IHT Lecture 18 2021-01-18

Localizations at K-theory

Goal: Compute $L_{KU}X$.

- $L_{KU}X \simeq L_{KU}\mathbb{S} \otimes X$, give an "explicit" description of $L_{KU}\mathbb{S}$.

Steps: Describing $(L_{KU}X)_{\mathbb{Q}}$, $(L_{KU}X)^\wedge_p$.

Lemma: Let E be a spectrum s.t. $E_{\mathbb{Q}} \neq 0$, then $\forall X \in Sp$

$$(L_E X)_{\mathbb{Q}} \simeq X_{\mathbb{Q}}.$$

i.e. the map $X \to L_E X$ is a rational equivalence.

Proof: Let $A := \mathrm{fib}(X \to L_E X)$, this is E-acyclic, we need to prove it is $\mathbb{Q}$-acyclic.

$$E \otimes A = 0 \Rightarrow (E\otimes A)_{\mathbb{Q}} = 0 \Leftrightarrow E_{\mathbb{Q}} \otimes A = 0$$

and $E_{\mathbb{Q}} \simeq \bigoplus \Sigma^{n_i} H\mathbb{Q} \neq 0$, so $E_{\mathbb{Q}} \otimes A = 0 \Leftrightarrow H\mathbb{Q} \otimes A = 0$. □

Lemma: Let $E, X \in Sp$. Then $X^\wedge_p$ is E-local $\Leftrightarrow$ X/p is E-local.

Proof: Since $X/p \simeq X^\wedge_p/p$ $\Rightarrow$ is clear.

($\Leftarrow$) Remember, we had a fiber sequence

$$\mathbb{S}/p \to \mathbb{S}/p^n \to \mathbb{S}/p^{n-1}$$

$\Rightarrow X/p \to X/p^n \to X/p^{n-1}$ $\Rightarrow$ if X/p is E-local, X/p^n is E-local $\forall n$.

$\Rightarrow X^\wedge_p = \lim_n X/p^n$ is E-local. □

Lemma: Let E be a spectrum, $X \in Sp$. Then $(L_E X)^\wedge_p \simeq L_{E/p} X$.

Proof: The map $X \to (L_E X)^\wedge_p$ is an E/p-eq. because it factors as

$$X \xrightarrow{} L_E X \xrightarrow{} (L_E X)^\wedge_p$$

(first map: E-eq. $\Rightarrow$ E/p-eq.; second map: $\mathbb{S}/p$-eq.)

Note that by the previous lemma $(L_E X)^\wedge_p$ is E-local because $(L_E X)^\wedge_p/p \simeq (L_E X)/p$.

Now let A E/p-acyclic $\Rightarrow E \otimes A/p = 0 \Rightarrow A/p$ is E-acyclic

$\Rightarrow \mathrm{map}(A/p, (L_E X)^\wedge_p) = 0$ (with $(L_E X)^\wedge_p$ E-local)

$\Rightarrow p : \mathrm{map}(A, (L_E X)^\wedge_p) \xrightarrow{\sim} \mathrm{map}(A, (L_E X)^\wedge_p)$ is an eq.

But $\mathrm{map}(A, (L_E X)^\wedge_p)$ is p-complete

$$\lim_n \mathrm{map}(A, L_E X/p^n) = \lim_n \mathrm{map}(A, L_E X)/p^n.$$

$\Rightarrow \mathrm{map}(A, (L_E X)^\wedge_p) = 0$. □

Summing up: we have a pullback square

$$\begin{array}{ccc} L_{KU} X & \longrightarrow & \prod_p L_{KU/p} X \\ \downarrow & & \downarrow \\ X_{\mathbb{Q}} & \longrightarrow & \left(\prod_p L_{KU/p} X\right)_{\mathbb{Q}} \end{array}$$

Our goal now is to understand $L_{KU/p}X$.

FACT 1: $\exists$ a map of ring spectra $\psi^{\ell} : KU^\wedge_p \to KU^\wedge_p$ $\forall \ell$ coprime w/ p.

