E spectrum, $A \in Sp$ <u>E-acyclic</u> if $E \otimes A \simeq 0$
$X \in Sp$ <u>E-local</u> if $\forall A$ E-acyclic $\mathrm{map}(A, X) \simeq 0$

<u>Thm</u>: $\forall X \in Sp$ $\exists X \to L_E X$ E-equivalence s.t. $L_E X \in Sp_E$ (E-local spectra). In particular $\forall Y \in Sp_E$
$$\mathrm{Map}(L_E X, Y) \simeq \mathrm{Map}(X, Y)$$
i.e. $Sp_E \subseteq Sp$ has a left adjoint.

$L_E X = X_E$

<u>Thm</u>: $F: \mathcal{C} \to \mathcal{D}$ functor of ∞-cat. s.t. $\forall d \in \mathcal{D}$ $\mathrm{Map}(d, F-)$ is corepresentable $\Rightarrow F$ has a left adjoint sending d to the corepresenting element.

<u>Prop</u>: Let κ regular cardinal s.t. $\pi_* E < \kappa$, then every E-acyclic spectrum is a κ-filtered colimit of κ-small E-acyclic spectra.

<u>Corollary</u>: $\exists A$ E-acyclic spectrum s.t. X is E-local iff $\mathrm{map}(A, X) \simeq 0$.

<u>Proof</u>: Let $\{A_i\}_{i \in I}$ be the set of κ-small E-acyclic spectra, we let $A = \bigoplus_{i \in I} A_i$
$$\mathrm{map}(A, X) = 0 \iff \mathrm{map}(A_i, X) \simeq 0 \ \forall i.$$
If A' E-acyclic spectrum $\mathrm{map}(A', X) \simeq \lim_i \mathrm{map}(A_i, X)$
If $\mathrm{map}(A, X) = 0 \Rightarrow \mathrm{map}(A', X) = 0$. $\square$

<u>Proof (of the theorem)</u>: X, we want to construct $X \to L_E X$. Fix A as in the corollary.
Note $A \in Sp^{\kappa}$ for some regular cardinal κ.
$$X_0 = X, \quad X_1 = \mathrm{cofib}\Big(\bigoplus_{\substack{f: \Sigma^n A \to X \\ n \in \mathbb{Z}}} \Sigma^n A \to X\Big), \ \dots$$
$$\begin{cases} X_{\alpha+1} = \mathrm{cofib}\Big(\bigoplus_{\substack{f: \Sigma^n A \to X_\alpha \\ n \in \mathbb{Z}}} \Sigma^n A \to X\Big) & \alpha \text{ ordinal} \\ X_\lambda = \mathrm{colim}_{\alpha < \lambda} X_\alpha & \lambda \text{ limit ordinal} \end{cases}$$

<u>Claim</u>: ① $X \to X_\alpha$ E-eq. $\forall \alpha \leq \kappa$.
② X_κ E-local (so $X_\kappa = L_E X$)

① $\begin{cases} \cdot\ X_\alpha \to X_{\alpha+1} \text{ is an } E\text{-eq. by construction} \\ \cdot\ E\text{-eq. are stable under colimits, so } X \to X_\lambda = \mathrm{colim}_{\alpha<\lambda}(X \to X_\alpha) \text{ is an } E\text{-eq.} \end{cases}$

$X_\kappa = \mathrm{colim}_{\alpha < \kappa} X_\alpha$ $\quad \{\alpha \mid \alpha < \kappa\}$ is a κ-filtered poset.

$$\mathrm{map}(A, X_\kappa) \simeq \mathrm{colim}_{\alpha < \kappa} \mathrm{map}(A, X_\alpha)$$
$f \in \pi_n \mathrm{map}(A, X_\kappa) = [\Sigma^n A, X_\kappa] \Rightarrow$ this factors through X_α for some $\alpha < \kappa$.
but then its image in $X_{\alpha+1}$ is $0 \Rightarrow$ its image in $\pi_n \mathrm{map}(A, X_\kappa)$ is 0. $\square$

<u>Prop</u>: Let κ regular cardinal s.t. $\pi_* E < \kappa$, then every E-acyclic spectrum is a κ-filtered colimit of κ-small E-acyclic spectra.

Let's fix X E-acyclic spectrum

<u>Prop</u>: $\forall f \in \pi_n X$ $\exists W$ κ-small acyclic spectrum, $W \xrightarrow{g} X$ s.t. f lifts to $\pi_* W$.

