ISHT Lecture 17 2021-01-14

§ p-completion

Let p be a prime number, we let

$\mathbb{S}/p = \mathrm{cof}(\mathbb{S} \xrightarrow{p} \mathbb{S})$, $X/p = X \otimes \mathbb{S}/p = \mathrm{cof}(X \xrightarrow{p} X)$

⚠ $\pi_* \mathbb{S}/p$ are not p-torsion (but they are p^2-torsion)

Def: A spectrum X is <u>p-complete</u> if it is $\mathbb{S}/p$-local. We write $X^\wedge_p$ for $L_{\mathbb{S}/p} X$.

Def: $\mathbb{S}/p^\infty = \mathrm{cof}(\mathbb{S} \to \mathbb{S}[\tfrac{1}{p}])$

$$\mathbb{S} = \mathrm{colim}\left(\begin{array}{ccccccc} \mathbb{S} & \xrightarrow{1} & \mathbb{S} & \xrightarrow{1} & \mathbb{S} & \xrightarrow{1} & \cdots \\ \downarrow 1 & & \downarrow p & & \downarrow p^2 & & \end{array}\right)$$

Recall $\mathbb{S}[\tfrac{1}{p}] = \mathrm{colim}(\mathbb{S} \xrightarrow{p} \mathbb{S} \xrightarrow{p} \mathbb{S} \xrightarrow{p} \cdots)$

$\Rightarrow \mathbb{S}/p^\infty = \mathrm{colim}(0 \to \mathbb{S}/p \xrightarrow{p} \mathbb{S}/p^2 \xrightarrow{p} \cdots)$

$$H\mathbb{Z}_* \mathbb{S}/p^\infty = \begin{cases} \mathrm{colim}(0 \to \mathbb{Z}/p \xrightarrow{p} \mathbb{Z}/p^2 \xrightarrow{p} \mathbb{Z}/p^3 \to \cdots) =: \mathbb{Z}/p^\infty \cong \mathbb{Z}[\tfrac{1}{p}]/\mathbb{Z} & * = 0 \\ 0 & \text{oth.} \end{cases}$$

$\mathbb{T}_p$ $\quad \mathbb{Q}_p/\mathbb{Z}_p \cong \mathbb{Z}[\tfrac{1}{p}]/\mathbb{Z}$

$\Sigma^{-1}\mathbb{S}/p^\infty \to \mathbb{S}$

Thm: $\forall X \in \mathrm{Sp}$ $\exists$ a nat. eq. $X^\wedge_p \simeq \mathrm{map}(\Sigma^{-1}\mathbb{S}/p^\infty, X) \leftarrow X$

Corollary: $X^\wedge_p \simeq \lim_n X/p^n$

Proof: $\mathrm{map}(\Sigma^{-1}\mathbb{S}/p^\infty, X) \simeq \mathrm{map}(\mathrm{colim}_n \Sigma^{-1}\mathbb{S}/p^n, X) \simeq \lim_n \underbrace{\mathrm{map}(\Sigma^{-1}\mathbb{S}/p^n, X)}_{X/p^n}$

$\Sigma^{-1}\mathbb{S}/p^n = \mathrm{fib}(\mathbb{S} \xrightarrow{p^n} \mathbb{S})$ [in Sp fiber $= \Sigma^{-1}$(cofiber) from stability]

$\mathrm{map}(\Sigma^{-1}\mathbb{S}/p^n, X) = \mathrm{cof}(\mathrm{map}(\mathbb{S}, X) \xrightarrow{p^n} \mathrm{map}(\mathbb{S}, X)) = X/p^n$

$\Sigma^{-1}\mathbb{S}/p^n \to \Sigma^{-1}\mathbb{S}/p^{n+1}$ induced on fibers by the square

$$\begin{array}{ccc} \mathbb{S} & \xrightarrow{1} & \mathbb{S} \\ \downarrow p^n & & \downarrow p^{n+1} \\ \mathbb{S} & \xrightarrow{p} & \mathbb{S} \end{array}$$

$\Rightarrow X/p^{n+1} \to X/p^n$ is the map induced on cofibers

$$\begin{array}{ccc} X & \xrightarrow{p} & X \\ \downarrow p^{n+1} & & \downarrow p^n \\ X & \xrightarrow{1} & X \end{array}$$

□

Rmk: We have a fiber sequence $X/p \xrightarrow{p^n} X/p^{n+1} \to X/p^n$.

