Introduction to Stable Homotopy Theory
Exercise Sheet 4

1. Let D_* be a chain complex of $\mathbb{Z}$-modules. Show that the functors

   $$H^n(X; D_*) : X \mapsto H^n(\mathrm{Hom}_*(\tilde{C}_*(X), D_*))$$

   assemble into a cohomology theory (here Hom_* is the internal Hom in chain complexes). The corresponding spectrum HD_* is called a **generalized Eilenberg-MacLane spectrum**.

2. For every pointed CW-complex X, let $\mathrm{Cov}_n^0(X)$ be the set of isomorphism classes of pairs $(p : \tilde{X} \to X, \sigma)$ where p is a covering space of degree n and σ is a bijection of the fiber of p over the basepoint with the set $\{1, \ldots, n\}$. Show that the functor

   $$\mathrm{Cov}_n^0 : h(\mathscr{S}_*^{\geq 0})^{\mathrm{op}} \to \mathrm{Set}$$

   is representable by a pointed space $B\Sigma_n$ such that

   $$\pi_m(B\Sigma_n) = \begin{cases} \Sigma_n & \text{if } m = 1 \\ 0 & \text{otherwise.} \end{cases}$$

3. Let X be a pointed space. Show that $\pi_0 \Sigma^\infty X$ is the free abelian group on the set $\pi_0 X$ modulo the subgroup generated by the connected component of the basepoint (hint: use the Freundenthal suspension theorem and the Hurewicz theorem).