ISHT Lecture 23 2021-02-04

Steenrod algebra & the homotopy type of MO

Def: Let p be a prime, the Steenrod algebra is the graded abelian group

$$A^* = H\mathbb{F}_p^* H\mathbb{F}_p = \pi_{-*}\operatorname{map}(H\mathbb{F}_p, H\mathbb{F}_p)$$

it is given an ($\mathbb{F}_p$-) algebra structure by composition

$A^0 = \mathbb{F}_p$ (Whitehead)

$A^i = 0 \quad \forall i < 0 \qquad \left(\operatorname{Map}(H\mathbb{F}_p, H\mathbb{F}_p) = \operatorname{Map}_{E_\infty\text{-grp}}(\mathbb{F}_p, \mathbb{F}_p) \text{ discrete} \right)$

Let's consider the abelian group extension

$$0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$$

It induces a fiber sequence

$$H\mathbb{Z}/p \to H\mathbb{Z}/p^2 \to H\mathbb{Z}/p \Rightarrow \text{homotopy map } \beta : H\mathbb{Z}/p \to \Sigma H\mathbb{Z}/p \quad (\text{Bockstein})$$

$A^1 = \mathbb{F}_p \cdot \beta.$

Note: β is 0 on homotopy groups, but it's not nullhomotopic:

$\ker \beta : H\mathbb{Z}/p^* X \to H\mathbb{Z}/p^{*+1} X$ is the image of $H\mathbb{Z}/p^{2\,*} X \to H\mathbb{Z}/p^* X$.

so to show $\beta \neq 0$ it's enough to find a space X & a class $H\mathbb{Z}/p^* X$ not in the image of $H\mathbb{Z}/p^{2\,*} X$.

$\boxed{p=2}$ $X = \overbrace{\mathbb{RP}^\infty}^{\Omega^\infty \Sigma H\mathbb{Z}/2}$, $H\mathbb{Z}/4^1\, \mathbb{RP}^\infty = 0$, $H\mathbb{Z}/2^1\, \mathbb{RP}^\infty = \mathbb{Z}/2$.

$\alpha \in A^n \quad \alpha : \underset{[X, \Sigma^i H\mathbb{F}_2]}{H\mathbb{F}_2^i X} \to \underset{[X, \Sigma^{i+n} H\mathbb{F}_2]}{H\mathbb{F}_2^{i+n} X} \quad \forall X \in Sp \quad$ by postcomposition

Thm: ($p=2$) $\forall i \geq 0 \ \exists\, Sq^i : H\mathbb{F}_2 \to \Sigma^i H\mathbb{F}_2$ w/ the following properties

① $Sq^0 = 1$, $Sq^1 = \beta$

② Under the iso $H\mathbb{F}_2^*(X \otimes Y) \simeq H\mathbb{F}_2^* X \otimes H\mathbb{F}_2^* Y \quad \forall X, Y \in Sp$

$$Sq^n(x \otimes y) = \sum_{i+j=n} Sq^i x \otimes Sq^j y \quad \text{(Cartan formula)}$$

i.e.: $Sq^\bullet = \sum_{i \geq 0} Sq^i \qquad Sq^\bullet(x \otimes y) = Sq^\bullet x \otimes Sq^\bullet y.$

③ If $X \in$ Space $\forall x \in H^n(X; \mathbb{F}_2)$ $(= H\mathbb{F}_2^n \Sigma^\infty X_+)$ we have

$Sq^n x = x^2$, $Sq^i x = 0 \ \forall i > n.$ [unstability of the action]

Moreover, ② implies $Sq^\bullet$ is a ring map.

④ [Adem relations] $\forall i < 2j$

$$\underset{i<2j}{Sq^i Sq^j} = \sum_{k=0}^{[i/2]} \binom{j-k-1}{i-2k} \underbrace{Sq^{i+j-k}\, Sq^k}_{i+j-k \geq 2k}$$

Corollary: The subalg generated by Sq^i's is spanned by monomials of the form

$Sq^I = Sq^{i_1} \cdots Sq^{i_n} \qquad i_j \geq 2 i_{j+1}. \qquad I = (\underbrace{i_1, \ldots}_{\text{admissible sequence}})$

Thm: The collection $\{Sq^I\}_{I \text{ adm seq.}}$ is a basis for $A^*/\mathbb{F}_2$.

Prove this: $H\mathbb{F}_2^* H\mathbb{F}_2 = \lim_n \widetilde{H\mathbb{F}}_2^{*+n} K(\mathbb{F}_2, n)$ because $H\mathbb{F}_2 = \operatorname{colim}_n \Sigma^{-n} \Sigma^\infty K(\mathbb{F}_2, n)$.

