ISHT Lecture 12 2020-12-14

§ Vector bundles

Def: Let X be a topological space. A vector bundle on X is a triple (E, s, μ) where $p: E \to X$ is a map of topological spaces, $s: E \times_X E \to E$, $\mu: \mathbb{C} \times E \to E$ are continuous maps over X s.t.
$\forall x \in X$ $\exists U \ni x$, V a (fd) $\mathbb{C}$-v. space & a homeom.
$$p^{-1}U \simeq X \times V \qquad [\text{trivialization of } E]$$
over X s.t. under this identification $s(x, v, w) = (x, v+w)$, $\mu(\lambda, x, v) = (x, \lambda v)$.
In particular each fiber E_x is given a fd $\mathbb{C}$-vector space structure by s & μ.

Lemma: X paracompact Hausdorff top. space $\forall \{U_i\}_{i \in I}$ open covering $\exists$ a countable open covering $\{V_n\}_{n \geq 0}$ s.t. each $V_n \simeq \coprod_{i \in I} V_{n,i}$ $V_{n,i} \subseteq U_i$ open subset of U_i.

Lemma: X paracompact Hausdorff top. space $\forall \{U_i\}_{i \in I}$ open covering $\exists \{\varphi_i\}_{i \in I}$ partition of unity wrt. to the covering, i.e. $X \to [0,1]$
- $\forall x \notin U_i$ $\varphi_i(x) = 0$
- $\forall x \in X$ $\exists V$ nbhd of x s.t. $\#\{i \mid \varphi_i|_V \neq 0\} < \infty$
- $\sum_{i \in I} \varphi_i = 1$.

Corollary: $p: E \to X$ v. bundle, X pcpt Hausdorff $\Rightarrow \exists \{V_n\}$ countable cover of trivializations

A map of v. bundles $f: E \to E'$ over X resp. the vector space structure
$$s(f \times f) = f \circ s \quad , \quad \mu(\lambda, f) = f \circ \mu(\lambda, -)$$
[f $\mathbb{C}$-linear when restricted to each fiber]

We will write $\pi_0 \mathrm{Vect}(X)$ for the set of isoclasses of vector bundles on X.
The rank of a vector bundle $E \to X$ is the function $X \to \mathbb{Z}_{\geq 0}$ $x \mapsto \dim_{\mathbb{C}} E_x$ (this is locally constant). We'll write $\pi_0 \mathrm{Vect}_d(X) \subseteq \pi_0 \mathrm{Vect}(X)$ for the subset of v. bundles of rank d.

Theorem: Let X be a paracompact Hausdorff space. There's a natural bijection
$$\pi_0 \mathrm{Vect}_d(X) = [X, BU_d]$$
where U_d is the topological group of $d \times d$ unitary matrices.
$\mathrm{Vect}_d(X) := \mathrm{Map}(X, BU_d)$

Proposition: Let $E \to X \times [0,1]$ be a v. bundle. Then if we let $E' = E|_{X \times \{0\}}$ $\exists$ an isom. $E \simeq E' \times [0,1]$.

$f: X \to Y$ cont. map, $E \to Y$ v. bundle $f^*E = E \times_Y X$ is a v. bundle (exercise)

Corollary: $\pi_0 \mathrm{Vect}(-)$ is homotopy invariant.

Proof: $f, g: X \to Y$ maps that are homotopic. We want to show $f^*E \simeq g^*E$ $\forall E$.
Let $H: X \times [0,1] \to Y$ the homotopy, $H^*E \to X \times [0,1]$ is a v. bundle.
$H^*E|_{X \times \{0\}} = f^*E$, $H^*E|_{X \times \{1\}} = g^*E$. But by the proposition, they need to be iso.
$\left(H^*E \simeq f^*E \times [0,1] \Rightarrow H^*E|_{X \times \{1\}} \simeq f^*E\right)$. □

[$E' \to X$ $\quad E' \times [0,1] \to X \times [0,1]$ $\quad (E' \times [0,1]) \times_{X \times [0,1]} (E' \times [0,1]) = (E' \times_X E') \times [0,1]$
$\mathrm{pr}: X \times [0,1] \to X$ $\quad \mathbb{C} \times E' \times [0,1] = \mathbb{C} \times (E' \times [0,1])$
$E' \times [0,1] = \mathrm{pr}^* E'$.]