(this gives an action of $\mathbb{Z}_p^\times$ on $KU^\wedge_p$)

FACT 2: $\exists\, v_1 : \Sigma^{2(p-1)} \mathbb{S}/p \to \mathbb{S}/p$ ($v_1^4 : \Sigma^8 \mathbb{S}/2 \to \mathbb{S}/2$) which is a KU-equivalence.

v_1 is constructed so that it fits in a diagram

$$\begin{array}{ccc} \Sigma^{2(p-1)} \mathbb{S}/p & \xrightarrow{v_1} & \mathbb{S}/p \\ \uparrow & & \downarrow \\ \Sigma^{2(p-1)} \mathbb{S} & \xrightarrow{\alpha_p} & \Sigma \mathbb{S} \end{array}$$

$\alpha_p \in \pi_{2p-3}\mathbb{S}$ is the element in smallest degree of order p

(Adams' On the groups $J(X)$ IV)

$\psi^\ell : KU^\wedge_p \to KU^\wedge_p$ is a map of rings, $\exists$ a nullhomotopy of the composition

$$\mathbb{S}^\wedge_p \to KU^\wedge_p \xrightarrow{\psi^\ell - 1} KU^\wedge_p \qquad \begin{array}{ccc} \mathbb{S} & \to & KU^\wedge_p \\ & \searrow & \downarrow \psi_\ell \\ & & KU^\wedge_p \end{array}$$

(unit inclusion)

$\Rightarrow$ a nullhomotopy $\mathbb{S}/p \to KU/p \xrightarrow{\psi^\ell - 1} KU/p$

$$\begin{array}{ccccc} \Sigma^{2(p-1)} \mathbb{S}/p & \to & \Sigma^{2(p-1)} KU/p & \to & \Sigma^{2(p-1)} KU/p \\ v_1 \downarrow & & \wr\downarrow v_1 & & \wr\downarrow v_1 \\ \mathbb{S}/p & \to & KU/p & \to & KU/p \end{array}$$

$\Rightarrow$ define $\mathbb{S}/p[v_1^{-1}] := \mathrm{colim}\left(\mathbb{S}/p \xrightarrow{v_1} \Sigma^{-2(p-1)}\mathbb{S}/p \xrightarrow{v_1} \Sigma^{-4(p-1)}\mathbb{S}/p \xrightarrow{v_1} \Sigma^{-6(p-1)}\mathbb{S}/p \to \dots\right)$

we get a sequence

$$\mathbb{S}/p[v_1^{-1}] \to KU/p \xrightarrow{\psi^\ell - 1} KU/p \qquad \left[p=2:\ \mathbb{S}/2[v_1^{-4}] \to KO/2 \to KO/2\right]$$

FACT 3: This is a fiber sequence (Adams-Baird, Miller, Bousfield) for ℓ a top. gen. of $\mathbb{Z}_p^\times$.

Remark: E homotopy ring spectrum (i.e. monoid in hSp), M is a module over E in hSp ($E\otimes M \xrightarrow{\mu} M$, $M \to E \otimes M \to M$ is id), then M is E-local.

Proof: A E-acyclic: $\mathrm{map}(A,M)$ is a retract of $\mathrm{map}(E\otimes A, M) = 0$

$$\mathrm{map}(A,M) \leftarrow \mathrm{map}(E\otimes A, M)$$

($A \to E\otimes A$)

$$\mathrm{map}(A,M) \to \mathrm{map}(E\otimes A, E\otimes M) \xrightarrow{\mathrm{map}(E\otimes A, \mu)} \mathrm{map}(E\otimes A, M) \quad □$$

$\Rightarrow KU/p$ is KU-local & so from FACT 3 we get $\mathbb{S}/p[v_1^{-1}]$ is KU-local

$\Rightarrow$ $\boxed{\mathbb{S}/p[v_1^{-1}] \simeq L_{KU}\mathbb{S}/p}$. More generally

$\boxed{X/p[v_1^{-1}] \simeq L_{KU} X/p}$ because the lhs is the fib of a map $KU/p \otimes X \to KU/p \otimes X$ & so KU-local.