To compute the Steenrod algebra, you need to compute $H\mathbb{F}_2^* K(\mathbb{F}_2, n)$. This is computed by an inductive procedure using the Serre spectral sequence for the fibration

$\underset{\Omega K(\mathbb{F}_2, n)}{K(\mathbb{F}_2, n-1)} \to * \to K(\mathbb{F}_2, n)$ (Mosher - Tangora)

We start w/ $H^*(\mathbb{RP}^\infty; \mathbb{F}_2) = \mathbb{F}_2[x] \quad |x| = 1.$

$\left(\begin{array}{ll} H^* \mathbb{CP}^\infty & S^1 \to * \to \mathbb{CP}^\infty \\ H^* \mathbb{RP}^\infty & S^1 \to \mathbb{RP}^\infty \to \mathbb{CP}^\infty \end{array} \right)$

$H\mathbb{Z} \to \Sigma H\mathbb{Z}/2 \to \Sigma H\mathbb{Z}/2$

Q: Action of Sq^* on $H^*(\mathbb{RP}^\infty)$?

$Sq^0 x = x$, $Sq^1 x = x^2$, $Sq^i x = 0 \ \forall i > 1$

$Sq^\bullet x = x + x^2 \Rightarrow Sq^\bullet x^n = (Sq^\bullet x)^n = (x + x^2)^n = \sum_{i=0}^n \binom{n}{i} \underbrace{(x^2)^i (x)^{n-i}}_{x^{n+i}}$ $\quad n - i + 2i$

$\Rightarrow \boxed{Sq^i x^n = \binom{n}{i} x^{n+i}}$

Next step: Compute the cohomology of BO_d w/ the action of A^*.

$O_1^d \hookrightarrow O_d$ inclusion of diagonal matrices $\Rightarrow \underset{\circlearrowleft \Sigma_d}{BO_1^d} \to BO_d$ classifying $\eta_1 \times \cdots \times \eta_1 \to BO_1^d$

$\Rightarrow H^* BO_d \longrightarrow H^* BO_1^d = \mathbb{F}_2[x_1, \ldots, x_d] \quad \circlearrowleft \Sigma_d$

Claim: This map is injective w/ image the symmetric polynomials (Splitting principle)

(Proof: Induction on d using the fiber seq. $\underset{O_{d+1}/O_d}{S^d} \to BO_d \to BO_{d+1}$ $\quad V_{d+1}$ $\quad O_d \hookrightarrow O_{d+1}$ $\quad \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$)

$H^*(BO_d) \simeq \mathbb{F}_2[w_1, \ldots, w_d] \quad |w_i| = i \qquad w_i =$ Stiefel-Whitney classes

$w_i = \sum x_{j_1} \cdots x_{j_i}$

$\Rightarrow H^*(BO) \simeq \mathbb{F}_2[w_1, w_2, \ldots]$

From this you can deduce the action of $Sq^\bullet$ onto w_i

$Sq^\bullet w_i = \sum Sq^\bullet x_{j_1} \cdots Sq^\bullet x_{j_i} = \sum (x_{j_1} + x_{j_1}^2) \cdots (x_{j_i} + x_{j_i}^2)$

$\Rightarrow \boxed{Sq^i w_j = \sum_{t=0}^{i} \binom{j+t-i-1}{t} w_{i-t} w_{j+t}}$ (Wu formula)

$\binom{n}{i} = \dfrac{n(n-1)\cdots(n-i+1)}{i!}$ makes sense also for $n < 0$.

Recall by the Thom isomorphism: $H\mathbb{F}_2^* MO$ is a free module of rank 1 on $H\mathbb{F}_2^* \Sigma^\infty BO_+$.

So elements of $H^* MO$ will be written as $p \cdot \theta \quad p \in \mathbb{F}_2[w_1, \ldots, \ldots]$, $\theta \in H\mathbb{F}_2^0 MO$ is the Thom class.

$\Rightarrow$ by the Cartan formula $Sq^n(p\theta) = \sum_{i+j} Sq^i p \cdot Sq^j \theta.$

$MO \to MO \otimes \Sigma^\infty BO_+$

$BO \to BO \times BO$

We reduced to compute $Sq^j \theta$.