Proof: $p: E \to X \times [0,1]$

X
$[0,1]$

$\forall t \in [0,1]$ U_t nbhd of x & $\varepsilon_t > 0$ s.t.
$\exists$ a triv. on $U_t \times [t - \varepsilon_t, t + \varepsilon_t]$. Since $\{x\} \times [0,1]$ is compact we can find $t_1, \dots, t_n$ s.t. $U_{t_1} \times [t_1 - \varepsilon_{t_1}, t + \varepsilon_{t_1}], \dots$ cover $\{x\} \times [0,1]$.
Let $U = \bigcap_{i=1}^{n} U_{t_i}$, $\varepsilon = \min(\varepsilon_{t_i})$
We have $t_0 = 0 < t_1 < \dots < t_n = 1$ & triv. of E on $U \times [t_i, t_{i+1}]$.
But then we can glue the trivial. inductively.
$\varphi_1: p^{-1}(U \times [0, t_1]) \simeq V \times U \times [0, t_1]$ $\quad V = E_{(x, t_1)}$
$\varphi_2: p^{-1}(U \times [t_1, t_2]) \simeq V \times U \times [t_1, t_2]$
In part. we get a map
$$F: V \times U \simeq V \times U \times \{t_1\} \overset{\varphi_2^{-1}}{\simeq} p^{-1}(U \times \{t_1\}) \overset{\varphi_1}{\simeq} V \times U \times \{t_1\} \simeq V \times U$$
$(F \times \mathrm{id})\varphi_2 : p^{-1}(U \times [t_1, t_2]) \simeq V \times U \times [t_1, t_2]$
$\varphi_2' \quad \varphi_2'|_{p^{-1}(U \times \{t_1\})} = \varphi_1|_{p^{-1}(U \times \{t_1\})}$
$\Rightarrow$ I can glue them together & get $\varphi: p^{-1}(U \times [0, t_2]) \simeq V \times U \times [0, t_2]$.

Step 1: $\forall x \in X$ $\exists U$ nbhd of x in X & trivializations of $E|_{U \times [0,1]}$
$\Rightarrow \{U_i\}_{i \in I}$ cov. of X & triv. of $E|_{U_i \times [0,1]}$.
By the lemma, we can find $\{V_n\}_{n \geq 0}$ count. cover of X & triv. $\varphi_n: E|_{V_n \times [0,1]} \simeq V_n \times U_n \times [0,1]$.
Let $\{\psi_n\}_{n \geq 0}$ a part. of unity wrt. to this cover & let $\Psi_n = \psi_0 + \dots + \psi_n$.
$0 \leq \Psi_0 \leq \Psi_1 \leq \dots \leq 1$
Want an iso $E \simeq E' \times [0,1]$ over $X \times [0,1]$. So we want $E \to E'$ over X which is an iso on fibers. $E_{(x,t)} \xrightarrow{\sim} E'_x$ $\forall t, x$.

x_n

$X_n = \{(x,t) \in X \times [0,1] \mid t \leq \Psi_n x\} \subseteq X \times [0,1]$
Locally finite closed cover of $X \times [0,1]$.
$f_n: E|_{X_n} \to E'$ w/ the above property $\forall n$
$X_{-1} = X \times \{0\}$ $\quad E|_{X_{-1}} = E'$ $\quad f_{-1} := \mathrm{id}$.
Suppose we have f_n, we need to construct f_{n+1}.
- The inclusion $X_n \hookrightarrow X_{n+1}$ has a retraction $X_{n+1} \to X_n$ over X
$$(x,t) \mapsto (x, \min(\Psi_n x, t))$$
- $X_{n+1} \setminus X_n \subseteq V_n \times [0,1]$
(x,t) $\Psi_{n+1} x \geq t > \Psi_n x \Rightarrow \psi_{n+1} x \neq 0 \Rightarrow x \in V_{n+1}$

We have a triv. $E|_{V_{n+1} \times [0,1]} \simeq V \times \underbrace{V_{n+1} \times [0,1]}$ $\to E|_{V_{n+1}} \xrightarrow{r_{n+1}} E|_{V_n}$
retraction above $X_{n+1} \to X_n$ $\quad \Rightarrow E|_{X_{n+1}} \xrightarrow{r_{n+1}} E|_{X_n}$
We let f_{n+1} to be the composite
$$E|_{X_{n+1}} \to E|_{X_n} \xrightarrow{f_n} E' \quad . \ \square$$

$$\mathbb{C}^\infty = \bigcup_n \mathbb{C}^n = \{(x_n) \in \mathbb{C}^{\mathbb{N}} \mid x_n = 0 \text{ for almost all } n\}$$

Lemma Let $p: E \to X$ v. bundle, X pct Hausdorff $\exists f: E \to \mathbb{C}^\infty$ $\mathbb{C}$-linear and injective on each fiber. Moreover, if X compact we can find $f: E \to \mathbb{C}^N \subseteq \mathbb{C}^\infty$.

