ISHT Lecture 20 — 2021-01-25

Recall: X space, a <u>stable spherical fibration</u> is a functor
$$V : X \to Sp \quad \text{s.t. } \forall x\ \exists n \quad V(x) \simeq \Sigma^n \mathbb{S}.$$

Ex: M top. manifold, $M \to Sp$, $x \mapsto \Sigma^\infty M/M \setminus x$ is a stable spherical fibration.
If M smooth manifold, this comes from the vector bundle TM.

The Thom spectrum X^V of a s.s. fibration V is just its colimit.

Recall: E homotopy ring spectrum, an <u>orientation</u> or <u>Thom class</u> on V is a map
$$\theta : X^V \to \Sigma^n E \quad \text{s.t. } \forall x \in X \text{ the map}$$
$$E \otimes S^{V_x} \xrightarrow{1 \otimes i_x} E \otimes X^V \xrightarrow{1 \otimes \theta} E \otimes \Sigma^n E \xrightarrow{\mu} \Sigma^n E$$
is an equivalence.

$$O_n \curvearrowright H_n(S^n, R) \qquad \{\pm 1\} \hookrightarrow GL_1 R$$

Ex: · Every s.s. fibration is $H\mathbb{F}_2$-orientable.
· A v. bundle V is HR-orientable iff the composite $X \to BO_n \to BGL_1 R$ is nullhomotopic.

Thm (Thom isomorphism): Let $V : X \to Sp$ s.s. fibration, E homotopy ring spectrum, $\theta : X^V \to \Sigma^n E$ an orientation. Then $\exists$ equivalences of E-modules (in fact $\mathrm{map}(\Sigma^\infty_+ X, E)$-mod!)
$$E \otimes X^V \simeq \Sigma^n E \otimes \Sigma^\infty X_+, \quad \mathrm{map}(X^V, E) \simeq \mathrm{map}(\Sigma^n \Sigma^\infty X_+, E).$$
In particular we have $E_* X^V \simeq E_{*-n} X$, $E^* X^V \simeq E^{*+n} X$.

Proof: $X^V \simeq \operatorname{colim}_{x \in X} V(x)$, $\theta : X^V \to E \iff$ a nat. transf. $V(x) \xrightarrow{\theta} \underline{E}$ where $\underline{E}$ is the const. functor at E.
$$E \otimes X^V \simeq E \otimes \operatorname{colim}_{x \in X} V(x) \simeq \operatorname{colim}_{x \in X} E \otimes V(x) \xrightarrow{\theta} \operatorname{colim}_{x \in X} \underline{E} \otimes \Sigma^n \underline{E} \xrightarrow{\mu} \operatorname{colim}_{x \in X} \Sigma^n \underline{E} \simeq \Sigma^n E \otimes \Sigma^\infty_+ X$$
$$\mathrm{map}(X^V, E) \simeq \lim_{x \in X} \mathrm{map}(V(x), E) \xleftarrow{\theta^*} \lim_{x \in X} \mathrm{map}(\Sigma^n E, E) \leftarrow \lim_{x \in X} \Sigma^{-n} E \simeq \mathrm{map}(\Sigma^\infty_+ X, \Sigma^{-n} E).$$
$E \to \mathrm{map}(E, E)$ adjoint to $E \otimes E \xrightarrow{\mu} E$
$$\Sigma^{-n} E \to \mathrm{map}(\Sigma^n E, E) \to \mathrm{map}(V(x), E) \quad \text{eq. } \forall x : V(x) \otimes - \text{ detects eq.}$$
$$V(x) \otimes \Sigma^{-n} E \to V(x) \otimes \mathrm{map}(V(x), E) \simeq E \quad \text{\& that's an eq. by hypothesis.} \qquad \square$$

