Let Y be a spectrum s.t. $\dim H_n(Y;\mathbb{F}_p) < \infty$. Then there's a spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(H^*Y, H^*X) \Rightarrow \pi_{t-s}\,\mathrm{map}(X, Y_p^\wedge) \qquad \left(F^s \mathrm{map}\,\pi_{t-s}(X,Y_p^\wedge)/F^{s+1} \simeq E_\infty^{s,t} \right)$$

Moreover this sequence converges conditionally if Y connective and strongly if in addition X finite.

$f: X \to Y$ it's in Adams filtration s

$\rightsquigarrow [f] \in F^s/F^{s+1}$ lives in a subquotient of $E_2^{s,s}$

$s=0$ $\rightsquigarrow$ need to give a class in $\mathrm{Hom}_A(H^*Y, H^*X)$, this is just H^*f.

$s=1$ We have $f: X \to Y$ s.t. $H^*f = 0$. We need to give an element of $\mathrm{Ext}_A^{1,1}(H^*Y, H^*X)$, i.e. the isoclass of an extension

$$0 \to H^*X \to H^*Cf \to H^*Y[1] \to 0$$

Look at les in cohomology for f

$$\cdots H^*X \overset{0}{\leftarrow} H^*Y \leftarrow H^*Cf \leftarrow H^{*-1}X \overset{0}{\leftarrow} H^{*-1}Y \cdots$$

General s: $f: X = Z^0 \xrightarrow{f_0} Z^1 \to Z^2 \to \cdots \xrightarrow{f_s} Z^s = Y$, $H^*f_i = 0$. Choose such a factorization

$$0 \to H^*Z^i \to H^*Cf_{i-1} \to H^{*-1}Z^{i-1} \to 0$$

$$\left[0 \to H^*X \to H^*Cf_0 \to H^*Cf_1[1] \to \cdots \to H^*Cf_{s-1}[s-1] \to H^*Y[s] \to 0 \right]$$

$\in \mathrm{Ext}^{s,s}(H^*Y, H^*X)$

From now on: $X = Y = \mathbb{S}$; $p=2$: $\boxed{\mathrm{Ext}_A^{s,t}(\mathbb{F}_2,\mathbb{F}_2) \Rightarrow \pi_{t-s}\mathbb{S}_2^\wedge}$

To compute Ext, the first naive idea is to find a free res. for $\mathbb{F}_2$.

$$0 \leftarrow \mathbb{F}_2 \leftarrow A \leftarrow A\otimes\bar{A} \leftarrow A\otimes\bar{A}^{\otimes 2} \leftarrow A\otimes\bar{A}^{\otimes 3} \leftarrow \cdots$$

($\bar{A}$, 0; $xy\otimes z - x\otimes yz \leftarrow\!\shortmid x\otimes y\otimes z$)

$$d: A\otimes\bar{A}^{\otimes n+1} \to A\otimes\bar{A}^{\otimes n}$$
$$x_0\otimes\cdots\otimes x_{n+1} \mapsto \sum (-1)^i x_0 \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1}$$

$\Rightarrow$ Ext is the cohomology of this complex

$$0 \to \mathrm{Hom}_A(A,\mathbb{F}_2) \to \mathrm{Hom}_A(A\otimes\bar{A},\mathbb{F}_2) \to \mathrm{Hom}_A(A\otimes\bar{A}^{\otimes 2},\mathbb{F}_2) \to \cdots$$

$$0 \to \mathbb{F}_2 \xrightarrow{0} \mathrm{Hom}_{\mathbb{F}_2}(\bar{A},\mathbb{F}_2) \to \mathrm{Hom}_{\mathbb{F}_2}(\bar{A}^{\otimes 2},\mathbb{F}_2) \to \cdots \qquad \text{(bar complex)}$$

$$\mathrm{Ext}_A^{0,t}(\mathbb{F}_2,\mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & t=0 \\ 0 & \text{otherwise} \end{cases} \qquad \mathbb{F}_2 \overset{0}{\leftarrow} \bar{A} \overset{m}{\leftarrow} \bar{A}\otimes\bar{A}$$

$$\mathrm{Ext}_A^{1,t}(\mathbb{F}_2,\mathbb{F}_2) = \mathrm{Hom}_{\mathbb{F}_2}(\underbrace{\bar{A}/\bar{A}^2}_{\text{module of indecomposables for } A}, \mathbb{F}_2)$$

