ISHT Lecture 13

2020-12-17

Gr_d = space of d-dimensional vector subspaces of $\mathbb{C}^\infty$

We have shown that for every space X $\quad [X, Gr_d] \simeq \pi_0 \mathrm{Vect}(X)$.

Next step: we want to see $Gr_d \simeq BU_d$

Lemma: $p: E \to B$ (any) fibration, G topological group acting on E over B s.t.

$$G \times E \xrightarrow{\sim} E \times_B E \quad \text{homeom.}$$
$$(g,e) \longmapsto (ge, e)$$

Suppose E contractible. Then $\exists$ a can. eq. of E_1-groups

$$G \simeq \Omega B.$$

(in part. $BG \simeq B$)

Let us consider $\mathrm{Emb}(\mathbb{C}^d, \mathbb{C}^\infty) =: V_d \; (= V_d\, \mathbb{C}^\infty)$ the space of isom. embeddings of $\mathbb{C}^d$ into $\mathbb{C}^\infty$

i.e. $\{ \underbrace{(x_1, \dots, x_d)}_{\in (\mathbb{C}^\infty)^d} \mid |x_i|^2 = 1, \; x_i \perp x_j \; i \neq j \}$

$$p: V_d \to Gr_d$$
$$f \longmapsto \mathrm{Im} f \quad (\leftrightarrow f f^T)$$

We have an U_d-action on V_d by precomposition $\quad U_d$ = isometries $\mathbb{C}^d \to \mathbb{C}^d$.

p is a fiber bundle, even more if $W \subseteq Gr_d$ std open subsets we have a U_d-eq. homeom.

$$U_d \times W \xrightarrow{\sim} p^{-1} W$$

$S \in Gr_d \quad W_S = \{ S' \in Gr_d \mid P_S S' = S \} \simeq \mathrm{Hom}(S, S^\perp)$

$p^{-1} W_S = \{ f \in \mathrm{Emb}(\mathbb{C}^d, \mathbb{C}^\infty) \mid \mathrm{Im}(P_S f) = S \}$ $\qquad P_S : \mathbb{C}^\infty \to S$ orth. proj. onto S.

Let us choose $f_0 : \mathbb{C}^d \xrightarrow{\sim} S$.

$$p^{-1} W_S \to U_d \times W_S$$
$$f \longmapsto (\underbrace{f_0^{-1} P_S f}_{\mathbb{C}^d \to S \to \mathbb{C}^d}, \mathrm{Im} f)$$

$\Rightarrow p$ fibration, & in fact a principal bundle. So to prove $Gr_d \simeq BU_d$ we are only missing one step.

Prop: V_d is contractible.

Proof: $\tilde{V}_d = \{ f : \mathbb{C}^d \to \mathbb{C}^\infty \mid \text{injective } \mathbb{C}\text{-linear maps} \} \supseteq V_d$

Step 1: The inclusion $V_d \subseteq \tilde{V}_d$ is a htpy eq.

$$V_d := V_d^{(d)} \subseteq V_d^{(d-1)} \subseteq \dots \subseteq V_d^{(0)} = \tilde{V}_d$$

$V_d^{(i)} = \{ f : \mathbb{C}^d \to \mathbb{C}^\infty \mid \text{inj}, \mathbb{C}\text{-lin}, f|_{\mathbb{C}^i} \text{ isometry} \}$

We're going to show $V_d^{(i)} \subseteq V_d^{(i-1)}$ is a htpy eq.

$$\underset{\substack{x_1, \dots, x_i \\ \text{orthon.}}}{V_d^{(i)}} \subseteq \underset{\substack{x_1, \dots, x_{i-1} \\ \text{ort.} \\ x_i \perp x_j \; j<i}}{V_d^{(i-1)'}} \subseteq \underset{\substack{x_1, \dots, x_{i-1} \\ \text{ort.}}}{V_d^{(i-1)}} \qquad x_i := f e_i$$

We have a def. retraction $V_d^{(i-1)'} \to V_d^{(i)}$ $\qquad V_d^{(i-1)'} \times [0,1] \to V_d^{(i-1)'}$ ✓

$$t \cdot (x_1, \dots, x_n) \longmapsto \left(x_1, \dots, x_{i-1}, \left(t + \frac{1-t}{|x_i|}\right) x_i, x_{i+1}, \dots, x_n\right)$$

$$V_d^{(i-1)} \times [0,1] \to V_d^{(i-1)'}$$

$$t \cdot (x_1, \dots, x_n) \longmapsto (x_1, \dots, x_{i-1}, x_i - t\langle x_i, x_1 \rangle x_1 - \dots - t \langle x_i, x_{i-1} \rangle x_{i-1}, x_{i+1}, \dots, x_n)$$

Upshot: It's enough to show $\tilde{V}_d$ is contractible.