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§ Spanier-Whitehead duality

Def: Let $\mathcal{C}$ be a symm. monoidal ∞-cat., $X \in \mathcal{C}$. $Y \in \mathcal{C}$ is called a (strong) dual if $\exists$ maps
$$ev : X \otimes Y \to \mathbb{1}, \qquad coev : \mathbb{1} \to Y \otimes X$$
& commutative diagrams
$$\begin{tikzcd} X \arrow[r, "id_X \otimes coev"] \arrow[dr, equal] & X \otimes Y \otimes X \arrow[d, "ev \otimes id_X"] \\ & X \end{tikzcd} \quad \& \quad \begin{tikzcd} Y \arrow[r, "coev \otimes id_Y"] \arrow[dr, equal] & Y \otimes X \otimes Y \arrow[d, "id_Y \otimes ev"] \\ & Y \end{tikzcd}$$

Rmk: $Y \in \mathcal{C}$ is a dual for $X \iff$ it is a dual in $h\mathcal{C}$.

Ex: k field, $\mathcal{C} = Vect_k$, the dual of X is $\mathrm{Hom}_k(X, k)$ iff X is finite dim.
$$ev : X \otimes \mathrm{Hom}_k(X, k) \to k, \quad (x, \varphi) \mapsto \varphi x$$
$$coev : k \to \mathrm{Hom}(X, k) \otimes X \simeq \mathrm{Hom}(X, X), \quad 1 \mapsto id_X$$
$$1 \mapsto \textstyle\sum_i e_i^\vee \otimes e_i \quad \{e_i\} \text{ basis of } X.$$

Rmk: · Y is dual to $X \iff X$ is dual to Y.
· If X has a dual it's called <u>(strongly) dualizable</u> or <u>rigid</u>.

Prop: $\mathcal{C}$ symm. mon. ∞-cat., $X \in \mathcal{C}$. $Y \in \mathcal{C}$ is a dual to X iff $\exists$ an adjunction
$$X \otimes - \dashv Y \otimes -$$
In particular if X has a dual, it is uniquely determined as the value of the right adjoint of $X \otimes -$ on $\mathbb{1} \in \mathcal{C}$.

Proof: Suppose Y is a dual to X. Then we have nat. transf.
$$id_{\mathcal{C}} \simeq \mathbb{1} \otimes id \xrightarrow{coev} Y \otimes X \otimes id_{\mathcal{C}}, \qquad X \otimes Y \otimes id_{\mathcal{C}} \xrightarrow{ev \otimes id} \mathbb{1} \otimes id_{\mathcal{C}} \simeq id_{\mathcal{C}}$$
that satisfy the triangular identities $\Rightarrow Y \otimes -$ is right adjoint to $X \otimes -$.
Viceversa, if $\exists$ an adjunction $X \otimes - \dashv Y \otimes -$, you can get ev, $coev$ by evaluating unit & counit at $\mathbb{1}$ & the triangular identity force Y to be a dual to X. $\square$

Corollary: $X \in Sp$ is dualizable iff $\exists\, \mathbb{D}X \in Sp$ & a natural eq.
$$\mathbb{D}X \otimes Z \simeq \mathrm{map}(X, Z).$$
In particular $\mathbb{D}X \simeq \mathrm{map}(X, \mathbb{S})$ ($\Rightarrow \mathbb{D} : (Sp^{dual})^{op} \to Sp^{dual}$ is a functor)

Exercise: A functor $F : Sp \to Sp$ preserves all colimits iff it is of the form $F\mathbb{S} \otimes -$.
Therefore X is dualizable iff $\mathrm{map}(X, -)$ preserves all colimits.
(Hint: use the std presentation of a spectrum)

Prop: A spectrum is dualizable iff it is finite.