Proof of the thm: $\Sigma^{-1}\mathbb{S}/p^\infty \to \mathbb{S} \to \mathbb{S}[1/p] \to \mathbb{S}/p^\infty$

$\Rightarrow X = \mathrm{map}(\mathbb{S}, X) \to \mathrm{map}(\Sigma^{-1}\mathbb{S}/p^\infty, X)$ is p-completion map.

- This map is a $\mathbb{S}/p$-eq. $\Leftrightarrow$ the cofiber $\mathrm{map}(\mathbb{S}[\tfrac{1}{p}], X)$ is $\mathbb{S}/p$-acyclic.

  Rmk: A is $\mathbb{S}/p$-acyclic iff $A/p \simeq 0$ iff $p: A \xrightarrow{\sim} A$ i.e. A is $\mathbb{S}[\tfrac{1}{p}]$-local.

  and $\mathrm{map}(\mathbb{S}[\tfrac{1}{p}], X)$ is $\mathbb{S}[1/p]$-local ✓

- $\mathrm{map}(\Sigma^{-1}\mathbb{S}/p^\infty, X)$ is $\mathbb{S}/p$-local

A $\mathbb{S}/p$-acyclic $\quad \mathrm{map}(A, \mathrm{map}(\Sigma^{-1}\mathbb{S}/p^\infty, X)) \overset{?}{\simeq} 0$

$\qquad\qquad\qquad\qquad \mathrm{map}(\Sigma^{-1}\mathbb{S}/p^\infty \otimes A, X)$

$\mathbb{S}/p^\infty \otimes A \overset{?}{\simeq} 0$

$= \mathrm{colim}_n(\mathbb{S}/p^n \otimes A)$

$\mathbb{S}/p^n \otimes A = A/p^n \overset{?}{\simeq} 0$ But A $\mathbb{S}/p$-acyclic $\Rightarrow p: A \to A$ eq.

$\Rightarrow p^n: A \simeq A$. □

Remark: Let X be a spectrum s.t. $X_\mathbb{Q} = 0$, $\underbrace{X/p = 0}_{p: X \simeq X \Rightarrow X \text{ rational}}$ $\forall p$ $\Rightarrow X = 0$.

In fact we will show X can be reconstructed from $X_\mathbb{Q}$, $X^\wedge_p$. $(\pi_* X_\mathbb{Q} = H\mathbb{Q}_* X)$

Prop: Let $X \in \mathrm{Sp}$. Then $\exists$ a canonical short exact sequence

$$0 \to \mathrm{Ext}(\mathbb{Z}/p^\infty, \pi_* X) \to \pi_*(X^\wedge_p) \to \mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_{*-1} X) \to 0$$

Def: A abelian group $\mathrm{Ext}(\mathbb{Z}/p^\infty, A)$ is called the <u>derived p-completion</u> of A.

Exercise: Show that the derived p-completion is the 0-th left derived functor of ordinary p-completion.

Proof: $\exists$ a resolution of $\mathbb{Z}/p^\infty$

$$\begin{array}{ccccccc} 0 \leftarrow \mathbb{Z}/p^\infty & \leftarrow & \bigoplus_{n \geq 1} \mathbb{Z} \cdot e_n & \leftarrow & \bigoplus_{n \geq 1} \mathbb{Z} \cdot e'_n & \leftarrow & 0 \\ \frac{1}{p^n} & \leftarrow\!\shortmid & e_n \quad p e_{n+1} - e_n & \leftarrow\!\shortmid & e'_n & & \end{array}$$

($\mathbb{Z}/p^\infty = \mathbb{Z}[\tfrac{1}{p}]/\mathbb{Z}$)

We want to construct a spectral version of that, that is a fiber seq.

