15HT Lecture 15 2021-01-07

Bousfield localisations

$Sp = \infty$-category of spectra

Intuition: $\mathbb{Z}[\frac{1}{p}]$-modules $\subseteq \mathbb{Z}$-modules, p-complete (derived) $\mathbb{Z}$-modules $\subseteq \mathbb{Z}$-modules

Recall: If E spectrum, $\forall X \in Sp$ $E_* X = E$-homology of $X = \pi_*(E \otimes X)$

(analogy w/ $H_*(Y;A) = \pi_*(HA \otimes \Sigma^\infty(Y_+))$)

Def: A map $f: X \to Y$ of spectra is an <u>E-equivalence</u> if it is an equivalence in E-homology or, eq., if $E \otimes f: E \otimes X \xrightarrow{\sim} E \otimes Y$.

A spectrum $X \in Sp$ is <u>E-acyclic</u> if $X \to 0$ is an E-eq. or, eq., $E \otimes X \simeq 0$.

Remark: Since $E \otimes -$ is exact, f is an E-equivalence iff cof f (or eq. fib f) is E-acyclic (from the les in homotopy groups for $E \otimes f$).

Def: A spectrum X is <u>E-local</u> if for every E-equivalence $f: Y \to Z$
$map(f, X): map(Z, X) \xrightarrow{\sim} map(Y, X)$
is an equivalence.

Exercise: X is E-local $\iff$ $Map(f, X)$ is an eq.

We will write the subcat. of E-local spectra as $Sp_E \subseteq Sp$.

Ex: Let $p: \mathbb{S} \to \mathbb{S}$ the "multiplication by p" map

$$\mathbb{S} \xrightarrow{\Delta} \mathbb{S}^{\oplus p} \xrightarrow{\nabla} \mathbb{S}$$

(on π_* this is literally mult by p), let $\mathbb{S}/p$ be its cofiber ("mod p Moore spectrum")

$$H_*(\mathbb{S}/p; \mathbb{Z}) = H\mathbb{Z}_* \mathbb{S}/p = \begin{cases} \mathbb{Z}/p & * = 0 \\ 0 & \text{otherwise} \end{cases} \qquad H\mathbb{Z}_* \mathbb{S} = \pi_* H\mathbb{Z} = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{oth} \end{cases}$$

A spectrum X is $\mathbb{S}/p$-acyclic if $p: \pi_* X \to \pi_* X$ is an iso:
from the les $X \xrightarrow{p} X \to X \otimes \mathbb{S}/p$ we get

$$\pi_{*+1}(X \otimes \mathbb{S}/p) \to \pi_* X \xrightarrow{p} \pi_* X \to \pi_*(X \otimes \mathbb{S}/p) \to \pi_* X \to \dots$$

0 iff p is iso

$X \otimes \mathbb{S}/p = 0 \iff p: X \to X$ is an eq. We will see that being $\mathbb{S}/p$-local is a "completeness" condition on $\pi_* X$, i.e. $X \xrightarrow{\sim} \lim_n X/p^n$.

Ex: $\mathbb{S}[p^{-1}] = \mathrm{colim}(\mathbb{S} \xrightarrow{p} \mathbb{S} \xrightarrow{p} \mathbb{S} \xrightarrow{p} \dots)$ $\quad \pi_* \mathbb{S}[p^{-1}] = (\pi_* \mathbb{S})[p^{-1}]$

& X is $\mathbb{S}[p^{-1}]$-acyclic iff $\pi_* X$ is locally p-power torsion.

$$\pi_* X = \bigcup_n (\pi_* X)[p^n]$$
p^n-torsion in $\pi_* X$

X is $\mathbb{S}[p^{-1}]$-local iff $p: X \xrightarrow{\sim} X$.

Thm: $E \in Sp$. Then the inclusion $Sp_E \subseteq Sp$ has a left adjoint $L_E: Sp \to Sp_E$ called <u>E-localisation</u> or <u>localisation at E</u>, i.e. $\forall X \in Sp$ $\exists L_E X \in Sp_E$ and a map $X \to L_E X$ s.t. $\forall Y \in Sp_E$

$$Map(L_E X, Y) \xrightarrow{\sim} Map(X, Y).$$

From now on κ is going to be a <u>regular cardinal</u>.

Def: A simplicial set I is <u>κ-small</u> if it has less than κ nondeg. simplices

Def: A simplicial set S is <u>κ-filtered</u> if $\forall I$ κ-small, $\forall f: I \to S$ map of sets $\exists I^{\triangleright} \to S$

Ex: When $\kappa = \omega$ $(= \aleph_0)$, κ-filtered = filtered.

