Last time: Bott periodicity: $\Omega U \simeq BU \times \mathbb{Z}$

$\Rightarrow$ defines the spectrum KU

Recall: If X is a finite space

$[\Sigma^\infty X_+, KU] = [X_+, \Omega^\infty KU]_* = [X, BU\times\mathbb{Z}]$ = Grothendieck group of vector bundles on X

- $KU^* X := [\Sigma^\infty X_+, \Sigma^* KU]_{Sp}$
- $\widetilde{KU}^* X := [\Sigma^\infty X, \Sigma^* KU]_{Sp}$

$KU^* X = \widetilde{KU}^* X \oplus KU^*$

- $\pi_* KU = \begin{cases} \mathbb{Z} & * \text{ even} \\ 0 & * \text{ odd} \end{cases}$

$\Rightarrow \exists \beta \in \pi_2 KU$ corresp. to 1 $\qquad \pi_2 KU = \widetilde{KU}^0(S^2) \subseteq KU^0(S^2)$

First problem: Let's identify β (as a v. bundle on S^2)

§ Clutching functions

Q: Can we describe vector bundles on ΣX?

$\mathrm{Vect}_n(\Sigma X) = [\Sigma X, BU_n]$

$[\Sigma X, BU_n]_*$ = vector bundles on ΣX together w/ a trivialization over x.

(Recall: We have a fib. seq. $\mathrm{Map}_*(Y, BU_n) \to \mathrm{Map}(Y, BU_n) \to BU_n$ $\forall Y \in \mathrm{Spaces}_*$)

$[\Sigma X, BU_n]_* = [X, \Omega BU_n]_* = [X, U_n]_*$

$\{\Sigma X \to BU_n\} \leftrightarrow$ {pushout diagram in Spaces} $= \{\varphi: X\times\mathbb{C}^n \to X\times\mathbb{C}^n \mid \varphi = \mathrm{id}$ over $x\} = \{\varphi: X \to U_n\}$

$\Sigma X = I^+ \times X \cup_X I^- \times X$

$\xi_\varphi = (I^+\times X)\times\mathbb{C}^n \cup_{X\times\mathbb{C}^n, \varphi} (I^-\times X)\times\mathbb{C}^n$

$\downarrow$

$\Sigma X = I^+\times X \cup_X I^-\times X$

$\beta \in \widetilde{KU}^0(S^2) \qquad \beta: S^2 \to BU\times\mathbb{Z}$; $S^1 \xrightarrow{\beta} \Omega BU \simeq U$ is the generator of $\pi_1 U$.

$U_1 = \{z \in \mathbb{C} \mid |z| = 1\} \hookrightarrow U$ is the generator of $\pi_1 U$. $\quad [U_n \to U_{n+1} \to S^{2n+1}]$

$U_n \to U_{n+1}$ iso on π_1, by ind. $U_1 \to U$ iso on π_1, & we are done ($\pi_1 S^1$ gen. by id_{S^1})

$\Rightarrow \beta: S^2 \to BU\times\mathbb{Z}$ corresponds to the map $S^1 \to U$ given by $z \mapsto \begin{pmatrix} z & 0 \\ 0 & \ddots \end{pmatrix}$

Claim: We have a homeomorphism $S^2 \cong \mathbb{CP}^1$ ($\cong \Sigma S^1$), the vector bundle is the tautological line bundle on $\mathbb{CP}^1$.

$\xi = \{(x, L) \in \mathbb{C}^2 \times \mathbb{CP}^1 \mid x \in L\}$

$S^1 \subseteq \Sigma S^1 \cong \mathbb{CP}^1 \qquad \mathbb{RP}^1$

Exercise: Compute the clutching function & see that it is $\mathrm{id}: S^1 \to S^1$.

$S^2 \xrightarrow{\beta} BU\times\{0\}$ ptd map adjoint to $S^1 = U_1 \hookrightarrow U$ $\qquad S^2 \to BU_1\times\{0\} \hookrightarrow BU\times\{0\} \to BU\times\mathbb{Z}$

($BU\times\{0\} \subseteq BU\times\mathbb{Z}$)

Vector bundles of rank 1 come from $BU_1 \times\{1\} \subseteq BU\times\mathbb{Z}$

β corresponds to the virtual v. bundle $\xi - 1$ (ξ = this v. bundle of rank 1 on $\mathbb{CP}^1$).

$\coprod_n BU_n \xrightarrow{\oplus 1} \coprod BU_n \xrightarrow{\oplus 1} \coprod_n BU_n$ over $BU\times\mathbb{Z} \overset{\mathrm{id}}{=} BU\times\mathbb{Z} = BU\times\mathbb{Z}$

$\beta \oplus 1 \in BU\times\{1\} \supseteq BU_1 \times\{1\} \ni \xi$

$$\boxed{KU^0(S^2) \simeq \mathbb{Z} \oplus \widetilde{KU}^0(S^2) \simeq \mathbb{Z}\cdot 1 \oplus \mathbb{Z}\cdot[\xi - 1]}$$

Def: A homotopy ring spectrum, is a spectrum E together w/ maps

$\eta: \mathbb{S} \to E$, $\mu: E\otimes E \to E$ s.t. $\mu(\eta \otimes \mathrm{id}_E) \simeq \mu(\mathrm{id}_E \otimes \eta) \simeq \mathrm{id}_E$

$\mu(\mu\otimes \mathrm{id}_E) = \mu(\mathrm{id}_E \otimes \mu)$.

