ISHT Lecture 22 2021-02-01

$\xi : B \to BO$

<u>Theorem</u>: $\Omega^{\xi}_n \to \pi_n M\xi$ sending $(M,\varphi) \mapsto [\Sigma^n \mathbb{S} \xrightarrow{\text{PT}} M^{n-TM} \xrightarrow{\varphi} M\xi]$ is an isomorphism

<u>Corollary</u>: X space $\pi_n(M\xi \otimes \Sigma^\infty X_+)$ = bordism group of closed n-manifolds w/ ξ-structure equipped w/ a map to X

<u>Proof</u>: The rhs $\Omega^{\xi_X}_n$ where $\xi_X : B \times X \to B \to BO$.
But $M\xi_X = M\xi \otimes \Sigma^\infty X_+$. $\square$

<u>Proof</u>:

<u>Theorem</u> (Smooth approximation theorem): If $f : M \to N$ is a <u>continuous</u> map of smooth manifolds then f is homotopic to a smooth map. Moreover if $A \subseteq M$ closed subset s.t. f is smooth in a nbhd of A we can choose the homotopy relative to A.

<u>Theorem (Transversality theorem)</u>: Let $f : M \to M'$ smooth map of smooth manifolds, $N \subseteq M'$ compact neatly embedded submanifold, then f is homotopic to a map $\pitchfork N$. Moreover if $A \subseteq M$ closed subset s.t. $f \pitchfork N$ in a nbhd of A we can choose the homotopy relative to A.
In particular, we can choose the htpy s.t. it's constant outside of a nbhd of $f^{-1}N$.

<u>Recall</u>: $N \subseteq M$ embedded (neatly embedded) if locally around every pt of N it looks like

$\mathbb{R}^k \subseteq \mathbb{R}^n$ $(x \in \operatorname{Int} N)$ $\qquad [0,\infty) \times \mathbb{R}^{k-1} \subseteq [0,\infty) \times \mathbb{R}^{n-1}$ $(x \in \partial N)$

<u>Recall</u>: $f \pitchfork N \iff \forall x \in M$ $fx \in N$ $df(T_x M) + T_{fx} N = T_x M'$.

<u>Surjectivity</u>: $\Omega^{\xi}_n \twoheadrightarrow \pi_n M\xi$

$$\begin{tikzcd} B \times_{BO} BO_k \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d] \\ BO_k \arrow[r, "n-(k)"] & BO \end{tikzcd}$$

$\Sigma^n \mathbb{S} \xrightarrow{f} M\xi = \operatorname{colim}_k \Sigma^{-k} \Sigma^\infty \operatorname{Th}(\xi_k)$

$\xi_k : \underbrace{B \times_{BO} BO_k}_{B_k} \to BO_k$ $\quad \operatorname{Th}(\xi_k) = \operatorname{Th}(\xi_n^* \eta_k)$

since $BO = \operatorname{colim}_n BO_n \Rightarrow B = \operatorname{colim}_n B_n \Rightarrow$ The Thom spectrum of ξ_n can be realised as the colim of the Thom spectra of ξ_k.

By compactness of $\mathbb{S}$, f factors through some $\Sigma^{-n}\Sigma^\infty \operatorname{Th}(\xi_k)$

$\Sigma^{n+k} \mathbb{S} \xrightarrow{f} \Sigma^\infty \operatorname{Th}(\xi_k)$ i.e. a class in $\underset{\ni f}{\pi_{n+k} \Sigma^\infty \operatorname{Th}(\xi_k)} \to \underset{\ni f}{\pi_n M\xi}$

But now, $\operatorname{Th}(\xi_k)$ is a colimit of S^k's $\Rightarrow (k-1)$-connected.

$\pi_{n+k} \operatorname{Th}(\xi_k) \to \underset{\ni f}{\pi_{n+k} \Sigma^\infty \operatorname{Th}(\xi_k)}$ this is surjective for $n+k \leq 2k + \text{const.}$ (Freudenthal Σ thm)

So if I take k large enough, f is in the image of $\pi_{n+k}(\operatorname{Th}\xi_k)$.

