§ Thom spectra & orientations

Def: Let X be a space, $V \xrightarrow{p} X$ a vector bundle. The Thom space of V, $Th(V)$ is the pointed space given by the cofiber of the map
$$V \setminus 0 \to V$$
(0 = 0 section)

$$\begin{array}{ccc} V\setminus 0 & \to & V \\ \downarrow & & \downarrow \\ * & \to & Th(V) \end{array}$$

Lemma: (X compact) $Th(V)$ is equivalent to the 1-pt compactification of V (based at the pt at ∞)

Proof: Let's choose a metric on V $|\cdot| : V \to \mathbb{R}_{\geq 0}$ induced by a scalar product on each fibre.

$$\begin{array}{ccc} V & \to & \xi_n \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{f} & Gr_n \end{array}$$
(ξ_n has a metric) $\xi_n = \{(v,x) \in \mathbb{R}^\infty \times X \mid v \in f x\}$

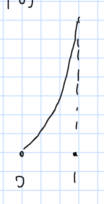
$v \mapsto \frac{v}{|v|}$

$$V\setminus 0 \xrightarrow{\simeq} S(V) = \{v \in V \mid |v| = 1\}$$
$$V \xleftarrow[0]{\simeq} X$$

$$cof(V\setminus 0 \to V) \simeq cof(S(V) \to X) \simeq \{*\} \cup_{S(V)\times\{0\}} S(V)\times[0,1] \cup_{S(V)\times\{1\}} X$$
(pt'd here; $\cong$ 1-pt comp of V)

$$S(V)\times(0,1] \to V$$
$$(v,t) \longmapsto \frac{1-t}{t} v$$

$\frac{1-t}{t}$:



$\Rightarrow$ gives a homeom. $S(V)\times(0,1] \cup_{S(V)\times\{1\}} X \simeq V$
$$\left(\frac{v}{|v|}, w\right) \longleftarrow v$$

& so these spaces have the same 1-pt compact. □

Lemma: Let $p: V \to X$ be a vector bundle classified by some $f: X \to \coprod_{n\geq 0} BO_n$. Then $Th(V)$ is the colimit of the composite
$$X \to \coprod_{n\geq 0} BO_n \to Space_*$$
$$x \longmapsto S^{V_x}$$
O_n = n-dim orth group $= \{A \in M_n\mathbb{R} \mid A^TA = AA^T = I\}$

where $BO_n \to Space_*$ is given by the O_n-action on $S^n \simeq$ 1-pt comp. of $\mathbb{R}^n$

Notation: V real v. space, S^V = 1-pt compactification of V

$Space_* = N^\Delta(Kan_*)$ $Sing\, O_n \to Map_*(S^n, S^n)$

$BO_n \simeq N^\Delta$(Kan-cx. cat. w/ one object & $Sing\, O_n$ as end.)

Proof: Let's consider $V\setminus 0 \to X$ this a fiber bundle, in particular a fibration.
$(V\setminus 0)\times_X \{x\} \simeq V_x \setminus 0$. Now notice that $X = \mathrm{colim}_{x\in X} \{x\}$

Therefore by descent $\mathrm{colim}_{x\in X} \underbrace{(V\setminus 0)\times_X \{x\}}_{V_x\setminus 0} \simeq (V\setminus 0)\times_X \mathrm{colim}_{x\in X}\{x\} \simeq V\setminus 0$

So we want to compute
$$cof\left(\mathrm{colim}_{x\in X} V_x\setminus 0 \to \mathrm{colim}_x V_x\right) = ?$$

cof: $Fun([1], Space) \to Space_*$ is a left adjoint of the functor $X \in Space_*$ to the arrow $* \to X$

$$\begin{array}{ccc} Y & \xrightarrow{g} & Z \\ \downarrow & & \downarrow \\ * & \to & X \end{array} \iff cof\, g \to X \text{ map of pt spaces}$$

$$Th(V) = \mathrm{colim}_{x\in X}\left(\overbrace{cof(V_x\setminus 0 \to V_x)}^{S^{V_x}}\right) \simeq \mathrm{colim}_{x\in X} S^{V_x}. \quad □$$

Def: A rank n spherical fibration is a functor $X \xrightarrow{S} Space_*$ s.t. $S(x) \simeq S^n$ $\forall x$.
$Th(S) := \mathrm{colim}_{x\in X} S_x$ (Thom space)