Remark: To get a good theory one would need to specify higher homotopies, the corresponding object is called a E_1-ring, but we won't have time to talk about those.

Def: E homotopy ring spectrum is commutative if $\mu \simeq \mu\circ\sigma$ $\quad \sigma: E\otimes E \xrightarrow{\sim} E\otimes E$ "swap" map

Remark: If E is a (commutative) hom. ring spectrum, the associated cohomology theory is multiplicative.

- $E^* X \otimes_{\mathbb{Z}} E^* Y \to E^*(X\wedge Y)$

$[\Sigma^\infty X, \Sigma^n E]_* \ni f$, $\quad [\Sigma^\infty Y, \Sigma^m E]_* \ni g$ $\qquad \Sigma^\infty(X\wedge Y) \simeq \Sigma^\infty X \otimes \Sigma^\infty Y \xrightarrow{f\otimes g} \Sigma^n E \otimes \Sigma^m E \simeq \Sigma^{n+m}(E\otimes E) \to \Sigma^{n+m} E$.

- $1 \in E^0(S^0)$ $\quad$ (i.e. $\eta: \Sigma^\infty S^0 = \mathbb{S} \to E$)

In particular $E^*(X_+)$ is a graded ring, (graded) commutative if E is commutative.

$E^*(X_+) \otimes E^*(X_+) \to E^*(\underbrace{X_+ \wedge X_+}_{(X\times X)_+}) \xrightarrow{\Delta^*} E^*(X_+)$.

Proposition: KU is a homotopy com. ring spectrum.

Proof: Let us choose an isomorphism $\varphi: \mathbb{C}^\infty \otimes_{\mathbb{C}} \mathbb{C}^\infty \simeq \mathbb{C}^\infty$ (i.e. a bij. $\mathbb{N}\times\mathbb{N} \overset{\varphi}{\to} \mathbb{N}$) $\quad e_i \otimes e_j \mapsto e_{\varphi(i,j)}$

$\mathcal{V}(I_+) \wedge \mathcal{V}(J_+) \to \mathcal{V}(I_+ \wedge J_+)$

$(\{V_i\}_{i\in I}, \{W_j\}_{j\in J}) \mapsto \{\varphi(V_i \otimes W_j)\}_{(i,j)\in I\times J}$

nat. trans. in I_+, J_+. Moreover this yields $\qquad (B\mathcal{V})(I_+) = \operatorname{colim}_n \mathcal{V}(I_+ \wedge \mathrm{Gap}(n)_+)$

$B^n\mathcal{V}(I_+) \wedge B^m\mathcal{V}(J_+) \to B^{n+m}\mathcal{V}(I_+\wedge J_+)$

$\Rightarrow \operatorname{colim} \Omega^n \Sigma^\infty B^n\mathcal{V} = B^\infty\mathcal{V} = ku$

$\operatorname{colim} \Omega^m \Sigma^\infty B^m\mathcal{V} = B^\infty\mathcal{V} = ku$

$B^\infty\mathcal{V} \otimes B^\infty\mathcal{V} = \operatorname{colim}_{n,m} \Omega^{n+m}\Sigma^\infty(B^n\mathcal{V} \wedge B^m\mathcal{V}) \to \operatorname{colim}_{n,m} \Omega^{n+m}\Sigma^\infty B^{n+m}\mathcal{V} \simeq B^\infty\mathcal{V}$

ku is the connective cover of KU (by definition).

We have shown that ku is a homotopy ring spectrum.

Since $ku = \tau_{\geq 0} KU$ $\quad \pi_2 ku = \pi_2 KU = \mathbb{Z}\cdot\beta$. $\quad \beta: \Sigma^2\mathbb{S} \to ku$.

Claim: $KU \overset{?}{=} \operatorname{colim}(ku \to \Sigma^{-2} ku \to \Sigma^{-4} ku \to \dots)$

$ku \simeq \mathbb{S}\otimes ku \to \Sigma^{-2} ku \otimes ku \xrightarrow{\mu} \Sigma^{-2} ku$

$KU\otimes KU = \operatorname{colim}(ku\otimes ku \to \Sigma^{-4} ku\otimes ku \to \dots)$

$\downarrow$

$KU = \operatorname{colim}(ku \to \Sigma^{-4} ku \to \Sigma^{-8} ku \to \dots)$

(This is like the proof that $R[\frac{1}{x}]$ is a ring

Proof of the claim: $ku = (\dots, \overset{-2}{BU\times\mathbb{Z}}, \overset{-1}{U}, \overset{0}{BU\times\mathbb{Z}}, \overset{1}{U}, \overset{2}{BU}, \overset{3}{SU}, \overset{4}{BSU}, \dots)$