$\bar{A}/\bar{A}^2$ = "elements that cannot be written as $\sum a_i b_i$, $a_i, b_i \in \bar{A}$".

$\bar{A}$ has a basis of the form $Sq^{i_0}\cdots Sq^{i_n}$, $2i_j \geq i_{j+1}$.
The only ones that could be indecomposable (i.e. not in $\bar{A}^2$) are classes of the form Sq^i.

$Sq^1 Sq^2 = Sq^3$ (Adem relation)

In fact using a suitable Adem relation you can show $Sq^i \in \bar{A}^2$ whenever $i \neq 2^j$.

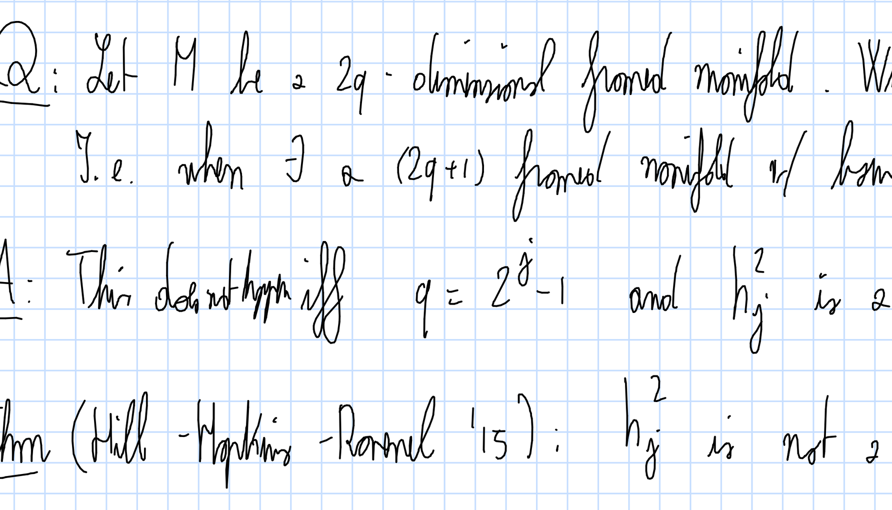
$\bar{A}/\bar{A}^2 = \bigoplus_{j\geq 0} \mathbb{F}_2[Sq^{2^j}]$. Let h_j be the dual basis

$\mathrm{Ext}_A^{1,*}(\mathbb{F}_2,\mathbb{F}_2) = \bigoplus_{j\geq 0} h_j$ $\qquad |h_j| = (1, 2^j)$

$$\left[0 \to \mathbb{F}_2 \to \mathbb{F}_2 x \oplus \mathbb{F}_2[2^j]y \to \mathbb{F}_2[2^j] \to 0 \right] = h_j$$

(A-module st. $Sq^{2^j}x = y$)

$\mathrm{Ext}_A^{2,*}(\mathbb{F}_2,\mathbb{F}_2)$ is spanned by $h_i h_j$ for $j \neq i \pm 1$



($q \geq 3$)

Q: Let M be a $2q$-dimensional framed manifold. When is M cobordant to a sphere?
I.e. when $\exists$ a $(2q+1)$ framed manifold w/ boundary W s.t. $\partial W = M \amalg S^{2q}$?

A: This doesn't happen iff $q = 2^j - 1$ and h_j^2 is a permanent cycle.

Thm (Hill-Hopkins-Ravenel '15): h_j^2 is not a permanent cycle for $j \geq 7$.

It is known that h_j^2 is a perm. cycle for $j \leq 5$.