Ex: If $S = NP$, S is κ-filtered $\iff$ $\forall$ subset of P of cardinality less than κ $\exists$ an upper bound.

Ex: If $\mathcal{C}$ ∞-cat w/ all κ-small colimits, $\mathcal{C}$ is κ-filtered.

Def: Let $Sp^\kappa \subseteq Sp$ generated by $\{\Sigma^n \mathbb{S}\}_{n \in \mathbb{Z}}$ under κ-small colimits.

Lemma: Sp^κ has only a set of eq. classes.

Proof: We're going to write $Sp^\kappa = \bigcup_{\alpha < \kappa} \mathcal{C}_\alpha$ s.t. $\mathcal{C}_\alpha$ is a (essentially) small category

$\mathcal{C}_0 = \{\Sigma^n \mathbb{S}\}_{n \in \mathbb{Z}}$. If we have $\mathcal{C}_\alpha$,

$\mathcal{C}_{\alpha+1} = \{$full subcat of Sp spanned by all colimits of functors $f: I \to \mathcal{C}_\alpha$, I κ-small$\}$

λ limit ordinal $\mathcal{C}_\lambda = \bigcup_{\alpha < \lambda} \mathcal{C}_\alpha$.

Goal: $Sp^\kappa = \mathcal{C}_\kappa$. $\supseteq$ clear, since if $\mathcal{C}_\alpha \subseteq Sp^\kappa \Rightarrow \mathcal{C}_{\alpha+1} \subseteq Sp^\kappa$ (Sp^κ is closed under κ-small colimits)

$\subseteq$ $\mathcal{C}_\kappa \supseteq \mathcal{C}_0 = \{\Sigma^n \mathbb{S}\}_{n \in \mathbb{Z}}$, so it's enough to show $\mathcal{C}_\kappa$ is closed under κ-small colimits. $f: I \to \mathcal{C}_\kappa$ **Claim:** f factors through $\mathcal{C}_\alpha$ for some α.

$\forall i \in I$ $\exists \alpha_i < \kappa$ $f_i \in \mathcal{C}_{\alpha_i}$. Moreover $\# I < \kappa \Rightarrow \alpha = \sup_{i \in I} \alpha_i < \kappa$. ($\kappa$ regular cardinal)

$\Rightarrow f$ factors through $\mathcal{C}_\alpha$. But then $\mathrm{colim} f \in \mathcal{C}_{\alpha+1} \subseteq \mathcal{C}_\kappa$. $\square$

Thm: If I κ-filtered, $\mathrm{colim}: Fun(I, Space) \to Space$ commutes w/ κ-small limits.

Lemma: If $X \in Sp^\kappa$, $Map_{Sp}(X, -): Sp \to Space$ commutes w/ κ-filtered colimits

Proof: It's true for $X = \Sigma^n \mathbb{S}$, so it's enough to show that the subcat of X s.t.

$$Map_{Sp}(X, -) \text{ commutes w/ } \kappa\text{-filtered colimits} \qquad (*)$$

is closed under κ-small colimits. But if $F: I \to Sp$ is a functor w/ I κ-small & F_i has the property $(*)$ $\forall i$

$$Map_{Sp}(\mathrm{colim}_I F_i, -) = \lim_I Map_{Sp}(F_i, -)$$

& the thesis follows from the above theorem. $\square$

Exercise: This is an iff, i.e. if $Map_{Sp}(X, -)$ commutes w/ κ-filtered colimits, then X is κ-small.

Proposition: Let κ be uncountable, $X \in Sp$, $\# \pi_* X < \kappa \iff X$ is κ-small.
In particular $\forall X \in Sp$ $\exists \kappa$ s.t. $X \in Sp^\kappa$.

Thm (Serre): $\pi_i S^n$ is finite if $i \neq n, 2n-1$ $\Rightarrow$ $\pi_i \mathbb{S}$ is finite for $i \neq 0$.