Corollary (Telescope conjecture in height 1): Let X be a spectrum. Then X is KU-local if and only if $\forall p$ prime the map $v_1 : \Sigma^{2(p-1)} X/p \to X/p$ ($v_1^4 : \Sigma^8 X/2 \to X/2$) is an equivalence.

In particular, note that if X is KU-local $\pi_*(X/p)$ are periodic with period $2(p-1)$.

In fact $\pi_*(L_{KU}X/p) \simeq \pi_*(X/p)[v_1^{-1}]$.

So L_{KU} sees the v_1-periodic part of mod p homotopy groups.

Proof: v_1 is a KU-equivalence, so if X is KU-local

$$v_1 : \Sigma^{2(p-1)} X/p \xrightarrow{\sim} X/p \quad \forall p.$$

The problem is ($\Leftarrow$). But we know $L_{KU} X/p \simeq X/p[v_1^{-1}]$, so if v_1 is an eq.

X/p is KU-local $\Rightarrow X^\wedge_p$ is KU-local, $X_{\mathbb{Q}}$ is KU-local, $\left(\prod_p X^\wedge_p\right)_{\mathbb{Q}}$ is KU-local

$\Rightarrow X$ is KU-local. □

Corollary: $L_{KU}X \simeq L_{KU}\mathbb{S} \otimes X$ (i.e. L_{KU} is smashing)

Proof: Note that $X \to L_E\mathbb{S} \otimes X$ is always an E-eq.

So it suffices to prove $L_{KU}\mathbb{S} \otimes X$ is KU-local. We need to check

$$v_1 : \Sigma^{2(p-1)} (L_{KU}\mathbb{S} \otimes X)/p \to (L_{KU}\mathbb{S} \otimes X)/p \quad \text{is an eq.}$$

$$\Sigma^{2(p-1)} \underbrace{L_{KU}\mathbb{S}/p}_{\mathbb{S}/p[v_1^{-1}]} \otimes X \to \underbrace{L_{KU}\mathbb{S}/p}_{\mathbb{S}/p[v_1^{-1}]} \otimes X$$

& this is an eq. by definition. □

To understand $L_{KU}\mathbb{S}$ it's enough to understand $(L_{KU}\mathbb{S})^\wedge_p = L_{KU/p}\mathbb{S}$.

Recall we have a sequence

$$\mathbb{S} \to KU^\wedge_p \xrightarrow{\psi^\ell - 1} KU^\wedge_p$$

$\Rightarrow$ we get a seq. $(L_{KU}\mathbb{S})^\wedge_p \to KU^\wedge_p \to KU^\wedge_p$ & we claim this is a fiber sequence. Indeed everything is p-complete, so it's enough to check it's a fiber sequence mod p.

$$L_{KU}\mathbb{S}/p \to KU/p \to KU/p$$

but this was the fiber sequence constructed using FACT 3.

$$J_p := \text{fiber of } KU^\wedge_p \xrightarrow{\psi^\ell - 1} KU^\wedge_p \quad \left(J_p \simeq (KU^\wedge_p)^{h\mathbb{Z}_p^\times}\right)$$

$$\begin{array}{ccc} L_{KU}\mathbb{S} & \longrightarrow & \prod_p J_p \\ \downarrow & & \downarrow \\ H\mathbb{Q} & \longrightarrow & \left(\prod_p J_p\right)^\wedge_p \end{array}$$

Remark: $\pi_*\mathbb{S} \to \pi_* L_{KU}\mathbb{S}$ is the projection onto the image of J for $* \geq 1$.