<u>Remark</u>: F finite spectrum, $\# \underbrace{E_* F}_{\pi_*(E \otimes F)} < \kappa$. (such spectra are closed under $\oplus$'s & cofibers)

<u>Proof</u>: We'll construct W as the colimit
$$\Sigma^n \mathbb{S} = W_0 \to W_1 \to \dots \to X, \quad f = g_0$$
$\# E_* W_i < \kappa$

s.t. the map $E_* W_i \to E_* W_{i+1}$ is the 0 map
$\Rightarrow E_* W = \mathrm{colim}_i E_* W_i = 0$ $\quad \overline{W}_i = \mathrm{fib}(W_i \to X) \Rightarrow E_* \overline{W}_i \xrightarrow{\sim} E_* W_i$

We
$$\begin{tikzcd} W_i \arrow[r, "g_i"] \arrow[d] & X \\ W_{i+1} \arrow[ur, dashed] & \end{tikzcd}$$

<u>Note</u>: $\overline{W}_i = \mathrm{colim}_{j \in J} F_j$ $\quad F_j$ finite spectra, J filtered
$\Rightarrow E_* \overline{W}_i = \mathrm{colim}_{j \in J} E_* F_j$

$\Rightarrow \forall x \in E_* \overline{W}_i$ $\exists F_x$ finite w/ a map $F_x \to \overline{W}_i$ s.t. x is in the image of $E_* F_x \to E_* \overline{W}_i$.

$$\begin{tikzcd} \underbrace{\bigoplus_{x \in E_* \overline{W}_i} F_x}_{\kappa\text{-small}} \arrow[r] \arrow[rrr, bend right] & \overline{W}_i \arrow[r] & W_i \arrow[r] \arrow[d] & X \\ & & \mathrm{cof}\big(\bigoplus F_x \to W_i\big) = \underbrace{W_{i+1}}_{\kappa\text{-small}} \arrow[ur] & \end{tikzcd}$$
canonically nullhomotopic

$$\begin{tikzcd} E_*\Big(\bigoplus_{x \in E_* \overline{W}_i} F_x\Big) \arrow[r, two heads] & E_* \overline{W}_i \arrow[r, "\sim"] & E_* W_i \arrow[d, "0"] \\ & & E_* W_{i+1} \arrow[d] \\ & & E_{*-1}\Big(\bigoplus_{x \in E_* \overline{W}_i} F_x\Big) \end{tikzcd}$$
0 map by the les in E_*

& $\# E_* W_{i+1} < \kappa$ because it fits between two groups of card $< \kappa$. $\square$

<u>Proof (of main proposition)</u>: Let $\mathcal{J}$ be the ∞-cat. of κ-small E-acyclic spectra w/ a map to X.
$$\mathcal{J} = Sp^\kappa \times_{Sp} Sp_{/X}.$$
$$\mathrm{colim}_{W \in \mathcal{J}} W \to X$$
<u>Claim</u>: $\cdot$ $\mathcal{J}$ has all κ-small colimits $\Rightarrow \mathcal{J}$ κ-filtered
$\cdot$ This map is an equivalence.

As before, it's enough to prove that
$$\mathrm{colim}_{W \in \mathcal{J}} \pi_* W \xrightarrow{\sim} \pi_* X$$
is an equivalence. We know it's surjective, by the proposition we just proved.
Now let's take $[g: W \to X] \in \mathcal{J}$ $\quad f \in \pi_n W$ s.t. $gf = 0$.
$$\begin{tikzcd} & \Sigma^n \mathbb{S} \arrow[dl, "\bar{f}"'] \arrow[d, "f"] & \\ F \arrow[r] & W \arrow[r, "g"] & X \end{tikzcd} \qquad F = \mathrm{fib}(g)$$

Note that F is E-acyclic, since W, X are. So, by the proposition, we can find $\overline{W}$ κ-small E-acyclic, w/ a map $\bar{g}: \overline{W} \to F$ s.t. $\bar{f}$ lifts to $\bar{\bar{f}}$

$$\begin{tikzcd} & \Sigma^n \mathbb{S} \arrow[dl, "\bar{\bar{f}}"'] \arrow[d, "f"] & \\ \overline{W} \arrow[r, "h"] & W \arrow[r, "g"] \arrow[d, "h'"] & X \\ & W' \arrow[ur, "g'"'] & \end{tikzcd}$$
κ-small

$\overline{W}$ κ-small, gh has a canonical nullhomotopy $\Rightarrow g: W \to X$ factors through $W' = \mathrm{cof}(\overline{W} \to W)$

$$g' h' f = g' \underbrace{h' h}_{= 0} \bar{\bar{f}} = 0. \ \square$$

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§ Inverting primes

Let $p \in \mathbb{Z}$ be a prime number
$$\mathbb{S}[1/p] = \mathrm{colim}(\mathbb{S} \xrightarrow{p} \mathbb{S} \xrightarrow{p} \mathbb{S} \xrightarrow{p} \dots)$$
"$\mathbb{Z}[1/p]$ - Moore spectrum"
$$H_* \mathbb{S}[1/p] = \begin{cases} \mathbb{Z}[\frac{1}{p}] & * = 0 \\ 0 & \text{otherwise} \end{cases}$$

<u>Goal</u>: Study $L_{\mathbb{S}[1/p]}$.