Recall: $MO = \operatorname{colim}_n \Sigma^{-n} \Sigma^\infty Th(\eta_n) \Rightarrow H^* MO \simeq \lim \tilde{H}^*(Th(\eta_n))$

$\tilde{H}^*(Th(\eta_n))$ is free module of rank 1 on $H^*(BO_n)$

$\tilde{H}^*(Th(\eta_n)) \simeq H^*(BO_n) \cdot \theta$

Trick: $Th(\eta_1) = \mathbb{RP}^\infty$. $\qquad Th(\eta_1) = \operatorname{cof}(\eta_1 \setminus 0 \to \eta_1) =$

$= \operatorname{cof}(S(\eta_1) \to BO_1) = BO_1$

$S(\eta_1) = \{x \in \eta_1 \mid |x| = 1\} \stackrel{!}{=} V_1 = \{(x, L) \mid L \subseteq \mathbb{R}^\infty \ \dim L = 1,\ x \in L,\ |x| = 1\} \sim *$

$\tilde{H}^*(Th(\eta_1)) = \tilde{H}(\mathbb{RP}^\infty) = \overset{\theta}{x} \cdot \mathbb{F}_2[x].$

$Th(\eta_1)^{\wedge d} \to Th(\eta_d)$

$BO_1^d \to BO_d$

$\theta \longmapsto \theta \otimes \cdots \otimes \theta = x_1 \cdots x_d$

$$\begin{array}{ccc} \tilde{H}^*(Th(\eta_d)) & \hookrightarrow & \tilde{H}^*(Th(\eta_1))^{\otimes d} = x_1 \cdots x_d\, \mathbb{F}_2[x_1, \ldots, x_d] \\ \wr\wr\, Th & & \wr\wr\, Th \\ H^*(BO_d) & \longrightarrow & H^*(BO_1)^{\otimes d} \end{array}$$

$Sq^\bullet \theta = Sq^\bullet(x_1 \cdots x_d) = Sq^\bullet x_1 \cdots Sq^\bullet x_d = (x_1 + x_1^2) \cdots (x_d + x_d^2)$

$\stackrel{!}{=} x_1 \cdots x_d (1 + x_1) \cdots (1 + x_d)$

$\stackrel{!}{=} \theta \cdot \sum_S x^S = \theta(\overset{1}{w_0} + w_1 + w_2 + \cdots + w_d)$

$\Rightarrow \boxed{Sq^i \theta = w_i \theta} \quad i \leq d. \qquad H^*(MO) = \mathbb{F}_2[w_1, \ldots, \ldots] \cdot \theta$

Now: Homotopy type of MO!

Remark: $\pi_* MO$ is 2-torsion: $\Omega_n = \pi_n MO$ is 2-torsion

because if $[M] \in \Omega_n \qquad 2[M] = [M \sqcup M] = [\partial(M \times [0,1])] = 0.$

Let $\omega = \{a_1, a_2, \ldots, a_d\}$ be a partition of m i.e. an unordered family of integers $\sum a_i = m$.

$x_\omega \in H^* BO \qquad x_\omega \in H^*(BO_d) \simeq \mathbb{F}_2[x_1, \ldots, x_d]^{\Sigma_d}$

$x_\omega = \sum_{\text{sym}} x_1^{a_1} \cdots x_d^{a_d}.$

x_ω form an additive basis of $H^* BO \Rightarrow x_\omega \theta$ form an additive basis for $H^* MO$

Def: ω is non-dyadic if none of the a_i is of the form $2^m - 1$.

Thm: $H^* MO$ is a free A^*-module on $\{x_\omega \theta\}$ where ω runs through all non-dyadic partitions.

Proof: Filter $H^* MO$ by the subspaces $F^s H^* MO$ spanned by x_ω has $\geq s$ dyadic elements. This is a filtration by A^*-submodules & the ass. graded is free □

Thm (Thom): The map $MO \xrightarrow{\oplus (x_\omega \theta)} \bigoplus_{\omega \text{ non-dyadic}} \Sigma^{|\omega|} H\mathbb{F}_2$ is a homotopy eq.

Proof: By the previous result, this is an eq. on $H\mathbb{F}_2^*$.

$\bigoplus A^* \cdot x_\omega \theta \xrightarrow{\sim} H^*(MO)$

Moreover, it is an iso on $H\mathbb{Z}[\tfrac{1}{2}]$-(co)homology (because both sides are 0).

$\Rightarrow$ It is an iso on $H\mathbb{Z}$ (co)homology using the fracture square $\Rightarrow$ it is an eq. since both sides are connective. □

Corollary: The map $MO \to H\mathbb{F}_2$ has a section.

Remark: A more careful analysis gives MO a (homotopy) ring structure coming from $BO_d \times BO_{d'} \to BO_{d+d'}$ which encodes the cartesian product of manifolds

$\Rightarrow \pi_* MO = \mathbb{F}_2[\alpha_i]_{i \neq 2^m - 1} \qquad |\alpha_i| = i.$

$\pi_* MU$ can be computed

$\pi_* MSO$ can also be "computed"

$H\mathbb{F}_2^* BO = \mathbb{F}_2[w_1, w_2, \ldots] \quad |w_i| = i \qquad BO \to BU$

$H\mathbb{Z}^* BU = \mathbb{Z}[c_1, c_2, \ldots] \quad |c_i| = 2i \qquad BU \to BO$