**Recall:** $M$ closed smooth manifold, want to prove $\mathbb{D}\Sigma^\infty M_+ \simeq M^{-TM}$.

$\eta_M: \mathbb{S} \to M^{-TM}$ fundamental class, constructed by embedding $M \hookrightarrow \mathbb{R}^n \subseteq S^n$, and taking the PT map $\Sigma^n \mathbb{S} \simeq \Sigma^\infty S^n_+ \to M^\nu$ & desuspending $M^{\nu - n} \simeq M^{-TM}$ $(\nu \oplus TM \simeq \underline{1}^{\oplus n})$

$\Sigma^n \mathbb{S} \oplus \mathbb{S}$

**Ex:** $M$ $H\mathbb{Z}$-oriented, then we get $[\eta_M] \in \pi_0 M^{-TM} \to \pi_0(H\mathbb{Z} \otimes M^{-TM}) \simeq \pi_n(H\mathbb{Z} \otimes \Sigma^\infty_+ M) = H_n M$, $[M]$

& the image is the "classical" notion of fund. class.

(the composition $\mathbb{S} \to M^{-TM} \xrightarrow{PT} \{x\}^{T_xM - T_xM} \simeq \mathbb{S}$ is the identity), $\{x\} \hookrightarrow M$; Thom spectrum of the triv. v. bundle of rank 0 over $\{x\}$

**Want** ev: $\Sigma^\infty M_+ \otimes M^{-TM} \to \mathbb{S}$, coev: $\mathbb{S} \to M^{-TM} \otimes \Sigma^\infty M_+$ satisfying some diagram

ev is the composition $\Sigma^\infty M_+ \otimes M^{-TM} \simeq (M \times M)^{(0, -TM)} \xrightarrow{PT} \Sigma^\infty M_+ \xrightarrow{p_*} \mathbb{S}$ ($M \xrightarrow{\Delta} M \times M$, $p: M \to *$)

coev is the composition $\mathbb{S} \xrightarrow{\eta_M} M^{-TM} \xrightarrow{\Delta_*} (M \times M)^{(-TM, 0)} \simeq M^{-TM} \otimes \Sigma^\infty M_+$

**Thm (Atiyah duality):** ev and coev exhibit $M^{-TM}$ as the SW dual of $\Sigma^\infty M_+$.

**Def:** Let $f: M \to M'$ smooth map, $N \subseteq M'$ an embedded submanifold. Then we say $f$ is transverse to $N$ ($f \pitchfork N$) if $\forall x \in M$, $fx \in N$

$df(T_x M) + T_x N = T_x M'$

**Intuition:** [crossing curves] transverse, [tangent curves] not transverse

**Lemma:** $f \pitchfork N$, $f^{-1}N$ is an embedded submanifold of $M$ & $df$ induces an iso between the normal bundle of $f^{-1}N$ in $M$ & the pullback of the normal bundle of $N$ in $M'$.

**Proof:** This is a local statement, so pick charts around $x \in f^{-1}N$, $fx \in M'$

$\underbrace{\mathbb{R}^m}_{M} \xrightarrow{(f', f'')} \underbrace{\mathbb{R}^n \times \mathbb{R}^k}_{M'}$ $\quad N = \mathbb{R}^n \times \{0\}$, $f0 = 0$

$f \pitchfork N \iff d_0 f''$ is full rank $\Rightarrow$ the thm follows from the implicit function theorem. □

**Lemma:** Let $f: M_1 \to M_2$ smooth map, $N_2 \overset{i_2}{\subseteq} M_2$ submanifold $f \pitchfork N_2$ ($M_1, M_2, N_2$ compact)

$N_1 = f^{-1}N_2$, $i_1: N_1 \subseteq M_1$. Then $\exists$ a commutative diagram $\forall V$ vector bundle $/M_2$

$$\begin{array}{ccc} M_1^{f^*V} & \xrightarrow{f_*} & M_2^V \\ \downarrow PT(i_1) & & \downarrow PT(i_2) \\ N_1^{f^*(V \oplus \nu)} & \xrightarrow{f_*} & N_2^{\nu} \end{array}$$