$\bigoplus_{n \geq 1} \mathbb{S} \to \bigoplus_{n \geq 1} \mathbb{S} \to \mathbb{S}/p^\infty$

giving the previous SES in $H\mathbb{Z}_*(-)$. Assuming we have that, we can apply $\mathrm{map}(\Sigma^{-1}-, X)$

$$\underbrace{\mathrm{map}(\Sigma^{-1}\mathbb{S}/p^\infty, X)}_{X^\wedge_p} \to \prod_{n \geq 1} \Sigma X \to \prod_{n \geq 1} \Sigma X \qquad (\mathrm{map}(\bigoplus -, -) = \prod \mathrm{map}(-, -))$$

$\Sigma X \xrightarrow{(p, -1)} \Sigma X_n \times \Sigma X_{n+1}$

We can look at the les in π_{*+1}

$$\cdots \to \pi_{*+1} X^\wedge_p \to \prod_{n \geq 1} \pi_* X \xrightarrow{\circledast} \prod_{n \geq 1} \pi_* X \to \pi_* X^\wedge_p \to \cdots$$

So now the goal is to understand ker & coker of $\circledast$. But it is the map induced on Hom by

$\circledast \quad \mathrm{Hom}(\bigoplus_{n \geq 1} \mathbb{Z}, \pi_* X) \to \mathrm{Hom}(\bigoplus_{n \geq 1} \mathbb{Z}, \pi_* X)$

$\bigoplus_{n \geq 1} \mathbb{Z} \to \bigoplus_{n \geq 1} \mathbb{Z}$ in the resolution of $\mathbb{Z}/p^\infty$ & so its kernel is $\mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_* X)$ & its coker is $\mathrm{Ext}(\mathbb{Z}/p^\infty, \pi_* X)$ by definition of Ext.

To construct the spectral lift, note that the resolution of $\mathbb{Z}/p^\infty$ is obtained as a colim

$$\begin{array}{ccccccc} 0 & \to & \mathbb{Z} & \to & \mathbb{Z} \oplus \mathbb{Z} & \to & \cdots \\ \downarrow & \binom{1}{0} & \downarrow \binom{-1}{p} & & \downarrow \begin{pmatrix} -1 & 0 \\ p & -1 \\ 0 & p \end{pmatrix} & & \\ \mathbb{Z} & \to & \mathbb{Z} \oplus \mathbb{Z} & \hookrightarrow & \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & \to & \cdots \\ \downarrow & & \downarrow (p, 1) & & \downarrow (p^2, p, 1) & & \\ \mathbb{Z} & \xrightarrow{p} & \mathbb{Z} & \xrightarrow{p} & \mathbb{Z} & \xrightarrow{p} & \cdots \end{array}$$

$$\begin{array}{ccccc} 0 & \to & \mathrm{colim} & \simeq & \bigoplus_{n \geq 1} \mathbb{Z} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Z} & \to & \mathrm{colim} & \simeq & \bigoplus_{n \geq 0} \mathbb{Z} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Z} & \to & \mathrm{colim} & \simeq & \mathbb{Z}[1/p] \end{array} \qquad \text{take coker} \Rightarrow \qquad \begin{array}{c} \bigoplus_{n \geq 1} \mathbb{Z} \\ \downarrow \\ \bigoplus_{n \geq 1} \mathbb{Z} \\ \downarrow \\ \mathbb{Z}/p^\infty \end{array}$$

you can replace $\mathbb{Z}$ w/ $\mathbb{S}$ here & nothing changes.

□

$$0 \to \mathrm{Ext}(\mathbb{Z}/p^\infty, \pi_* X) \to \pi_*(X^\wedge_p) \to \mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_{*-1} X) \to 0$$

Recall: $A_0 \xleftarrow{p_0} A_1 \xleftarrow{p_1} A_2 \xleftarrow{p_2} \cdots$ tower of abelian groups, then we have a left exact se.

$$0 \to \lim_n A_n \to \prod_n A_n \xrightarrow{\circledast} \prod_n A_n$$

$$(a_n) \mapsto (a_n - p_n a_{n+1})$$

The cokernel of this map is called $\lim^1 A_n$ (related to the "completeness" of $\ker(\lim A_n \to A_n)$)

If all p_n's are surjective $\lim^1 A_n = 0$ (exercise)

Lemma: A abelian group. $\exists$ SES

$$0 \to \lim\nolimits^1 \underbrace{\mathrm{Hom}(\mathbb{Z}/p^n, A)}_{A[p^n] = \{a \in A \mid p^n a = 0\}} \to \mathrm{Ext}(\mathbb{Z}/p^\infty, A) \to \lim A/p^n \to 0$$

Therefore if A has bounded p-power torsion (i.e. $A[p^n] = A[p^{n+1}] = \cdots$ for some n), the derived p-completion coincides w/ the ordinary p-completion.