Let $f_0 : \mathbb{C}^d \to \mathbb{C}^\infty$ the std embedding $x \mapsto (x, 0, 0, 0, \dots)$

Let $W_d = \{ f \in \tilde{V}_d \mid \mathrm{Im} f \perp \mathrm{Im} f_0 \} = \{ f \in \tilde{V}_d \mid f x = (\overbrace{0, \dots, 0}^{\text{first } d \text{ coordinates}}, \dots) \}$

Step 2: $W_d \subseteq \tilde{V}_d$ is a homotopy eq.

Step 3: $W_d \subseteq \tilde{V}_d$ is nullhomotopic. $\qquad [0,1] \times W_d \to \tilde{V}_d$
$$(t, f) \longmapsto t f_0 + (1-t) f$$
injective map.

$$T^d : \mathbb{C}^\infty \to \mathbb{C}^\infty$$
$$x \longmapsto (\underbrace{0, \dots, 0}_{\text{first } d \text{ coordinates}}, x)$$

$$\tilde{V}_d \to W_d \qquad \qquad \tilde{V}_d \times [0,1] \to \tilde{V}_d$$
$$f \mapsto T^d f \qquad \qquad (f, t) \longmapsto t f + (1-t) T^d f$$

we need to check that for every t this is injective.

$$t x + (1-t) T^d x = 0 \overset{?}{\Rightarrow} x = 0$$

If $t = 0$, $T^d x = 0 \Rightarrow x = 0$.

$t \neq 0 \quad t x_i + (1-t) x_{i-d} = 0$ & go by induction on i, $t x_0 = 0 \Rightarrow x_0 = 0, \dots$ □

Special case: $V_1 = \{ x \in \mathbb{C}^\infty \mid |x| = 1 \} = S^\infty$ is contractible.

Upshot: $\pi_0 \mathrm{Vect}_d(X) = [X, BU_d]$

Exercise: If $p : E \to X$ v. bundle $\quad \Omega_p \mathrm{Map}(X, BU_d) \simeq$ top. group of autom. of p.

$\mathrm{Vect}_d(X) := \mathrm{Map}(X, BU_d)$.

Corollary: $\pi_0 \mathrm{Vect}(X) = [X, \coprod_{d \geq 0} BU_d]$

Proof: You can reduce to the case X connected, & then every v. bundle has a precise rank. □

Def: $\mathcal{V} : \mathrm{Fin}_* \to \mathrm{Top} \to \mathrm{Space}$
$$I_+ \longmapsto \{ (V_i)_{i \in I} \in Gr^I \mid V_i \perp V_j \; i \neq j \}$$
$$[f : I_+ \to J_+] \longmapsto (V_i)_{i \in I} \longmapsto \Big(\bigoplus_{f i = j} V_i \Big)_{j \in J}$$

$$\mathcal{V}(1_+) \simeq \coprod_{d \geq 0} BU_d$$

Lemma: $\mathcal{V}$ is an E_∞-space.

Proof: $\mathcal{V}(n_+) \to \prod_{i=1}^n \mathcal{V}(1_+)$ homotopy eq.
$$(V_1, \dots, V_n) \longmapsto \{ (V_i) \}_{i = 1, \dots, n}$$

$$\mathcal{V}(n_+) = \coprod_{d_1, \dots, d_n \geq 0} \underbrace{\{ (V_1, \dots, V_n) \in \mathcal{V}(n_+) \mid \dim V_i = d_i \}}_{\mathcal{V}_{\underline{d}}(n_+)}$$

$$\mathcal{V}_{\underline{d}}(n_+) \xrightarrow{\sim} Gr_{d_1} \times \dots \times Gr_{d_n}$$

$$V_{d_1 + \dots + d_n} \to \mathcal{V}_{\underline{d}}(n_+)$$
$$f : \mathbb{C}^{d_1 + \dots + d_n} \to \mathbb{C}^\infty \longmapsto (\mathrm{Im}(f(\mathbb{C}^{d_i})))_{i = 1, \dots, n}$$

Redo the proof as before w/ the group $U_{d_1} \times \dots \times U_{d_n}$. $\qquad \begin{pmatrix} \begin{array}{c|c|c} A_1 & 0 & 0 \\ \hline 0 & A_2 & \\ \hline 0 & 0 & \ddots \end{array} & 0 \end{pmatrix}$

$$\mathcal{V}_{\underline{d}}(n_+) \simeq B\Big(\prod_{i=1}^n U_{d_i}\Big) \simeq \prod_{i=1}^n BU_{d_i}. \qquad □$$