Proof: Let $\{U_n\}_{n \geq 0}$ be a countable triv. of E. Then $\forall n$ we have
$f_n: E|_{U_n} \to \mathbb{C}^{d_n}$ iso on each fiber of E. Let ψ_n part. of unity.
$$f: E \to \mathbb{C}^\infty \qquad fe = ((\psi_0 pe) \cdot f_0 e, (\psi_1 pe) \cdot f_1 e, \dots) \in \mathbb{C}^{d_0 + d_1 + \dots} \subseteq \mathbb{C}^\infty.$$
If X compact, I can choose finitely many U_i's. □
When X compact
$$\begin{tikzcd} E \arrow[rr, hook] \arrow[dr] & & X \times \mathbb{C}^N \arrow[dl] \\ & X & \end{tikzcd}$$

Exercise: If $Z \hookrightarrow X$ closed cofibration, you can choose f as above agreeing w/ a given $f_0: E|_Z \to \mathbb{C}^\infty$.

Let $M_\infty(\mathbb{C}) \subseteq \prod_{\mathbb{N} \times \mathbb{N}} \mathbb{C}$ be the space of $\mathbb{C}$-lin maps $\mathbb{C}^\infty \to \mathbb{C}^\infty$.
$\forall A \in M_\infty(\mathbb{C})$, $A^H \in M_\infty(\mathbb{C})$ (not always, but we'll write it only in this case).

Def: $\mathrm{Gr}_d := \{P \in M_\infty(\mathbb{C}) \mid \underset{(\mathrm{rk} P)}{\overset{\|}{\mathrm{rk}\, P}} = d, P^2 = P, P^H = P\}$
We think of this as the space of d-dim subspaces of $\mathbb{C}^\infty$.

Exercise: Let $V \in \mathrm{Gr}_d$ (i.e. a d-dim subspace of $\mathbb{C}^\infty$). We have a map
$$\mathrm{Hom}_{\mathbb{C}}(V, V^\perp) \to \mathrm{Gr}_d$$
$$f \longmapsto \text{graph of } f \text{ in } V \oplus V^\perp \simeq \mathbb{C}^\infty := \{(x, fx) \in V \oplus V^\perp \mid x \in V\}$$
Show that this is an open embedding, w/ image those $W \in \mathrm{Gr}_d$ s.t. $W \overset{P_V}{\twoheadrightarrow} V$.
$(\mathrm{rk}(P_V P_W) \geq d)$

Def: The tautological bundle $\xi_d \to \mathrm{Gr}_d$ is the space $\xi_d \subseteq \mathbb{C}^\infty \times \mathrm{Gr}_d$
$\xi_d = \{(v, V) \mid v \in V\}$. This is a v. bundle trivialized by the st. homs
$\xi_d|_{\{V\}} \simeq V$. $\quad \xi_d|_{\mathrm{Hom}(V, V^\perp)} \simeq V \times \mathrm{Hom}(V, V^\perp)$
$(\Gamma_f, (v, fv)) \longleftarrow\!\mapsto (v, f)$

Thm: The map
$$[X, \mathrm{Gr}_d] \to \pi_0 \mathrm{Vect}_d(X)$$
$$f \longmapsto f^* \xi_d$$
is a bijection.