Example: A where derived p-completion $\neq$ ord. p-completion $A = \bigoplus_n \mathbb{Z}/p^n$.

Proof (sketch): $\exists$ a SES $\quad 0 \to \bigoplus_n \mathbb{Z}/p^n \to \bigoplus_n \mathbb{Z}/p^n \to \mathbb{Z}/p^\infty \to 0$

$$(a_n) \mapsto (a_n - p a_{n+1})$$

you look at the les for $\mathrm{Hom}(-, A)$ & use that $\mathrm{Ext}(\mathbb{Z}/p^n, A) \simeq A/p^n$. □

Def: A abelian group is <u>derived p-complete</u> if the map $A \to \mathrm{Ext}(\mathbb{Z}/p^\infty, A)$ is an isomorphism

Lemma: A derived p-complete $\Rightarrow \mathrm{Hom}(\mathbb{Z}/p^\infty, A) = 0$.

Corollary: E spectrum is p-complete $\Leftrightarrow$ $\pi_* E$ is derived p-complete.

Proof (Lemma): $0 \to \mathbb{Z} \to \mathbb{Z}[1/p] \to \mathbb{Z}/p^\infty \to 0$ look at the les in $\mathrm{Hom}(-, A)$

$0 \to \mathrm{Hom}(\mathbb{Z}/p^\infty, A) \to \mathrm{Hom}(\mathbb{Z}[\tfrac{1}{p}], A) \to A \xrightarrow{\sim} \mathrm{Ext}(\mathbb{Z}/p^\infty, A) \to \mathrm{Ext}(\mathbb{Z}[\tfrac{1}{p}], A) \to 0$

$\Rightarrow \mathrm{Hom}(\mathbb{Z}/p^\infty, A) \simeq \mathrm{Hom}(\mathbb{Z}[\tfrac{1}{p}], A)$. In particular p acts invertibly on $\mathrm{Hom}(\mathbb{Z}/p^\infty, A)$.

$\mathrm{Hom}(\mathbb{Z}/p^\infty, A) \simeq \lim(\mathrm{Hom}(\mathbb{Z}/p^\infty, A) \xleftarrow{p} \mathrm{Hom}(\mathbb{Z}/p^\infty, A) \xleftarrow{p} \cdots)$

$\overset{!}{\simeq} \mathrm{Hom}(\underbrace{\mathbb{Z}/p^\infty[p^{-1}]}_{=0}, A) = 0$. □

Thm (p-adic arithmetic fracture square): Let $X \in \mathrm{Sp}$. Then the following square is cartesian

$$\begin{array}{ccc} X & \to & X^\wedge_p \quad (= X_{\mathbb{Z}_p}) \\ \downarrow & \lrcorner & \downarrow \\ X[\tfrac{1}{p}] & \to & X^\wedge_p[\tfrac{1}{p}] \quad (= X_{\mathbb{Q}_p}) \end{array}$$

Proof: Let F be the fiber of the map $X \to X[\tfrac{1}{p}] \times_{X^\wedge_p[\frac{1}{p}]} X^\wedge_p$. Note $F[\tfrac{1}{p}] = 0$

since
$$\begin{array}{ccc} X[\tfrac{1}{p}] & \to & X^\wedge_p[\tfrac{1}{p}] \\ \downarrow \wr & \lrcorner & \downarrow \wr \\ X[\tfrac{1}{p}] & \to & X^\wedge_p[\tfrac{1}{p}] \end{array}$$
is cartesian. Note also that $F/p = 0$ since ($- \otimes \mathbb{S}/p$ is exact)

$$\begin{array}{ccc} X/p & \xrightarrow{\sim} & X^\wedge_p/p \\ \downarrow & & \downarrow \\ 0 = X[\tfrac{1}{p}]/p & \xrightarrow{\sim} & X^\wedge_p[\tfrac{1}{p}]/p = 0 \end{array}$$

$\Leftarrow X \to X^\wedge_p$ is $\mathbb{S}/p$-eq.

is cartesian $\Rightarrow F = 0$. □

Def: $X \in \mathrm{Sp}$ is <u>profinitely complete</u> if it is local for $\bigoplus_{p \text{ prime}} \mathbb{S}/p$

Prop: The map $X \to \prod_p X^\wedge_p$ is a profinite completion.

