*(X; \mathbb{F}_p) \to H^*(F^0; \mathbb{F}_p) \to H^{*-1}(F^1; \mathbb{F}_p) \to H^{*-2}(F^2; \mathbb{F}_p) \to \cdots$$

is exact. In particular if $H_*(X; \mathbb{F}_p)$ is finitely generated in each degree, $H^*(F^s; \mathbb{F}_p)$ gives a free resolution of $H^*(X; \mathbb{F}_p)$ as a module over the Steenrod algebra.

Proof. Note that the map $X^{s+1} \to X^s$ is trivial in mod p homology (and therefore cohomology). Thus we have short exact sequences

$$0 \to H^{*-1}(X^{s+1}; \mathbb{F}_p) \to H^*(F^s; \mathbb{F}_p) \to H^*(X^s; \mathbb{F}_p) \to 0\,.$$

Splicing them together we obtain the required exact complex. □

Lemma 7.21. *Let F be a sum of $H\mathbb{F}_p$, finitely many in each degree. Then for any spectrum Y there is a natural equivalence*

$$\pi_* \operatorname{map}(Y, F) \cong \operatorname{Hom}_{\mathcal{A}}(H^*F, H^*Y)\,.$$

Proof. Exercise (H^*F is free as a $\mathcal{A}$-module). □

Proposition 7.22. *Let X be a spectrum such that $H_*(X; \mathbb{F}_p)$ is finitely generated in each degree*[1]*. The E_2-page of the Adams spectral sequence is given by*

$$E_2^{s,t} = \operatorname{Ext}_{\mathcal{A}}^{s,t}(H^*X, H^*Y) \Rightarrow \pi_{t-s} \operatorname{map}(Y, X_p^\wedge)\,.$$

Moreover this spectral sequence converges conditionally if X is connective and converges strongly if Y is finite.

Proof. Since $H\mathbb{F}_p \otimes X \cong H\mathbb{F}_p \otimes X_p^\wedge$ we can replace X with its p-completion without altering the spectral sequence. The only missing part is the identification of the associated graded. But we know this is the homology of the complex

Denis: We need to say something about strong convergence, how does Adams do it in his book?

$$0 \to \pi_* \operatorname{map}(Y, F^0) \to \pi_{*+1} \operatorname{map}(Y, F^1) \to \pi_{*+2} \operatorname{map}(Y, F^2) \to \cdots$$

By lemma 7.21 this is exactly the complex

$$0 \to \operatorname{Hom}_{\mathcal{A}}(H^*F^0, H^*Y) \to \operatorname{Hom}_{\mathcal{A}}(H^{*-1}F^1, H^*Y) \to \operatorname{Hom}_{\mathcal{A}}(H^{*-2}F^2, H^*Y) \to \cdots$$

and the thesis follows from lemma and the definition of Ext. □

This spectral sequence allows us to translate algebraic properties of H^*X as $\mathcal{A}$-module to topological properties of the spectrum X (or rather $X_p^\wedge$).

Theorem 7.23 (Margolis). *Let Y be a spectrum of finite type. Then*

$$\pi_* \operatorname{map}(H\mathbb{F}_p, Y) \cong \operatorname{Hom}_{\mathcal{A}}^*(H^*(Y, \mathbb{F}_p), \mathcal{A})$$

Proof. This is an immediate consequence of a theorem by Adams and Margolis that shows that $\mathcal{A}$ is injective as a module over itself. □

[1] This hypothesis is due to the fact that we're working with cohomology. If we worked with homology and the dual Steenrod algebra we wouldn't need it.

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