<u>Remark</u>: $\pi_*(\mathbb{S}[1/p] \otimes X) = \mathrm{colim}(\pi_* X \xrightarrow{p} \pi_* X \xrightarrow{p} \dots) = \pi_* X[1/p]$.

<u>Lemma</u>: A spectrum X is <u>$\mathbb{S}[1/p]$-acyclic</u> iff p acts locally nilpotently on $\pi_* X$, i.e. every element of $\pi_* X$ is p-power torsion.

<u>Lemma</u>: A spectrum X is <u>$\mathbb{S}[1/p]$-local</u> iff p acts invertibly on X (eq. on $\pi_* X$)

<u>Proof</u>: ($\Rightarrow$) Remark $\mathbb{S}/p$ is $\mathbb{S}[1/p]$-acyclic.
$$\mathbb{S}/p \otimes \mathbb{S}[1/p] = \mathbb{S}[1/p]/p \simeq 0.$$
If X is $\mathbb{S}[1/p]$-local, want to say $\mathbb{S}/p \otimes X \simeq 0$, i.e.
$X \xrightarrow{p} X$ is an eq.
$$\begin{matrix} \mathrm{map}(\mathbb{S}, X) & \to & \mathrm{map}(\mathbb{S}, X) \\ & \mathrm{map}(p, X) & \end{matrix}$$
$\Rightarrow$ the fiber of this map is $\mathrm{map}(\underbrace{\mathbb{S}/p}_{\mathbb{S}[1/p]\text{-acyclic}}, X) = 0$ because X is $\mathbb{S}[1/p]$-local

($\Leftarrow$) Suppose p acts invertibly on X. Then the map
$$\mathrm{map}(\mathbb{S}[1/p], X) \to X \qquad (\text{precomp. w/ } \mathbb{S}[1/p] \leftarrow \mathbb{S})$$
is an eq.
$$\mathrm{map}(\mathbb{S}[1/p], X) = \lim\big(\underset{\underset{X}{\uparrow}}{\mathrm{map}(\mathbb{S}, X)} \xleftarrow[\sim]{p} \mathrm{map}(\mathbb{S}, X) \xleftarrow[\sim]{p} \dots\big) \xrightarrow{\sim} X.$$
Now let W E-acyclic
$$\mathrm{map}(W, X) \simeq \mathrm{map}(W, \mathrm{map}(\mathbb{S}[\tfrac{1}{p}], X)) \simeq \mathrm{map}(\underbrace{W \otimes \mathbb{S}[\tfrac{1}{p}]}_{\simeq 0,\ W\ \mathbb{S}[\frac1p]\text{-acyclic}}, X) \simeq 0. \ \square$$

<u>Theorem</u>: The map $X \to \mathbb{S}[\frac{1}{p}] \otimes X =: X[\frac{1}{p}]$ exhibits $X[\frac{1}{p}]$ as the $\mathbb{S}[\frac{1}{p}]$-localization of X.

<u>Proof</u>: $\mathbb{S}[\frac{1}{p}] \otimes X$ is $\mathbb{S}[\frac{1}{p}]$-local, by the criterion above, so it's enough to show the map is a $\mathbb{S}[\frac{1}{p}]$-equivalence. We'll look at this map as a colimit of $\mathbb{S}[\frac{1}{p}]$-eq.

$$\begin{tikzcd} X = \mathrm{colim}\big(X \arrow[d] \arrow[r, equal] & X \arrow[d, "p"] \arrow[r, equal] & X \arrow[d, "p^2"] \arrow[r, equal] & X \arrow[d, "p^3"] \arrow[r, equal] & \dots \big) \\ X[\frac{1}{p}] = \mathrm{colim}\big(X \arrow[r, "p"] & X \arrow[r, "p"] & X \arrow[r, "p"] & X \arrow[r, "p"] & \dots \big) \end{tikzcd}$$
$$X \to X[\tfrac{1}{p}] = \mathrm{colim}\big(\underbrace{X \xrightarrow{p^n} X}_{\mathbb{S}[\frac{1}{p}]\text{-eq.}}\big)$$
Their cofiber is $X \otimes \mathbb{S}/p^n$, so it's enough to show $\mathbb{S}[\frac{1}{p}] \otimes \mathbb{S}/p^n \simeq \mathbb{S}[\frac{1}{p}]/p^n \simeq 0$. $\square$

<u>Warning</u>: We have seen that every element of $\pi_* \mathbb{S}/p$ is p-power torsion.
It's not p-torsion in general (one can show it is p^2 torsion)
$$\pi_1 \mathbb{S}/2 \simeq \mathbb{Z}/4.$$

<u>Remark</u>: There's always a map $X \to (L_E \mathbb{S}) \otimes X$, which is always an E-equivalence
When the target is E-local $\forall X \Rightarrow (L_E \mathbb{S}) \otimes X \simeq L_E X$.
In this case the localization is called <u>smashing</u>.