$S^{n+k} \xrightarrow{f} \operatorname{Th}(\xi_k) \to \Omega^\infty \Sigma^\infty \operatorname{Th}(\xi_k) \to \Omega^\infty \Sigma^k M\xi$

Recall $BO_d = Gr_d(\mathbb{R}^\infty) = \{ V \subseteq \mathbb{R}^\infty \mid \dim V = d \} = \bigcup_m Gr_d(\mathbb{R}^m) = \operatorname{colim}_m Gr_d(\mathbb{R}^m)$

Let $\xi_{k,m} : \underbrace{B_k \times_{BO_d} Gr_d(\mathbb{R}^m)}_{B_{k,m}} \to \underbrace{Gr_d(\mathbb{R}^m)}_{\text{closed smooth manifold}} = \{ P \in M_m(\mathbb{R}) \mid P = {}^tP = P^{-1}, \operatorname{rk} P = d \}$

compact because closed & lies in $\|P\| < M$

$\operatorname{Th}(\xi_k) = \operatorname{colim}_m \operatorname{Th}(\xi_{k,m})$

Again by finiteness of S^{n+k}, we can further restrict f to $\operatorname{Th}(\xi_{k,m})$ for some m.

$$\begin{tikzcd} S^{n+k} \arrow[r, "f"] \arrow[dr, "pf"'] & \operatorname{Th}(\xi_{k,m}) \arrow[r] \arrow[d, "p"] & \Omega^\infty \Sigma^k M\xi \\ & \operatorname{Th}(\eta_{k,m}) & \end{tikzcd}$$

$$\begin{tikzcd} \xi_{k,m} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \eta_{k,m} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \eta_k \arrow[d] \\ B_{k,m} \arrow[r] & Gr_k(\mathbb{R}^m) \arrow[r] & BO_k \end{tikzcd}$$

tautological bundle on $Gr_d(\mathbb{R}^m)$

$\operatorname{Th}(\eta_{k,m})$ = one pt compactification of $\eta_{k,m}$, which is a smooth manifold

moreover $\eta_{k,m} \supseteq Gr_k(\mathbb{R}^m)$ embedded closed submanifold as the zero section

<u>Goal</u>: Make $pf \pitchfork Gr_k(\mathbb{R}^m)$.

$\Sigma^{-k}\Sigma^\infty \operatorname{Th}(\xi_k) \to M\xi$

$\operatorname{Th}(\xi_{k,m}) \to \operatorname{Th}(\xi_k) \to \Omega^\infty \Sigma^\infty \operatorname{Th}(\xi_k) \to \Omega^\infty \Sigma^k M\xi$

$U = (pf)^{-1} \eta_{k,m} \subseteq \mathbb{R}^{n+k} \subseteq S^{n+k}$ $\quad U = f^{-1}\xi_{k,m}$

open subset $\Rightarrow$ it's a smooth manifold, therefore up to homotopy we can assume $pf|_U : U \to \eta_{k,m}$ is smooth. Furthermore, we can homotopy further to make it $\pitchfork$ to $Gr_k(\mathbb{R}^m)$.

$M := (pf)^{-1} Gr_k(\mathbb{R}^m)$ embedded submanifold of S^{n+k} of codim k. Moreover let ν the normal bundle of M in S^{n+k}, then $d(pf)|_\nu : \nu \xrightarrow{\sim} \eta_{k,m}$ = normal bundle of $Gr_k(\mathbb{R}^d)$ into $\eta_{k,m}$

$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[dr, "f"] & & B \arrow[d, "\xi"] \\ M \arrow[u] \arrow[r] \arrow[urr, dashed, "\varphi"] \arrow[rr, bend right, "n-TM"'] & Gr_k(\mathbb{R}^m) \arrow[r] & BO \end{tikzcd}$$

<u>Upshot</u>: (M,φ) is a manifold w/ ξ-structure. Moreover if you look at the construction we used the composite $\Sigma^n \mathbb{S} \simeq \Sigma^{-k}\Sigma^\infty S^{n+k} \to \Sigma^{-k}\Sigma^\infty \operatorname{Th}(\nu) = M^{n-TM} \to M\xi$ is exactly our original f.

<u>Injectivity</u>: (M,φ) in the kernel

$\Sigma^n \mathbb{S} \to M^{n-TM} \to M\xi$ is nullhomotopic.