($J : \pi_* O \to \pi_*\mathbb{S}$)

IDEA: Can we look at Bousfield localization at $\mathbb{S}/(p,v_1) = \mathrm{cof}\left(v_1 : \Sigma^{2(p-1)}\mathbb{S}/p \to \mathbb{S}/p\right)$

There exists a spectrum called MU that is the universal spectrum w/ a theory of Chern classes.

$$\pi_* MU = \mathbb{Z}[a_1, a_2, a_3, \dots] \qquad |a_i| = 2i$$

Def: R ring, a formal group law is a power series $F \in R[[x_1, x_2]]$ s.t.

- $F(x,y) = F(y,x)$
- $F(x,0) = x$, $F(0,x) = x$
- $F(x, F(y,z)) = F(F(x,y), z)$

Def: The Lazard ring L is the universal ring w/ a fgl

$$L = \mathbb{Z}[c_{ij}]/\text{relations so that } F(x,y) = \sum c_{ij} x^i y^j \text{ is a fgl}$$

Thm (Quillen): $\pi_* MU \simeq L$

$[p] \in L[[x]]$ $\qquad [p](x) := \underbrace{F(x, F(x, F(x, \dots)))}_{p \text{ factors}}$

$[p] = px + o(x)$ so we can use it to define some elements called v_n

$v_0 = p$, and in $L/(v_0, \dots, v_{n-1})$ $\quad [p] = v_n \cdot x^{p^n} + o(x^{p^n})$.

($v_n \in L/(v_0, \dots, v_{n-1})$) and v_n is a non-zero-div there.

$v_0 \in \pi_* MU$, $v_1 \in \pi_* MU/p$, $v_2 \in \pi_* MU/(p, v_1)$, ...

$|v_i| = 2(p^i - 1)$

$$\widetilde{E(n)} := MU/(v_{n+1}, v_{n+2}, \dots)[v_n^{-1}], \qquad \widetilde{K(n)} = \widetilde{E(n)}/(v_0, \dots, v_{n-1})$$

$$\widetilde{K(n)} = \bigoplus_{i=0}^{p^n - 1} \Sigma^{2i} K(n)$$

$$L_n := L_{\widetilde{E(n)}}, \qquad L_{K(n)} = L_{\widetilde{K(n)}}$$

Thm (chromatic fracture square): $\forall X \in Sp$ $\exists$ a pullback square

$$\begin{array}{ccc} L_n X & \longrightarrow & L_{K(n)} X \\ \downarrow & & \downarrow \\ L_{n-1} X & \longrightarrow & L_{n-1} L_{K(n)} X \end{array}$$

Thm (Chromatic convergence theorem):

$$X \in Sp^\omega \qquad X^\wedge_p \simeq \lim_n L_n X$$

$\forall n$ $\exists\, \mathbb{S}/(v_0^{i_0}, \dots, v_n^{i_n})$ finite spectrum w/ a map

$$v_{n+1}^{i_{n+1}} : \Sigma^? \mathbb{S}/(-) \longrightarrow \mathbb{S}/(-)$$

inducing $v_{n+1}^{i_{n+1}}$ on $MU_*(-)$ which is an L_{n+1}-eq. $\Rightarrow$ we have a map

$$\mathbb{S}/(v_0^{i_0}, \dots, v_n^{i_n})[v_{n+1}^{-1}] \to L_{K(n+1)} \mathbb{S}/(v_0^{i_0}, \dots, v_n^{i_n})$$

Conj (Telescope conjecture): This map is an equivalence

Corollary: X L_n-local $\Leftrightarrow$ v_{n+1} acts invertibly on X.

This subject is called "chromatic homotopy theory" because $L_n X$ sees the "v_n-periodic" part of $\pi_* X$

$\pi_*\left(L_n X/(v_0^{i_0}, \dots, v_{n-1}^{i_{n-1}})\right)$ are v_n-periodic

So you think of the chromatic tower

$$\to L_n X \to L_{n-1} X \to \dots \to L_0 X \;(= (X^\wedge_p)_{\mathbb{Q}})$$

as splitting $\pi_* X$ into "constituent frequencies"

Thm (Smashing localization thm): L_n is smashing.