Let $S = \{p_1, p_2, \dots\}$ a set of prime numbers
$$\mathbb{S}[S^{-1}] = \mathrm{colim}(\mathbb{S} \xrightarrow{p_1} \mathbb{S} \xrightarrow{p_1 p_2} \mathbb{S} \xrightarrow{p_1 p_2 p_3} \mathbb{S} \to \dots)$$
As before $\pi_*(\mathbb{S}[S^{-1}] \otimes X) \simeq \pi_* X[S^{-1}]$ $\quad \forall X \in Sp$.

<u>Prop</u>: A spectrum X is <u>$\mathbb{S}[S^{-1}]$-local</u> if all primes in S act invertibly on X.

<u>Prop</u>: The map $X \to \mathbb{S}[S^{-1}] \otimes X =: X[S^{-1}]$ exhibits the rhs as the $\mathbb{S}[S^{-1}]$-localiz. of X.

I can take S = set of all primes $X[S^{-1}] = X_{\mathbb{Q}}$ "rationalization" of X.
$$\pi_* \mathbb{S}_{\mathbb{Q}} = \pi_* \mathbb{S} \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & * = 0 \\ 0 & \text{everywhere else.} \end{cases}$$
$\Rightarrow \mathbb{S}_{\mathbb{Q}} = H\mathbb{Q}$.

<u>Prop</u>: Let $X \in Sp_{\mathbb{Q}}$. Then $\exists$ an eq. $X \simeq \bigoplus_{n \in \mathbb{Z}} \Sigma^n H \pi_n X$

<u>Proof</u>: The point is to construct maps $\Sigma^n H \pi_n X \to X$ iso on π_n.
$\pi_n X$ is a $\mathbb{Q}$-vector space, so it has a basis $\{e_i\}_{i \in I_n}$ & we have a map
$$\begin{tikzcd} \bigoplus_{i \in I_n} \Sigma^n \mathbb{S} \arrow[r, "(e_i)"] \arrow[d] & X \\ \Sigma^n H \pi_n X = \bigoplus_{i \in I_n} \Sigma^n H\mathbb{Q} \arrow[ur] & \end{tikzcd}$$
$\leftarrow$ rational!
$\leftarrow$ required map iso on π_n
$\square$

<u>Corollary</u>: $KU_{\mathbb{Q}} \simeq \bigoplus_{n \in \mathbb{Z}} \Sigma^{2n} H\mathbb{Q}$

$\Rightarrow$ We have a map $KU \xrightarrow{ch} \bigoplus_{n \in \mathbb{Z}} \Sigma^{2n} H\mathbb{Q} = KU_{\mathbb{Q}}$, which is called the <u>Chern character</u>

<u>Prop</u>: Let X be a finite space
$$KU^0(X)_{\mathbb{Q}} \simeq \bigoplus_{n \geq 0} H^{2n}(X; \mathbb{Q})$$

<u>Proof</u>: Since X is a finite space, $\mathrm{map}(\Sigma^\infty X_+, -)$ commutes w/ filtered colimits.
$$\begin{aligned} KU^0(X)_{\mathbb{Q}} &= \pi_0 \mathrm{map}(\Sigma^\infty X_+, KU)_{\mathbb{Q}} = \pi_0 \mathrm{map}(\Sigma^\infty_+ X, KU_{\mathbb{Q}}) \\ &\simeq \pi_0 \mathrm{map}(\Sigma^\infty X_+, \bigoplus_{n \in \mathbb{Z}} \Sigma^{2n} H\mathbb{Q}) = \bigoplus_{n \in \mathbb{Z}} \pi_0 \mathrm{map}(\Sigma^\infty X_+, \Sigma^{2n} H\mathbb{Q}) \\ &\simeq \bigoplus_{n \in \mathbb{Z}} H^{2n}(X; \mathbb{Q}) \ \square \end{aligned}$$

<u>Thm (Schwede-Shipley)</u>: The ∞-cat $Sp_{\mathbb{Q}}$ is equivalent to the derived ∞-cat. of $\mathbb{Q}$
$$\mathcal{D}(\mathbb{Q}) := Ch(\mathbb{Q})[q.iso^{-1}]$$