$M \subseteq \mathbb{R}^{n+k}$, ν normal bundle $\subseteq \mathbb{R}^{n+k}$

$\Sigma^{-k}\Sigma^\infty S^{n+k} \to \Sigma^{-k}\Sigma^\infty \operatorname{Th}(\nu) \to \Sigma^{-k}\Sigma^\infty \operatorname{Th}(\xi_{k,m}) \to M\xi$

where $\nu = f^* \xi_{k,m}$ for some map $M \xrightarrow{f} B_{k,m}$ $\quad (M \xrightarrow{\varphi} B = \operatorname{colim}_{k,m} B_{k,m})$

Moreover, we can choose $k \gg 0$ s.t. the composite

$S^{n+k} \xrightarrow{PT} \operatorname{Th}(\nu) \xrightarrow{\operatorname{Th}(f)} \operatorname{Th}(\xi_{k,m})$

is nullhomotopic (compactness of S^{n+k} + Freudenthal)

$\Rightarrow$ We can extend it to

$$\begin{tikzcd} S^{n+k} \arrow[r, "PT"] \arrow[d, hook] & \operatorname{Th}(\nu) \arrow[r, "\operatorname{Th}(f)"] & \operatorname{Th}(\xi_{k,m}) \arrow[r, "\operatorname{Th}(p)"] & \operatorname{Th}(\eta_{k,m}) \\ D^{n+k+1} \arrow[urr, "H"'] & & & \end{tikzcd}$$

$p : B_{k,m} \to Gr_k(\mathbb{R}^m)$

Wlog $H(tx) = H(x)$ $t \geq 1-\varepsilon$, $x \in \partial D^{n+k+1}$

<u>Idea</u>: Make $\operatorname{Th}(p) \circ H \pitchfork Gr_k(\mathbb{R}^m) \subseteq \operatorname{Th}(\eta_{k,m})$

<u>Rmk</u>: $A = \{ tx \in D^{n+k+1} \mid t \in [0, 1-\varepsilon/2], x \in \text{closed nbhd of the 0 section in } \nu \}$
$\operatorname{Th}(p) \circ H|_A$ is smooth & $\pitchfork Gr_k(\mathbb{R}^m)$ (after homotopying $pf : M \to Gr_k(\mathbb{R}^m)$ is smooth)

$\Rightarrow$ As before let $U = f^{-1}(\xi_{k,m})$, $U \cap \{ x \mid |x| \geq [1-\varepsilon] \} = \nu \times [1-\varepsilon]$.

So we can find a htpy of $\operatorname{Th}(p) \circ H$ relative to $U \cap A$ s.t. $\operatorname{Th}(p) \circ H$ becomes smooth on U & $\pitchfork$ to $Gr_k(\mathbb{R}^m)$.

$W = (\operatorname{Th}(p) H)^{-1} Gr_k(\mathbb{R}^m)$ neat submanifold of D^{n+k+1} w/ boundary M.

As before

$$\begin{tikzcd} U \arrow[r] & B_{k,m} \arrow[r] \arrow[d, "p"] & B \arrow[d] \\ W \arrow[u, phantom, "\cap"] \arrow[r] \arrow[ur, "\varphi"] & Gr_{k,m} \arrow[r] & BO \end{tikzcd}$$

φ gives a ξ-structure on W compatible w/ the one on M. $\square$

<u>Generalization</u>: $\Omega^{\xi}_n \simeq \pi_n M\xi$.

$\exists$ an ∞-category $\operatorname{Cob}^{\xi}_n$ whose objects are $(n-1)$-dimensional closed manifolds w/ ξ-structure & morphisms are bordisms i.e. $(M,\varphi) \to (M',\varphi')$ is $(W,\tilde{\varphi})$

$\partial(W,\tilde{\varphi}) = (M,\varphi) \amalg (M', -\varphi')$.

This has a sym. monoidal structure given by $\amalg$ of manifolds

Then we can consider the space $|\operatorname{Cob}^{\xi}_n|$, which is the ∞-groupoid obtained by inverting all arrows.

so $\pi_0 \underbrace{|\operatorname{Cob}^{\xi}_n|}_{E_\infty\text{-space} \Rightarrow E_\infty\text{-group}} \simeq \Omega^{\xi}_{n-1}$.