**Proof:** Pick a tub. nbd $\nu \subseteq M_2$ of $N_2$ s.t. $f^{-1}\nu$ tubular nbd for $N_1$. Then the comm. of the diagram is clear.

$$\begin{array}{ccc} M_1 & \xrightarrow{f} & M_2 \\ \downarrow & & \downarrow \\ Th(f^{-1}\nu) & \xrightarrow{f} & Th(\nu) \end{array} \qquad x \in M_1 \mapsto \begin{cases} * & x \notin f^{-1}\nu \\ fx & x \in f^{-1}\nu \end{cases}$$

□

**Proof of Atiyah duality:** $f(x,y) = (x,y,y)$, $g(x,y) = (x,y,y)$

$M^{-TM} \xrightarrow{\eta_M \otimes 1} M^{-TM} \otimes M^{-TM}$ [$= (M \times M)^{(-TM,-TM)}$] $\xrightarrow{f_*} M^{-TM} \otimes \Sigma^\infty_+ M \otimes M^{-TM}$ [$= (M \times M \times M)^{(-TM,0,-TM)}$] $\xrightarrow{PT(g)} M^{-TM} \otimes \Sigma^\infty_+ M \simeq (M \times M)^{(-TM,0)} \xrightarrow{(pr_1)_*} M^{-TM}$; also coev $\otimes 1$ and $1 \otimes$ ev to $M^{-TM}$

$PT(g) \circ f_*$:

$$\begin{array}{ccc} (M \times M)^{(-TM,-TM)} & \xrightarrow{f_*} & (M \times M \times M)^{(-TM,0,-TM)} \\ \downarrow PT(\Delta) & & \downarrow PT(g) \\ M^{-TM} & \xrightarrow{\Delta_*} & (M \times M)^{(-TM,0)} \end{array}$$

$g \pitchfork \Delta(M)$, $g^{-1}M = M \times M \xrightarrow{f} M \times M \times M$.

$M^{-TM} \xrightarrow{\eta_M \otimes 1} (M \times M)^{-TM} \xrightarrow{PT(\Delta)} M^{-TM} \xrightarrow{\Delta_*} (M \times M)^{(-TM,0)} \xrightarrow{(pr_1)_*} M^{-TM}$, $id = (pr_1 \Delta)_*$

$M \hookrightarrow \mathbb{R}^n \hookrightarrow S^n$ (composite $i$) $\quad i \times id: M \times M \hookrightarrow S^n \times M$

$\eta_M = \Sigma^{-n} PT(i)$ $\quad \eta_M \otimes 1 = \Sigma^{-n} PT(i \times id)$

$PT(\Delta) \circ (\eta_M \otimes 1) = PT((i \times id)\Delta) \simeq PT(i, id)$

**Point:** $(i, id): M \hookrightarrow S^n \times M$ is isotopic to $(0, id)$ (straight line homotopy)

$PT(0, id) \simeq \Sigma^n M^{-TM} \xrightarrow{id} \Sigma^n M^{-TM}$.

The other diagram is identical & in the notes. □

**Manifolds w/ boundary:** $(M, \partial M)$ compact mfd w/ boundary

$\partial M \hookrightarrow M \times \partial M$ (nor. bdle $1 + T\partial M$), $(\partial M)^{-1}$

$$\begin{array}{ccccc} M^{-TM} \otimes \Sigma^\infty \partial M_+ & \xrightarrow{PT} & \Sigma^{-1}\Sigma^\infty \partial M_+ & \xrightarrow{p_*} & \Sigma^{-1}\mathbb{S} \\ \downarrow & & \downarrow & & \downarrow \\ M^{-TM} \otimes \Sigma^\infty M_+ & \to & 0 & \to & 0 \\ \downarrow & & & & \downarrow \\ M^{-TM} \otimes \Sigma^\infty M/\partial M & & \xrightarrow{ev} & & \mathbb{S} \end{array}$$

vertical cofibers

You can embed $M$ into $D^n$ s.t. $M \cap \partial D^n = \partial M$ & it's "neat" (locally $[0,\infty) \times \mathbb{R}^k \subseteq [0,\infty) \times \mathbb{R}^n$)