Ex: M topological manifold of dimension n, then the tangent spherical fibration is the functor
$$M \to Space_* \qquad x \longmapsto cof(M\setminus\{x\} \to M)$$
$$M\setminus\{x\} := (M\times M \setminus \Delta)\times_M \{x\} \qquad \{x\} \to M \qquad \begin{array}{c} M\times M \\ \downarrow\uparrow\Delta \\ M \end{array}$$

Recall: U, V open subsets
$$\begin{array}{ccc} U\cap V & \to & U \\ \downarrow & \lrcorner & \downarrow \\ V & \to & U\cup V \end{array}$$
homotopy pushout

$U \subseteq M$ nbhd of x homeo to $\mathbb{R}^n$ $\Rightarrow$
$$\begin{array}{ccccc} U\setminus x & \to & U & \to & U/U\setminus x \simeq \mathbb{R}^n/\mathbb{R}^n\setminus 0 \simeq S^n \\ \downarrow & \lrcorner & \downarrow & & \wr\wr \\ M\setminus x & \to & M & \to & M/M\setminus x \end{array}$$

Exercise: If M is smooth, this is equivalent to $x \mapsto S^{T_xM}$.

Def: If $S: X \to Space_*$ is a spherical fibration, the Thom spectrum of S is the suspension spectrum of $Th(S)$
$$X^S := \Sigma^\infty Th(S) \simeq \mathrm{colim}_{x\in X} \Sigma^\infty S(x)$$

Def: A stable spherical fibration is a functor $X \xrightarrow{S} Sp$ $S(x) \simeq \Sigma^{n_x}\mathbb{S}$ $\forall x$.
The Thom spectrum of a stable spherical fibration is just its colimit. (X^S)

Ex: $BO_n \to Sp$ = composite $BO_n \to Space_* \xrightarrow{\Sigma^{-n}\Sigma^\infty} Sp$
$* \longmapsto \mathbb{S}$

$$\begin{array}{ccc} BO_n & \to & Sp \\ \downarrow & \nearrow & \\ BO_{n+1} & & \end{array}$$
$\Rightarrow$ we have a map $\boxed{BO \xrightarrow{S} Sp}$
i.e. if I have a virtual vector bundle $V: X \to BO$ I can form a stable spherical fibration $S^V: X \to Sp$.

Def: $MO := BO^S$, MU = Thom spectrum of the composite $BU \to BO \to Sp$.

Remark: $S: X \to Space_*$, $S': X' \to Space_*$ spherical fibrations
$S \wedge S' : X\times X' \to Space_*$ is also a spherical fibration
$(x,x') \longmapsto S_x \wedge S_{x'}$

$$Th(S\wedge S') \simeq Th S \wedge Th S' \qquad \mathrm{colim}_{x,x'\in X\times X'} S_x \wedge S_{x'} \simeq \mathrm{colim}_{x\in X} S_x \wedge \mathrm{colim}_{x'\in X'} S_{x'}$$

In particular V vector bundle
$$Th(V\oplus 1) \simeq \Sigma Th(V) \qquad (Th(\mathbb{R}\to *) \simeq S^1)$$

$BO \xrightarrow{S} Sp$ is the colimit of functors $BO_n \xrightarrow{S_n} Sp$
$$MO = \mathrm{colim}_n BO_n^{S_n}$$
But S_n is the n-th desuspension of the stable spherical fibration associated to the tautological vector bundle $\xi_n \to BO_n$
$$BO_n^{S_n} \simeq \Sigma^{-n}\Sigma^\infty Th(\xi_n) \to BO_{n+1}^{S_{n+1}}$$
induced $\Sigma Th(\xi_n) \simeq Th(\xi_n \oplus 1) \to Th(\xi_{n+1})$ coming
$$\begin{array}{ccc} \xi_n\oplus 1 & \to & \xi_{n+1} \\ \downarrow & \lrcorner & \downarrow \\ BO_n & \to & BO_{n+1} \end{array}$$
So MO is the spectrum associated to the prespectrum $\{Th(\xi_n)\}$.
$$\mathrm{colim}_n \Sigma^{-n}\Sigma^\infty Th(\xi_n) \simeq MO$$

Let E be a homotopy ring spectrum (alg. object in hSp: $\mathbb{S} \xrightarrow{\eta} E$, $E\otimes E \xrightarrow{\mu} E$, ...)