Recall: Bordism classes of framed manifolds are the homotopy groups of the Thom spectrum of $* \to BO$ i.e., they are just $\pi_*\mathbb{S} = \Omega_*^{fr}$

Thm (Kervaire-Milnor '63): $\exists$ a group homom. $c: \Omega_{2q}^{fr} \to \mathbb{Z}/2$ (Kervaire invariant) s.t. a $2q$-fram. manifold is bordant to a sphere iff $c(M) = 1$.

Present a proof by Browder (+ simpl. by Brown) connecting this c to h_j^2.

$\exists v_n \in H^n(BO;\mathbb{F}_2)$ w/ the property $\forall M$ (Wu classes)

$v_n x$ is Poincaré dual to $Sq^n x$ when x is in the right dim.

v_n can be defined as follows: in $H^n(MO;\mathbb{F}_2)$ $v_n\theta = \overline{Sq}^n\theta$ where $\overline{Sq}^n \in A^*$ is defined inductively by $\overline{Sq}^0 = Sq^0$, $\sum_{i+j=n} \overline{Sq}^i Sq^j = 0$ $n>0$.

$$v_n: BO \to K(\mathbb{F}_2, n)$$

$BO\langle v_n\rangle$ be the homotopy fiber of v_n. Let Br_n be the Thom spectrum of $BO\langle v_n\rangle \to BO$.

$\pi_d Br_n$ are bordism classes of d-manifolds w/ a nullhomotopy $M \xrightarrow{-TM} BO \xrightarrow{v_n} K(\mathbb{F}_2, n)$

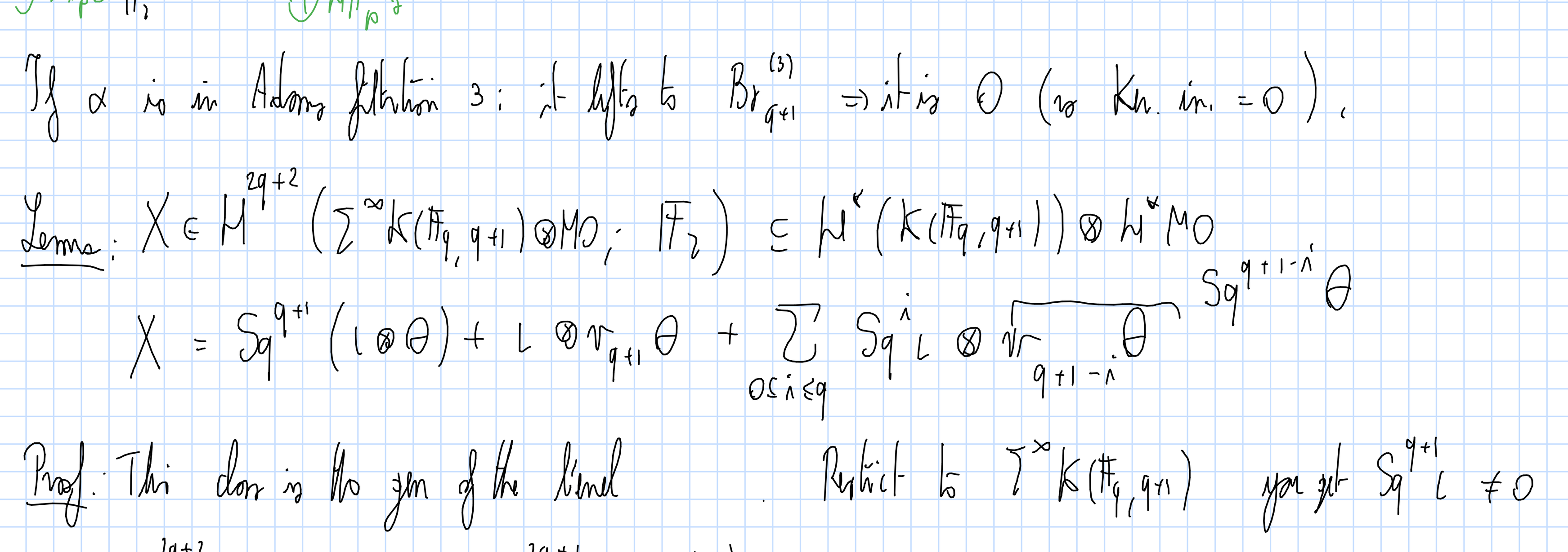
Thm (Brown): $\exists$ a map $c: \pi_{2q} Br_{q+1} \to \mathbb{Z}/8$ making the diagram commute