$\Rightarrow$ Get a PT collapse map

$$\begin{array}{ccc} S^{n-1} \simeq \partial D^n & \xrightarrow{PT} & Th(\nu|_{\partial M}) \\ \downarrow & & \downarrow \\ D^n & \xrightarrow{PT} & Th(\nu) \end{array}$$

$j: \partial M \hookrightarrow \partial M \times M$

$$\begin{array}{ccccc} \Sigma^{-1}\mathbb{S} & \to & \partial M^{-1-T\partial M} & \xrightarrow{j_*} & \Sigma^\infty \partial M_+ \otimes M^{-TM} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \to & M^{-TM} & \xrightarrow{\Delta_*} & \Sigma^\infty M_+ \otimes M^{-TM} \\ \downarrow & & & & \downarrow \\ \mathbb{S} & & \xrightarrow{coev} & & \Sigma^\infty M/\partial M \otimes M^{-TM} \end{array}$$

(vertical cofibers).

**Thm (Atiyah duality w/ $\partial$):** ev, coev exhibit $M^{-TM}$ as the SW dual of $\Sigma^\infty M/\partial M$.

**Exercise:** We have a cofiber seq. $\partial M_+ \xrightarrow{i} M_+ \to M/\partial M \Rightarrow$ a boundary map $\Sigma^\infty M/\partial M \to \Sigma\Sigma^\infty \partial M_+$. The dual of this map is $\partial M^{-1-T\partial M} \xrightarrow{i_*} M^{-TM}$ (Hint: $\Sigma \partial M_+ = W/\partial W$, $W = M \times [0,1]$)

---

## § Manifolds w/ structure & bordism

From now on we fix a map $\xi: B \to BO$.

**Def:** Let $V: X \to BO$ virtual vector bundle of rank 0, a $\underline{\xi\text{-structure}}$ on $V$ is just a lift of $V$ along $\xi$.

$$\begin{array}{ccc} & & B \\ & \nearrow^{V_\xi} & \downarrow \xi \\ X & \xrightarrow{V} & BO \end{array}$$

If $V$ has not rank 0, a $\xi$-str. on $V$ is a $\xi$-structure on $V - \operatorname{rk} V$.

**Ex:**
- $\xi = id$, no data.
- $\xi: BSO \to BO$ fiber ($\det: BO \to BC_2$), a $\xi$-structure is just an ($H\mathbb{Z}$)-orientation.
- $\xi: BU \to BO$ complex structure.
- $\xi: * \to BO$ stable framing (ex: $V$ vector bundle, a stable framing is an eq. class of isom $V \oplus \mathbb{R}^n \simeq \mathbb{R}^m$)
- $\xi: B \to BO$, $Y$ space, $\xi_Y: B \times Y \to B \to BO$

  A $\xi_Y$-structure $\iff$ $\xi$-structure + a map $X \to Y$.

**Def:** A $\underline{\xi\text{-structure}}$ on a smooth manifold $M$ is a $\xi$-structure on $\dim M - TM$.

**Construction:** Let $(M, \partial M)$ a smooth manifold w/ boundary

$TM|_{\partial M} \simeq \underline{1} \oplus T\partial M$ identified w/ the "inward pointing normal" [$T\partial M$, $e_1$, $\mathbb{R}$]

Via this identification, we can identify

$(\dim M - TM)|_{\partial M} = \dim \partial M - T\partial M$ $\qquad \varphi \quad \varphi_\partial$

and so we can "pullback" a $\xi$-structure from $M$ to $\partial M$.