Proof: $(\Leftarrow)$ By the theorem by Serre, $\Sigma^n \mathbb{S} \in A$ $\forall n \in \mathbb{Z}$.

where $A = \{X \in Sp \mid \# \pi_* X < \kappa\}$

So it's enough to prove A is closed under κ-small colimits. We're going to prove it is closed under cofibers & κ-small coproducts.

$$\begin{matrix} A & \to & B \\ \downarrow & & \downarrow \\ C & \to & D \end{matrix} \qquad A \to B \oplus C \to D$$

· κ-small coproducts are clear $\pi_*\left(\bigoplus_{i \in I} X_i\right) \simeq \bigoplus_{i \in I} \pi_* X_i$ so if $\# I < \kappa$, $\# \pi_* X_i < \kappa$

$\Rightarrow \# \bigoplus_{i \in I} \pi_* X_i < \kappa$ ($\bigoplus_{i \in I} \pi_* X_i$ is the union of the subgroups of those elems supported on a given finite subset)

· $X, Y \in A$ $f: X \to Y$ $Z = $ cof f. By the les on π_*

$$\pi_* X \to \pi_* Y \to \pi_* Z \to \pi_{*-1} X \to \pi_{*-1} Y$$

$$0 \to \underbrace{\mathrm{coker}\, \pi_* f}_{\# < \kappa \text{ (quotient of } \pi_* Y)} \to \pi_* Z \to \underbrace{\ker \pi_{*-1} f}_{\# < \kappa \text{ (subgroup of } \pi_{*-1} X)} \to 0 \quad \Rightarrow \# \pi_* Z < \kappa. \checkmark$$

$(\Rightarrow)$ X s.t. $\# \pi_* X < \kappa$.

$$X_0 = 0 \to X_1 \to X_2 \to \dots \to X$$

s.t. $X_i \in Sp^\kappa$ $\forall i$, $\underbrace{\mathrm{colim}\, X_i = X}$ (κ-small colimit). $f_i: X_i \to X$

Let $F_i \simeq$ fib f_i

$$\begin{tikzcd} F_i \arrow[r] \arrow[d, "h_i"] & X_i \arrow[r, "f_i"] \arrow[d] & X \arrow[d, equal] \\ F_{i+1} \arrow[r] & X_{i+1} \arrow[r, "f_{i+1}"] & X \end{tikzcd}$$

We're going to construct X_i s.t. h_i is 0 on π_* $\Rightarrow$ $\mathrm{colim}\, F_i \overset{0}{=} \mathrm{fib}(\mathrm{colim}\, X_i \to X)$

We're constructing X_i by induction, $X_0 = 0$ ✓. Suppose we have X_i

$$\begin{tikzcd} \bigoplus_{\alpha \in \pi_* F_i} \Sigma^{|\alpha|} \mathbb{S} \arrow[d] \arrow[dr] & & \\ F_i \arrow[r] \arrow[d] & X_i \arrow[r] \arrow[d] & X \\ F_{i+1} \arrow[r] & X_{i+1} \arrow[ur] & \end{tikzcd}$$

κ-small; onto on π_*; is 0 on π_*; cof seq; κ-small; cof seq!; κ-small

$\forall n \in \mathbb{Z}$ $\forall \alpha \in \pi_n F_i$ $n = |\alpha|$

note $\# \pi_* F_i < \kappa$

$\# \pi_n F_i < \kappa$ $\forall n$

$\square$

Prop: Let $X \in Sp$, then the map [$Sp^\kappa_{/X}$ has all κ-small colimits $\Rightarrow$ it is κ-filtered]

$$\mathrm{colim}_{Y \in Sp^\kappa_{/X}} Y \to X \qquad Sp = \mathrm{Ind}_\kappa Sp^\kappa$$

is an equivalence. In particular every spectrum is a κ-filtered colimit of κ-small spectra

Proof: Recall π_* commutes w/ filtered colimits $\Rightarrow$ it's enough to prove

$$\mathrm{colim}_{Y \in Sp^\kappa_{/X}} \pi_* Y \xrightarrow{\sim} \pi_* X.$$

It's onto because if we have $\alpha \in \pi_n X$, this comes from $[\alpha: \Sigma^n \mathbb{S} \to X] \in Sp^\kappa_{/X}$. ($1 \in \pi_n \Sigma^n \mathbb{S}$)

Suppose now we have $f: Y \to X$, $Y \in Sp^\kappa$, $\alpha \in \pi_n Y$ going to 0 in $\pi_n X$.

$$\begin{tikzcd} \Sigma^n \mathbb{S} \arrow[d, "\alpha"] \arrow[dr, "0"] & \\ Y \arrow[r, "f"] \arrow[d] & X \\ Y' \arrow[ur, "f'"] & \end{tikzcd}$$

cof α, still κ-small

$\Rightarrow [Y, \alpha] \mapsto 0$ in $Y' \xrightarrow{f'} X$

& so it is 0 in the colimit. $\square$

Prop: Let $E \in Sp$, κ uncountable s.t. $\# \pi_* E < \kappa$. Then every E-acyclic spectrum is a κ-filtered colimit of κ-small E-acyclic spectra.

($\ker(E \otimes -: Sp \to Sp)$ is κ-accessible)