Proof: This is well-defined by htpy invariance $(f \sim g \Rightarrow f^*\xi_d \simeq g^*\xi_d)$
Let $E \to X$ v. bundle. What does it mean to give $E \simeq f^*\xi_d$?
$f^*\xi_d \simeq \xi_d \times_X \mathrm{Gr}_d$, so a map $E \to f^*\xi_d$ is
$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[d] & \xi_d \subseteq \mathbb{C}^\infty \times \mathrm{Gr}_d \arrow[d] \arrow[dl] \\ X \arrow[r, "f"] & \mathrm{Gr}_d \end{tikzcd}$$
$\tilde{f}|_{E_x}$ inj.
So, giving φ is the same as giving $\tilde{f}: E \to \mathbb{C}^\infty$ s.t. $\tilde{f}(E_x) = fx$.
So a pair $(f, E \simeq f^*\xi_d) \leftrightarrow$ a map $\tilde{f}: E \to \mathbb{C}^\infty$ $\mathbb{C}$-lin embedding on each fiber.
$\Rightarrow [X, \mathrm{Gr}_d] \to \pi_0 \mathrm{Vect}(X)$ is surjective ✓
Injectivity boils down to show that if we have $\tilde{f}, \tilde{g}: E \to \mathbb{C}^\infty$, then they are htpic.
(You see this as extending a map $E \times |\partial \Delta^1| \to \mathbb{C}^\infty$ to $E \times |\Delta^1| \to \mathbb{C}^\infty$)
First of all we have a hpy through embedding
$$H: [0,1] \times \mathbb{C}^\infty \to \mathbb{C}^\infty$$
$$(t, x_\cdot) \longmapsto t(x_0, x_1, \dots) + (1-t)(x_0, 0, x_1, 0, \dots)$$
$$H': [0,1] \times \mathbb{C}^\infty \to \mathbb{C}^\infty$$
$$(t, x_\cdot) \longmapsto t(x_0, x_1, \dots) + (1-t)(0, x_0, 0, x_1, \dots)$$
By taking $H\tilde{f}$, $H'\tilde{g}$ we can ensure $\tilde{f}$ has values only on the ev. numb coord. & $\tilde{g}$ has values only on the odd numb coordinates.
But then $t\tilde{f} + (1-t)\tilde{g}$ is the htpy we were looking for. □

We have $\pi_0 \mathrm{Vect}_d(X) \simeq [X, \mathrm{Gr}_d]$
Exercise: $\pi_0 \mathrm{Vect}(X) = [X, \coprod_d \mathrm{Gr}_d]$
Thm: $\mathrm{Gr}_d \simeq BU_d$ $(\leftrightarrow \Omega \mathrm{Gr}_d \simeq U_d)$

Lemma: Let G be a top. group, $p: E \to B$ surj. fibration w/ E contractible. Assume $G \curvearrowright E$ over B i.e.
$$\begin{tikzcd} E \arrow[rr, "g"] \arrow[dr, "p"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$
& that the action is simply transitive on each fiber
$$G \times E \to E \times_B E \text{ is a homeom.}$$
$$(g, e) \mapsto (ge, e)$$
Then $\exists$ an eq. of E_1-spaces $G \simeq \Omega_b B$

Remark: If we wanted only an eq. of spaces, it would be easy
$G \simeq E_b$ by picking $e \in E_b$ $\quad G \to E_b$, $g \mapsto ge$
$$\Rightarrow \begin{tikzcd} G \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \sim * \arrow[d, "p"] \\ \{b\} \arrow[r] & B \end{tikzcd}$$
pullback in Top, but it's a htpy pullback $\Rightarrow G \simeq \Omega_b B$.

Proof:
$$(\Omega_b B)([n]) = \lim \begin{pmatrix} & \{b\} \cdots \{b\} \\ \{b\} & \to B \end{pmatrix} \simeq \{b\} \times^h_B \{b\} \times^h_B \cdots \times^h_B \{b\}$$
$$= E \times_B E \times_B \cdots \times_B E$$
By ind, using $G \times E \simeq E \times_B E$ we see $(\Omega_b B)([n]) = G \times G \times \dots \times G \times E \simeq \underbrace{G \times \dots \times G}_{G([n])}$ □

Next time we'll constr. a fiber bundle $V_d \to \mathrm{Gr}_d$ & a $U_d \curvearrowright V_d$.
Stiefel manifold
iso embeddings
$f \longmapsto \mathrm{Im} f$
$$V_d = \mathrm{Emb}(\mathbb{C}^d, \mathbb{C}^\infty) \to \mathrm{Gr}_d \qquad d = 1 \quad V_1 = S^\infty$$
$$f \longmapsto f f^H: \mathbb{C}^\infty \to \mathbb{C}^\infty$$
$U_d \curvearrowright V_d$ by precomposition.