**Remark:** If $(M, \varphi)$, $(M', \varphi')$ manifolds w/ $\xi$-structure $(M \amalg M', \varphi \amalg \varphi')$ manifold w/ $\xi$-structure.

**Def:** Let $n \geq 0$. The $\underline{n\text{-dimensional bordism group w/ } \xi\text{-structure}}$ $\Omega^\xi_n$ is the quotient of the monoid (under $\amalg$) of equivalence classes of closed manifolds w/ $\xi$-structures by the submonoid of the boundaries of compact manifolds w/ $\xi$-structure.

**Lemma:** $\Omega^\xi_n$ is a group.

**Proof:** $[M, \varphi] \in \Omega^\xi_n$, let us consider $M \times [0,1]$. This is a compact manifold w/ boundary

$T(M \times [0,1]) \simeq p^* TM \oplus \underline{1}$ ← left-to-right orientation

$T(M \times [0,1])|_{\partial(M \times [0,1])} = (TM \oplus \underline{1} \amalg TM \oplus \underline{1})$ (identity; $(1,-1)$)

Now we can lift $\varphi$ to $\tilde{\varphi}: M \times [0,1] \to B$

$$\begin{array}{ccc} & \xrightarrow{\tilde\varphi} & B \\ M \times [0,1] & \searrow & \downarrow \xi \\ \downarrow \wr & & \\ M & \to & BO \end{array}$$

$\tilde{\varphi}|_{M \times \{0\}} = \varphi$ $\qquad \partial(M \times [0,1], \tilde{\varphi}) = (M, \varphi) \amalg (M, -\varphi)$

$[M, \varphi] + [M, -\varphi] = 0$. □

Suppose $(M, \varphi)$ closed $n$-manifold w/ $\xi$-structure. Then we get $[M, \varphi] \in \pi_n M\xi$ (Thom spectrum of $\xi: B \to BO$)

$[M, \varphi]: \Sigma^n \mathbb{S} \xrightarrow{\eta_M} \Sigma^n M^{-TM} \simeq M^{n-TM} \xrightarrow{\varphi_*} M\xi$.

**Remark:**
- $[M \amalg M', \varphi \amalg \varphi'] = [M, \varphi] + [M', \varphi']$ $\qquad (M \amalg M')^{-T(-)} = M^{-TM} \oplus M'^{-TM'}$
- $(M, \varphi) = \partial(\tilde{M}, \tilde{\varphi})$ we can factor $[M, \varphi]$ as $(j: M \hookrightarrow \tilde{M})$

$\Sigma^n \mathbb{S} \xrightarrow{\eta_M} M^{n-TM} \xrightarrow{j_*} \tilde{M}^{n+1-T\tilde{M}} \xrightarrow{\tilde{\varphi}_*} M\xi$

($\eta_M = \mathbb{D}(M_+ \to S^0)$, $j_* = \mathbb{D}(\tilde{M}/M \to \Sigma M_+)$)

So $j_* \eta_M$ is the dual of $\tilde{M}/M \to \Sigma M_+ \to S^1$, and this is nullhomotopic.

$[\partial(\tilde{M}, \tilde{\varphi})] = 0$. Therefore $(M, \varphi) \mapsto [M, \varphi]$ induces a group homomorphism

$\Omega^\xi_n \to \pi_n M\xi$

**Thm (Pontryagin–Thom):** The homom. $\Omega^\xi_n \to \pi_n M\xi$ is an isomorphism.

**Corollary:** Let $X$ be any space. $\pi_n(M\xi \otimes \Sigma^\infty_+ X)$ is the group of bordism classes of manifolds w/ $\xi$-structure & a map to $X$.

**Proof:** $M\xi \otimes \Sigma^\infty_+ X \simeq M\xi_X$. □