Remark: $\pi_0 \mathcal{V} = \mathbb{N}$

$\mathrm{Map}(X, \mathcal{V}) : I_+ \longmapsto \mathrm{Map}(X, \mathcal{V}(I_+))$ is also an E_∞-space.

and $\pi_0 \mathrm{Map}(X, \mathcal{V})$ = isoclasses of v. bundles on X, w/ operation $\oplus$.

$$\pi_0 \mathrm{Map}(X, \mathcal{V}(1_+)) \times \pi_0 \mathrm{Map}(X, \mathcal{V}(1_+)) \simeq \pi_0 \mathrm{Map}(X, \mathcal{V}(1_+) \times \mathcal{V}(1_+)) \xleftarrow{\sim} \pi_0 \mathrm{Map}(X, \mathcal{V}(2_+))$$
$$\downarrow \oplus$$
$$\pi_0 \mathrm{Map}(X, \mathcal{V}(1_+))$$

$p : E \to X \quad , p : E' \to X$

$$X \xrightarrow{(f, f')} Gr_d \times Gr_{d'} \quad (fx, f x') \text{ s.t. } \forall x \quad fx \perp fx'.$$
$$\downarrow \qquad \qquad \downarrow$$
$$Gr_{d+d'} \quad fx \oplus f x'. \qquad □$$

Def: X space is finite if it can be obtained from $*$ by finite pushouts & pushouts.

Ex: $\cdot$ X the homotopy type of a finite CW complex $\Rightarrow X$ finite.

$\cdot$ The converse is also true (use $[S^n, S^m] = *$ $\; n < m$)

X finitely dominated, if it is a retract of a finite space.

Prop: $\cdot$ The underlying space of $\mathcal{V}^{gp} \simeq BU \times \mathbb{Z}$. $\qquad \left(U = \bigcup U_n \; ; \; \begin{pmatrix} \begin{array}{c|c} A & 0 \\ \hline 0 & I \end{array} \end{pmatrix} \right) \quad A \in U_d$

$\cdot$ If X finitely dominated $\mathrm{Map}(X, \mathcal{V})^{gp} \simeq \mathrm{Map}(X, \mathcal{V}^{gp})$.

in part. $[X, BU \times \mathbb{Z}]$ = Groth. group of v. bundles on X.

Proof: By the group completion theorem $(\pi_0 \mathcal{V} = \mathbb{N}, \; x = 1 \in \mathbb{N})$

$$\mathcal{V}^{gp} = (\mathrm{tel}_1 \mathcal{V})^+$$

$$\mathrm{tel}_1 \mathcal{V} = \mathrm{colim} \Big(\coprod_{n \geq 0} BU_n \xrightarrow{+1} \coprod_{n \geq 0} BU_n \to \dots \Big) \qquad \begin{array}{c} BU_1 \times BU_n \to BU_{n+1} \\ U_1 \times U_n \to U_{n+1} \\ (A, B) \longmapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \end{array}$$

$$BU_n \xrightarrow{+1} BU_{n+1} \to \dots$$
$$A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

so on the n-th component this gives exactly BU & then the con. components are $\mathbb{Z}$.

$$\mathrm{colim}(\mathbb{N} \xrightarrow{+1} \mathbb{N} \xrightarrow{+1} \mathbb{N} \xrightarrow{+1} \dots) = \mathbb{Z}$$

$U_n \to U_{n+1} \to S^{2n+1} \qquad U_1 = S^1$

$\Rightarrow \mathrm{tel}_1 \mathcal{V} = BU \times \mathbb{Z}. \qquad \pi_1 \mathrm{tel}_1 \mathcal{V} = \pi_1 BU = \pi_0 U = *$

$\Rightarrow \mathrm{tel}_1 \mathcal{V}^+ = \mathrm{tel}_1 \mathcal{V} = BU \times \mathbb{Z} = \mathcal{V}^{gp}$.

$\mathrm{Map}(X, \mathcal{V})^{gp} = ?$

$$\mathrm{tel}_1 \mathrm{Map}(X, \mathcal{V}) \overset{X \text{ finitely dominated}}{=} \mathrm{Map}(X, \mathrm{tel}_1 \mathcal{V}) = \mathrm{Map}(X, \mathcal{V}^{gp})$$

Remark: The property $\mathrm{Map}(X, -)$ commutes w/ filtered colimits is closed under finite colim's, pushouts & retracts.