$$\begin{array}{ccc} \pi_{2q}\mathbb{S} & \xrightarrow{c} & \mathbb{Z}/2 \\ \downarrow & & \downarrow 4 \\ \pi_{2q} Br_{q+1} & \xrightarrow{c} & \mathbb{Z}/8 \end{array}$$

In particular every class in the kernel of $\pi_{2q}\mathbb{S} \to \pi_{2q}Br_{q+1}$ has Kervaire inv. 0.

$$\begin{array}{ccc} \Sigma^{2q}\mathbb{S} & & \\ \downarrow & \searrow & \\ \mathbb{S} & & \\ \downarrow & & \\ Br_{q+1} & \leftarrow & Br_{q+1}^{(1)} \\ \downarrow & & \\ MO & & \end{array}$$

Thm: $\exists$ a map $Br_{q+1}^{(1)} \to \Sigma^{-1}\Sigma^\infty K(\mathbb{F}_q, q+1)\otimes MO$ which is an iso on $H^*(-;\mathbb{F}_2)$ for $* \leq 2q$ and whose kernel in $* = 2q+1$ is cyclic of order 2 generated by a class whose image in $H^{2q+2}(K(\mathbb{F}_q, q+1))$ is $Sq^{q+1}\iota$ $\quad (\iota \in H^{q+1}(K(\mathbb{F}_q,q+1)) \simeq \mathbb{F}_2$ gen.)



If α is in Adams filtration 3: it lifts to $Br_{q+1}^{(3)}$ $\Rightarrow$ it is 0 (so Ker. inv. $= 0$).

Lemma: $X \in H^{2q+2}(\Sigma^\infty K(\mathbb{F}_q, q+1)\otimes MO; \mathbb{F}_2) \subseteq H^*(K(\mathbb{F}_q,q+1))\otimes H^*MO$

$$X = Sq^{q+1}(\iota\otimes\theta) + \iota\otimes v_{q+1}\theta + \sum_{0\leq i\leq q} Sq^i\iota \otimes \overbrace{v_{q+1-i}\theta}^{Sq^{q+1-i}\theta}$$

Proof: This class is the gen. of the kernel $H^{2q+2}(-) \to H^{2q+1}(Br_{q+1}^{(1)})$. Restrict to $\Sigma^\infty K(\mathbb{F}_q,q+1)$ you get $Sq^{q+1}\iota \neq 0$ $\square$

$h_i h_j$

$$\pi_*\mathbb{S} \to \pi_* MO^{\otimes -} \to \pi_* (MO\otimes\Sigma^\infty K(\mathbb{F}_2,q+1))^{\otimes -} \to \pi_* H\mathbb{F}_2 \otimes - \quad (\oplus H\mathbb{F}_2\text{'s})$$

$$\mathrm{Hom}(F', \mathbb{F}_2) \to \mathrm{Hom}_A(F, \mathbb{F}_2) \to \mathrm{Hom}_A(A, \mathbb{F}_2)$$

$$\begin{array}{ccccc} F' & \leftarrow & F & \leftarrow & A \\ \uparrow & & \uparrow & & \uparrow \\ A\otimes\bar{A} & \leftarrow & A\otimes\bar{A}^{\otimes 2} & \leftarrow & A\otimes\bar{A}^{\otimes 3} \end{array} \quad h_i h_j$$

$h_i h_j$ cannot detect this class.