Def: An (E-)orientation or Thom class on a stable sph. fib $S: X \to Sp$ of rank n is a map
$$\theta: X^S \to \Sigma^n E$$
s.t. $\forall x \in X$ the map
$$\Sigma^n\mathbb{S} \simeq S(x) \to X^S \xrightarrow{\theta} \Sigma^n E$$
corresponds to a unit of $\pi_0 E$ (or, more canonically, to a gen of $[S(x), \Sigma^n E]$ as a $\pi_0 E$-module)

In particular, the map $E\otimes S(x) \to E\otimes\Sigma^n E \xrightarrow{\mu} \Sigma^n E$ is an equivalence.

Remark: Giving a map $X^S = \mathrm{colim}_{x\in X} S(x) \to \Sigma^n E$ is eq. to giving a natural transformation
$S(x) \to \Sigma^n E$ ← constant functor

Example: If $E = H\mathbb{F}_2$ every stable spherical fibration is orientable.
[$Map_{Sp}(H\mathbb{F}_2, H\mathbb{F}_2) \simeq Hom_{Ab}(\mathbb{F}_2,\mathbb{F}_2) \simeq \mathbb{F}_2$ by the recognition principle.]

$$Map_{Sp}(\mathbb{S}, H\mathbb{F}_2) \simeq \Omega^\infty H\mathbb{F}_2 \simeq \mathbb{F}_2$$
(discrete w/ 2 objects) $Map(\Sigma^n\mathbb{S}, \Sigma^n H\mathbb{F}_2) \simeq Map(\mathbb{S}, H\mathbb{F}_2) \simeq \{0,1\}$

$$Map(X^S, \Sigma^n H\mathbb{F}_2) \simeq \lim_{x\in X} Map(S(x), \Sigma^n H\mathbb{F}_2)$$
$$\cup \qquad\qquad \cup$$
$$\text{"Thom classes"} \simeq \lim_{x\in X} \{\text{"id"}\} \simeq *$$

Ex: $E = H\mathbb{Z}$. Let V be a vector bundle of rank n. An E-orientation is the same thing as a nullhomotopy of the composite
$$X \to BO_n \to BC_2 \quad (\text{induced by } \det: O_n \to C_2)$$
i.e. the same thing as a lift $X \to BO_n \to BC_2$ (through BSO_n)

$$Map(X^V, \Sigma^n H\mathbb{Z}) \simeq \lim_{x\in X} Map(\Sigma^\infty S^{V_x}, \Sigma^n H\mathbb{Z})$$
H($\mathbb{Z}$-module of rank 1) $\tilde{H}^n(S^{V_x})$; O_n acts on $\tilde{H}^n(S^{V_x})$ via the determinant
$$\cup \qquad\qquad \cup$$
$$\text{"Thom classes"} \simeq \lim_{x\in X} \underbrace{\{\text{generators of } \tilde{H}^n(S^{V_x})\}}_{\text{functor } X \to \{\text{free } \mathbb{Z}\text{-modules of rank 1 w/ a choice of generator}\}}$$

gen of $\tilde{H}^n(S^{V_x})$ ⟷ iso $\mathbb{Z} \xrightarrow{\simeq} \tilde{H}^n(S^{V_x})$

$$\simeq \lim_{x\in X} Iso(\mathbb{Z}, \tilde{H}^n(S^{V_x})) \simeq Nat^{iso}(\underline{\mathbb{Z}}, x \mapsto \tilde{H}^n(S^{V_x}))$$
($Fun(X$, free $\mathbb{Z}$-mod. of rk 1$)$)

{free $\mathbb{Z}$-mod. of rank 1} $\simeq BC_2$ a nullhom. of the functor $Fun(X, BC_2) \simeq Map(X, BC_2)$
$x \mapsto \tilde{H}^n(S^{V_x})$.

But this is given by the composite $X \to BO_n \to BC_2$.

$$Map(S^{V_x}, \Sigma^n H\mathbb{Z}) \simeq \tilde{H}^n(S^{V_x}) \quad \text{the space of generators}$$
is the same thing as $Hom^{\simeq}(\mathbb{Z}, \tilde{H}^n(S^{V_x}))$

$$Fun(X, \{\text{free } \mathbb{Z}\text{-mod. of rank 1}\}^{\simeq}) \simeq Fun(X, BC_2) \simeq Map(X, BC_2)$$
$x \longmapsto \tilde{H}^n(S^{V_x})$ ⟷ the composite $X \to BO_n \to BC_2$
$x \longmapsto \mathbb{Z}$ $\qquad X \to * \to BC_2$

Ex: R discrete ring, V is HR-orientable iff the map
$$X \to BO_n \to BGL_1 R$$
is nullhomotopic.