<u>Theorem (Galatius-Madsen-Tillmann-Weiss)</u>: $|\operatorname{Cob}^{\xi}_n| = \Omega^\infty \Sigma MT\xi_n$ where $MT\xi_n$ is the Thom spectrum of the composite

$T\xi_n : TB_n \to BO_n \xrightarrow{-\eta_n} Sp$ — picking up the O_n-action on $\mathbb{D}\Sigma^\infty S^n = \Sigma^{-n}\mathbb{S}$

$$\begin{tikzcd} TB_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d, "\xi"] \\ BO_n \arrow[r, "n-\eta_n"] & BO \end{tikzcd}$$

$\Omega^{\xi}_{n-1} = \pi_{-1} MT\xi_n$ & there's an n-connected map $\Sigma^n MT\xi_n \to M\xi$.

<u>Steenrod problem</u> $\qquad (\Sigma^n \mathbb{S} \to M^{-TM} \otimes H\mathbb{Z} \simeq \Sigma^n M \otimes H\mathbb{Z})$

<u>Recall</u>: $\forall$ smooth n-manifold M $\exists [M] \in H_n(M;\mathbb{F}_2)$
$\forall$ smooth oriented n-manifold M $\exists [M] \in H_n(M;\mathbb{Z})$

<u>Problem</u>: Let X be a space
- (unoriented) for which $\alpha \in H_n(X;\mathbb{F}_2)$ $\exists f : M \to X$ s.t. $f_*[M] = \alpha$? (M: closed smooth n-manifold)
- (oriented) for which $\alpha \in H_n(X;\mathbb{Z})$ $\exists f : M \to X$ s.t. $f_*[M] = \alpha$? (M: closed oriented smooth n-manifold)

<u>Claim</u>: This is equivalent to describing the image of

$\pi_*(MO \otimes \Sigma^\infty X_+) \to \pi_*(H\mathbb{F}_2 \otimes \Sigma^\infty X_+)$ and $\pi_*(MSO \otimes \Sigma^\infty X_+) \to \pi_*(H\mathbb{Z} \otimes \Sigma^\infty X_+)$

where $MO \to H\mathbb{F}_2$, $MSO \to H\mathbb{Z}$ are the Thom classes.

$M \xrightarrow{f} X$ $\dim M = n$. Its class in $\pi_*(MO \otimes \Sigma^\infty X_+)$ is the composite

$$\begin{tikzcd} \Sigma^n \mathbb{S} \arrow[r] \arrow[rrr, bend right=50, "{[M]}"'] & M^{-TM} \arrow[r] \arrow[d, "(M \subset M \times M)_*"] & MO \otimes \Sigma^\infty X_+ \arrow[r] & H\mathbb{F}_2 \otimes \Sigma^\infty X_+ \\ & M^{-TM} \otimes \Sigma^\infty M_+ \arrow[ur] \arrow[urr] \arrow[rr] & & H\mathbb{F}_2 \otimes \Sigma^\infty M_+ \arrow[u] \end{tikzcd}$$

$[M]$ by the Thom isom.

$MO = M(\mathrm{id}_{BO})$
$MSO = M(BSO \to BO)$

Moreover $MO \to H\mathbb{F}_2$, $MSO \to H\mathbb{Z}$ are isom on π_0. That's because

$\mathbb{S} \to MO \to H\mathbb{F}_2$, $\quad \mathbb{S} \to MSO \to H\mathbb{Z}$
are $\mathbb{Z} \twoheadrightarrow \mathbb{F}_2$ $\quad \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}$ by def. of Thom class.

Easy to see $\pi_0 MO = \mathbb{F}_2$ (bordism of pts) $\pi_0 MSO = \mathbb{Z}$

<u>FACT</u>: $MO \to H\mathbb{F}_2$ has a section $\Rightarrow \pi_*(MO \otimes \Sigma^\infty X_+) \to \pi_*(H\mathbb{F}_2 \otimes \Sigma^\infty X_+)$ is always surjective

<u>FACT</u>: $MSO \to H\mathbb{Z}$ has a section rationally ($\pi_0 MSO_{\mathbb{Q}} \simeq \mathbb{Q}$ & any rational spectrum is a sum of E-ML spectra)

$\Rightarrow \forall \alpha \in H_n(X;\mathbb{Z})$ $\exists n \geq 1$ $M \to X$ s.t. $f_*[M] = n\alpha$. $\square$

$\pi_*(\tau_{\leq n} M\xi \otimes \Sigma^\infty X_+)$ = "bordism classes of objects w/ sing. in codim $\geq n+1$"

$\pi_*(H\mathbb{Z} \otimes \Sigma^\infty X_+)$
$\quad \tau_{\leq 0} MSO$