$\downarrow\beta$

$\Sigma^{-2} ku = (\dots, BU\times\mathbb{Z}, U, BU\times\mathbb{Z}, U, BU\times\mathbb{Z}, U, BU, \dots)$

$\beta: \Sigma^2\mathbb{S} \to ku \quad 1 \in \pi_1 U = \mathbb{Z} \iff \mathbb{S} \xrightarrow{\beta} \Sigma^{-2} ku$

$ku \simeq \mathbb{S}\otimes ku \xrightarrow{\beta\otimes 1} \Sigma^{-2} ku \otimes ku \xrightarrow{\Sigma^{-2}\mu} \Sigma^{-2} ku$

The map at level 1 is the eq. $B(BU\times\mathbb{Z}) \xrightarrow{\sim} U$

$ku \to \Sigma^{-2} ku$ factors through the con. cover $ku \to \tau_{\geq 0}\Sigma^{-2} ku$ ($= ku$).

$\Omega^\infty \Sigma^{-2} ku = \Omega^2(BU\times\mathbb{Z}) \simeq BU\times\mathbb{Z} \simeq \Omega^\infty ku$.

So at the limit the spaces of $\operatorname{colim}(ku \to \Sigma^{-2} ku \to \Sigma^{-4} ku \to \dots)$ are exactly those we used to define KU. $\square$

Remark: Looking at the def, the ring structure on $KU^0(X)$ is given by $\otimes$ of v. bundles.

$p: E \to B \qquad p': E' \to B \qquad E\otimes E' \to B$ whose fiber over $b\in B$ $E_b \otimes E'_b$.

Prop: $\widetilde{KU}^0(S^{2n}) = \mathbb{Z}\cdot[\xi - 1]^n$

i.e. $\widetilde{KU}^0(S^2)\otimes\dots\otimes\widetilde{KU}^0(S^2) \xrightarrow{\sim} \widetilde{KU}^0(\overbrace{S^2\wedge\dots\wedge S^2}^{S^n})$

Prop: $\widetilde{KU}^0(S^2) \otimes_{\mathbb{Z}} \widetilde{KU}^* X \to \widetilde{KU}^*\Sigma^2 X$ is an iso $\forall X$.

Proof: This is a map of cohomology theories, so it corresponds to a map of spectra

$KU \xrightarrow[\sim]{\beta\cdot} \Sigma^{-2} KU$. $\square$ $\qquad [\Sigma^\infty\Sigma^2 X, \widetilde{KU}] = [\Sigma^\infty X, \Sigma^{-2}\widetilde{KU}]$ $\square$

Prop: $KU^*\mathbb{CP}^n \simeq KU^* \oplus KU^*[\xi - 1] \oplus KU^*[\xi-1]^2 \oplus \dots \oplus KU^*[\xi - 1]^n$.

as a KU^*-module.

(Analogue to $H^*\mathbb{CP}^n \simeq \mathbb{Z}\oplus\mathbb{Z}x\oplus\dots\oplus\mathbb{Z}x^n \simeq \mathbb{Z}[x]/x^{n+1}$)

Proof: By induction on n. Consider the cofiber sequence

$\mathbb{CP}^{n-1} \to \mathbb{CP}^n \to S^{2n}$

$x \mapsto [0:x]$

$\Rightarrow \underset{0}{\widetilde{KU}^{2i+1} S^{2n}} \leftarrow \widetilde{KU}^{2i}\mathbb{CP}^{n-1} \leftarrow \widetilde{KU}^{2i}\mathbb{CP}^n \leftarrow \underset{\mathbb{Z}}{\widetilde{KU}^{2i} S^{2n}} \leftarrow \underset{0}{\widetilde{KU}^{2i-1}\mathbb{CP}^{n-1}}$

$\widetilde{KU}^{2i}\mathbb{CP}^{n-1} = \mathbb{Z}[\xi - 1] \oplus\dots\oplus \mathbb{Z}[\xi - 1]^{n-1}$

$[\xi - 1]^n$ restricts to the generator of $\widetilde{KU}^0 S^{2n}$.

$S^2\times\dots\times S^2 \rightleftarrows S^{2n} = S^2\wedge\dots\wedge S^2 \ni$ generator

$\mathbb{CP}^1\times\dots\times\mathbb{CP}^1 \to \mathbb{CP}^n$

$([a_1:b_1], \dots, [a_n:b_n]) \mapsto$ [coeff. of the polyn. $(a_1 + b_1 x)\cdots(a_n + b_n x)$]

$[0:1]$ $\qquad$ const. term is trivial $\square$

$\widetilde{KU}^i X = \begin{cases} \widetilde{KU}^0 X = \text{v. vect. bundles w/ rank 0 over the basept.} & i \text{ even} \\ [X, U]_* = \bigcup_n [X, U_n]_* = \mathrm{Aut}_*(X\times\mathbb{C}^n) & i \text{ odd} \end{cases}$