$\Rightarrow \pi_0 \mathrm{tel}_1 \mathrm{Map}(X, \mathcal{V})$ is a group, so x has the property needed for the group compl. theorem
$$\pi_0 \mathrm{tel}_1 \mathrm{Map}(X, \mathcal{V}) = \pi_0 \mathrm{Map}(X, \mathcal{V})[x^{-1}]$$

$$\Rightarrow \mathrm{Map}(X, \mathcal{V})^{gp} \simeq \mathrm{tel}_1 \mathrm{Map}(X, \mathcal{V})^+ = \mathrm{Map}(X, \mathcal{V}^{gp})^+ = \mathrm{Map}(X, \mathcal{V}^{gp}). \quad □$$

Remark: X finitely dominated is essential here; However $[X, BU \times \mathbb{Z}]$ does have an int. in terms of vector bundles for a gen. X, but it's more complicated.

Thm (Bott periodicity): $\quad B\mathcal{V} \simeq U$

$$\Rightarrow \Omega U \simeq BU \times \mathbb{Z}, \quad \Omega^2 U \simeq U$$

(this approach is due to Bruno Harris)

Proof: $\quad B\mathcal{V} = \mathrm{colim}_{[n] \in \Delta^{op}} \mathcal{V}([n])$.

$$\mathrm{colim}_{[n] \in \Delta^{op}} \mathcal{V}([n]) \simeq \coprod_{n \geq 0} |\Delta^n| \times \mathcal{V}([n]) \big/ \sim \;\overset{\text{homeomorphic}}{\cong} U$$

$$[(t_0, \dots, t_n), V_1, \dots, V_n] \longmapsto P_{(\oplus V_i)^\perp} + e^{2\pi i t_0} P_{V_1} + e^{2\pi i (t_0 + t_1)} \cdot P_{V_2} + \dots + e^{2\pi i (t_0 + \dots + t_{n-1})} \cdot P_{V_n}$$

A is the matrix w/ eigenvalue $e^{2\pi i (t_0 + \dots + t_n)}$ on V_n.

This respects the simplicial relations
$$(\dots, t_i, t_{i+1}, \dots) \mapsto (\dots, t_i + t_{i+1}, \dots)$$
is the same as removing V_{i+1} & adding a 0
is the same as replacing V_i, V_{i+1} by $V_i \oplus V_{i+1}$.

The inverse works like this: We have $A \in U$. This has some eigenvalues

$1 = \lambda_0, \lambda_1, \dots, \lambda_n \qquad \underset{[0, 2\pi)}{\arg} \lambda_i > \arg \lambda_{i+1}$.

$V_0, V_1, \dots, V_n$ eigenspaces

$$A \longmapsto \left(\frac{\arg \lambda_1}{2\pi}, \frac{\arg \lambda_2 - \arg \lambda_1}{2\pi}, \dots, \frac{2\pi - \arg \lambda_n}{2\pi}, (V_1, \dots, V_n) \right) \qquad □$$

Remark: This is not enough over $\mathbb{R}$. Over $\mathbb{R}$, this shows $\Omega\, U/O \simeq BO \times \mathbb{Z}$.

$\Omega\, U/Sp \simeq BSp \times \mathbb{Z}$. $\quad$ Over $\mathbb{R}$ $\quad \Omega^8 O \simeq O$.

Def: KU topological complex K-theory spectrum

$$(BU \times \mathbb{Z}, U, BU \times \mathbb{Z}, U, BU \times \mathbb{Z}, \dots)$$

$$\underset{KU^0 X}{\overset{\parallel}{[\Sigma^\infty X_+, KU]}} \simeq [X_+, \Omega^\infty KU]_* \simeq [X, BU \times \mathbb{Z}] = \pi_0 \mathrm{Vect}(X)^{gp} \quad X \text{ finitely dominated}$$

$$KU^1 X \simeq [X, U]$$

$$KU^* X = \begin{cases} KU^0 X & * \text{ even} \\ KU^1 X & * \text{ odd} \end{cases} \qquad \pi_* KU = \begin{cases} \mathbb{Z} & * \text{ even} \\ 0 & * \text{ odd} \end{cases}$$

Using real Bott periodicity, one can define KO and

$$\pi_* KO = \begin{cases} \mathbb{Z} & \\ \mathbb{Z}/2 & * \equiv 1 \\ \mathbb{Z}/2 & * \equiv 2 \\ 0 & * \equiv 3 \\ \mathbb{Z} & * \equiv 4 \\ 0 & * \equiv 5 \\ 0 & * \equiv 6 \\ 0 & \\ \mathbb{Z} & * \equiv 8 \end{cases} \quad (8)$$

$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
1	2	3	4	5	6	7	8