Proof: $\prod_p X^\wedge_p$ is prof. complete because $X^\wedge_p$ is profinite complete.

A $\bigoplus \mathbb{S}/p$-acyclic $\Rightarrow$ $\mathbb{S}/p$-acyclic $\Rightarrow \mathrm{map}(A, X^\wedge_p) \simeq 0$.

$X \to \prod_p X^\wedge_p$ is an $\mathbb{S}/q$-eq. $\forall q$, i.e.

$$X/q \xrightarrow{\sim} \prod_p X^\wedge_p/q \simeq \underbrace{X^\wedge_q/q}_{X/q} \times \prod_{p \neq q} X^\wedge_p/q$$

$X^\wedge_p/q \simeq (X/q)^\wedge_p$ $\quad p$ acts invertibly on X/q because $\pi_* X/q$ is q^2-torsion. □

Thm (Arithmetic fracture square): Let $X \in \mathrm{Sp}$, then the following square is cartesian

$$\begin{array}{ccc} X & \to & \prod_p X^\wedge_p \\ \downarrow & \lrcorner & \downarrow \\ X_\mathbb{Q} & \to & (\prod_p X^\wedge_p)_\mathbb{Q} \end{array}$$

"adelic part"

$\mathbb{A} = (\prod \mathbb{Z}_p) \otimes \mathbb{Q}$ "finite adeles"

Proof: $F \simeq$ fiber of $X \to X_\mathbb{Q} \times_{(\prod X^\wedge_p)_\mathbb{Q}} \prod_p X^\wedge_p$. We're going to show $F_\mathbb{Q} = 0$

$F/p = 0$ $\forall p$,

$\underline{\mathbb{Q}}$:
$$\begin{array}{ccc} X_\mathbb{Q} & \to & (\prod_p X^\wedge_p)_\mathbb{Q} \\ \downarrow \wr & \lrcorner & \downarrow \wr \\ X_\mathbb{Q} & \to & (\prod_p X^\wedge_p)_\mathbb{Q} \end{array}$$

F/p:
$$\begin{array}{ccc} X/p & \xrightarrow{\sim} & (\prod_p X^\wedge_p)/p \\ \downarrow & \lrcorner & \downarrow \\ 0 = X_\mathbb{Q}/p & \to & (\prod_p X^\wedge_p)_\mathbb{Q}/p = 0 \end{array}$$

. □

To understand $X^\wedge_p$ the best known strategy is to filter it by simpler pieces

$X^\wedge_p \to \lim(\to L_{E_n}(X^\wedge_p) \to \cdots \to L_{E_0}(X^\wedge_p) = (X^\wedge_p)_\mathbb{Q})$ (<u>Chromatic homotopy theory</u>)

Next time $L_{KU^\wedge_p} X = L_{E_1} X$.

<u>Ex.</u> $(H\mathbb{Z})^\wedge_p = H(\mathbb{Z}^\wedge_p)$, $MO^\wedge_2 = MO$

$$\pi_* KU^\wedge_p = \begin{cases} \mathbb{Z}_p & * \text{ even} \\ 0 & * \text{ odd} \end{cases}$$

, $KU^\wedge_p$ can be described in terms of "arithmetic data".

$\mathbb{S}^\wedge_p \simeq \lim(\to L_{E_n}\mathbb{S} \to \cdots)$ $\qquad L_{E_n}\mathbb{S} \hookrightarrow E_n^{h\mathbb{G}_n} \simeq L_{K(n)}\mathbb{S}$, $L_{E_{n-1}}\mathbb{S}$

$\pi_* \mathbb{S}^\wedge_p$

$P, Q \in \mathcal{D}(\mathbb{Z})$ $\quad f: P \to Q$

$$\begin{array}{ccc} P & \to & \prod_p P^\wedge_p \\ \downarrow & & \downarrow \\ P_\mathbb{Q} & \to & (\prod_p P^\wedge_p)_\mathbb{Q} \end{array}$$

derived in $\mathcal{D}(\mathbb{Z})$