Proof: ($\Rightarrow$) If X is dualizable, $\mathrm{map}(X, -) \simeq \mathbb{D}X \otimes -$ preserves all colimits.
$\Rightarrow X$ finite (exercise in exercise sheet)

($\Leftarrow$) $\mathbb{S}$ is dualizable because it's the unit ($ev, coev = id_{\mathbb{S}}$), similarly $\Sigma^n \mathbb{S}$ is dualizable $\forall n$ w/ dual $\Sigma^{-n} \mathbb{S}$.
$$\Sigma^n \mathbb{S} \otimes - \simeq \Sigma^n - \dashv \Sigma^{-n} - \simeq \Sigma^{-n} \mathbb{S} \otimes -.$$
So it's enough to show dualizable spectra are closed under finite colimits.
X, X' dualiz. $\Rightarrow X \oplus X'$ dualiz. w/ dual $\mathbb{D}X \oplus \mathbb{D}X'$
$$\mathrm{map}(X \oplus X', Z) \simeq \mathrm{map}(X, Z) \oplus \mathrm{map}(X', Z) \simeq \mathbb{D}X \otimes Z \oplus \mathbb{D}X' \otimes Z \simeq (\mathbb{D}X \oplus \mathbb{D}X') \otimes Z.$$
X, X' dual., $f : X \to X' \Rightarrow \operatorname{cof} f$ dualiz. w/ dual $\operatorname{fib} : \mathbb{D}X' \xrightarrow{\mathbb{D}f} \mathbb{D}X$
$$\mathrm{map}(\operatorname{cof} f, Z) \simeq \operatorname{fib}(f^* : \mathrm{map}(X', Z) \to \mathrm{map}(X, Z))$$
$$\simeq \operatorname{fib}(\mathbb{D}X' \otimes Z \to \mathbb{D}X \otimes Z) \simeq \operatorname{fib}(\mathbb{D}X' \to \mathbb{D}X) \otimes Z. \quad \square$$

Remark: Let E be a spectrum, $\forall X$ finite spectrum
$$\begin{array}{ccc} E_* \mathbb{D}X & \simeq & E^{-*} X \\ \| & & \| \\ \pi_*(E \otimes \mathbb{D}X) & = & \pi_* \mathrm{map}(X, E) \end{array}$$

$(M, \partial M)$ compact manifold w/ boundary. Let $-TM$ to be the virtual vector bundle given by the opposite of TM tangent bundle in $KO^0 M$. It is called the <u>stable normal bundle</u>.

Thm (Atiyah duality): $\Sigma^\infty M/\partial M$ is dualizable w/ dual M^{-TM}. [Milnor-Spanier]

Corollary (Poincaré duality) (Lefschetz): Let $(M, \partial M)$ be a compact manifold w/ boundary s.t. it is E-orientable for E homotopy ring spectrum. Then $\exists$ isomorphism
$$E_*(M, \partial M) \simeq E^{\dim M - *}(M), \quad E^*(M, \partial M) \simeq E_{\dim M - *}(M)$$

Proof:
$$E_*(M, \partial M) = \pi_*(\Sigma^\infty M/\partial M \otimes E) \overset{\text{Atiyah duality}}{\simeq} \pi_* \mathrm{map}(M^{-TM}, E)$$
$$\overset{\text{Thom isom}}{\simeq} \pi_* \mathrm{map}(\Sigma^{-\dim M} \Sigma^\infty M_+, E)$$
$$\simeq E^{\dim M - *}(M) \qquad \square$$

Def / Thm: M, N closed manifolds, $f : N \hookrightarrow M$ embedding. Then $\exists$ a homotopy eq.
$$M / M \setminus N \simeq Th(\nu_f) \quad \text{normal bundle of } f = \operatorname{coker}(df : TN \to f^* TM)$$
The map $\Sigma^\infty M_+ \to N^{\nu_f}$ is called the <u>Pontryagin-Thom collapse map</u>
$(\Sigma^\infty(M_+ \to M/M \setminus N))$
This equivalence depends only on the isotopy class of f. (isotopy: $N \times [0,1] \hookrightarrow M$ s.t. $N \times \{t\} \hookrightarrow M$ embedding)

Proof: We can extend f to a tubular nbd.
$$N \hookrightarrow \nu_f \hookrightarrow M \quad \text{open embedding of the normal bundle}$$
$\forall H : f \sim f'$ isotopy can be extended to an isotopy of tubular neighborhoods.
$\Rightarrow$ We have an open covering $\{M \setminus N, \nu_f\}$ of $M \Rightarrow$ a homotopy pushout square
$$\begin{tikzcd} \nu_f \setminus 0 \arrow[r, hook] \arrow[d] & M \setminus N \arrow[d] \\ \nu_f \arrow[r] & M \end{tikzcd} \Rightarrow \text{an eq. on vertical cofibers, which is what we wanted.} \quad \square$$

Rmk: $M \to Th(\nu_f) =$ 1-pt compactification of ν_f. send $\nu_f \to \nu_f$ via the identity & everything else to the pt at ∞.

Let V be a v. bundle on M, we have a PT collapse map
$$M^V \to N^{f^* V \oplus \nu_f}$$
In fact, let $D(V)$ be the disc bundle of V, $N \hookrightarrow M \hookrightarrow D(V)$ w/ normal bundle $f^* V \oplus \nu_f$ & then we have $D(V) \to N^{f^* V \oplus \nu_f}$ PT collapse map sending $\partial D(V)$ to $*$.
$$\Rightarrow M^V = D(V)/\partial D(V) \to N^{f^* V \oplus \nu_f}$$
& in fact by desuspending this map, I can define it also for a virtual v. bundle.
We're going to denote this map by $PT(f)$ or $PT(f, V)$.

Remark: $W \overset{g}{\hookrightarrow} N \overset{f}{\hookrightarrow} M \Rightarrow PT(fg) = PT(g) \circ PT(f)$
$$M^V \to N^{V \oplus \nu_f} \to W^{V \oplus \nu_f \oplus \nu_g}$$

Def: $f : N \to M$, V (virtual) vector bundle on M, $f_* : N^{f^* V} \to M^V$ induced map on colimits.

Thm (Whitney embedding theorem): M closed manifold, $\exists n \gg 0$ s.t. M embeds into $\mathbb{R}^n$ & any two embeddings in $\mathbb{R}^m$ are isotopic $\forall m \geq n$

Proof sketch: For the $\exists$ cover M w/ charts $U_1, \dots, U_r$ s.t. $U_i \hookrightarrow \mathbb{R}^{\dim M}$ extend to a map $M \to \mathbb{R}^{\dim M}$
Take $M \to \prod_{i=1}^r \mathbb{R}^{\dim M}$ is an embedding. $\square$

Def: M closed manifold, we define the <u>fundamental map</u> $\mathbb{S} \to M^{-TM}$ as follows:
we embed $M \hookrightarrow \mathbb{R}^m \subseteq S^m$ so we have a PT collapse map
$$\mathbb{S} \simeq \Sigma^{-m} \Sigma^\infty S^m \to \Sigma^{-m} M^\nu \simeq M^{\nu - m} \simeq M^{-TM}.$$
This is well defined because any two embeddings are isotopic.

X lch top space $Sh(X; Sp)$, $f : X \to Y$, $f^* \dashv f_* : Sh(X; Sp) \rightleftarrows Sh(Y; Sp)$
$f_! \dashv f^!$
$$(f_! F)(U) = \operatorname*{colim}_{\substack{K \subseteq f^{-1} U \\ \text{proper}/Y}} \operatorname{fib}(F(f^{-1} U) \to F(f^{-1} U \setminus K))$$

Thm (Verdier duality): · $f_!$ commutes w/ all colimits $\Rightarrow$ it has a right adjoint $f^!$
· When f is smooth (locally $(0,1)^n \times U \to U$) $f^! \underline{\mathbb{S}}$ locally const. eq. $\Sigma^{-n} \underline{\mathbb{S}}$
In fact if $f : M \to *$, $f^! M$ locally const. sheaf corr. to the local system $x \mapsto \mathbb{D} M/M \setminus x$.
M compact manifold
$$\mathbb{D} M_+ = \Gamma(M; \underline{\mathbb{S}}) \simeq f_* \underline{\mathbb{S}} \simeq f_! \underline{\mathbb{S}} \simeq f_! f^! \underline{\mathbb{S}}$$