HSMT Lecture 9

2020-12-03

§ Associative monoids

$T \in \Delta \qquad Gap(T) = \{(t,t') \mid t < t', \nexists\, t < s < t'\}$

<u>Remark:</u> $Gap(T)$ has a canonical order $(t,t') < (s,s') \iff t < s$

$T = \{0 < 1 < 2 < \dots < n\}$

$\forall (t,t') \in Gap \quad [1] \xrightarrow{g_{(t,t')}} T \qquad g_{(t,t')}(0) = t, \; g_{(t,t')}(1) = t'$

$g_{(i-1,i)} =: g_i \qquad T = [n].$

Def: Let $\mathcal{C}$ be an ∞-cat w/ all finite products. Then an <u>associative monoid</u> is a functor

$$M : \Delta^{op} \to \mathcal{C}$$

s.t. $\forall T \in \Delta$

$$M(T) \xrightarrow[\sim]{\prod M(g_{(t,t')})} \prod_{(t,t') \in Gap(T)} M([1])$$

is an equivalence. An associative monoid in Space is called an E_1-space.

Def: Let $\mathcal{C}$ be a category w/ finite products, a <u>classical monoid</u> is a triple $(M, \eta : * \to M, \mu : M \times M \to M)$ s.t.

- $\mu \circ (\eta, id_M) = \mu \circ (id_M, \eta) = id_M$ [unitality]
- $\mu \circ (\mu, id_M) = \mu \circ (id_M, \mu)$ [associativity]

A map of classical monoids is $f : M \to M'$ s.t. $\eta' = f \circ \eta$, $\mu' \circ (f,f) = f \circ \mu$.

Prop: If $\mathcal{C}$ is a cat. there's an equivalence

$$Mon(\mathcal{C}) \simeq Mon_{cl}(\mathcal{C})$$

<u>Proof (sketch):</u> If M classical monoid, T finite totally ordered set

$\mu_T : M^T \to M$

as follows: $\mu_\emptyset : M^\emptyset = * \xrightarrow{\eta} M \qquad \mu_{[0]} = id_M,$

$\mu_T = \mu(id_M, \mu_{T \setminus \max T}) \qquad \mu_{[1]} = \mu, \dots$

$[\; \mu_{[n]}(x_0, \dots, x_n) = x_0 \cdots x_n \;]$

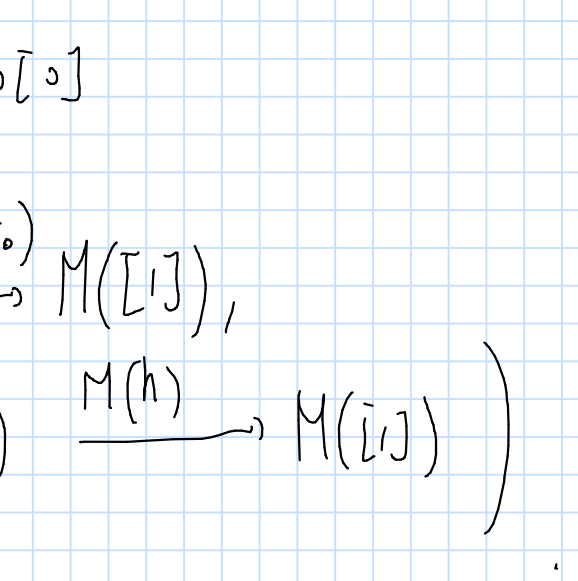
Associativity $\mu_{T \cup S} = \mu(\mu_T, \mu_S)$ where $T \cup S$ has the lexicographic order.

$f : T \to S$ in $\Delta \quad (t,t') \in Gap(T)$

$f(t,t') \subseteq Gap(S)$

$\qquad \| $

$\{(s,s') \mid ft \leq s < s' \leq ft'\}$



<u>Ex:</u> $f : [1] \to [2]$, $0 \mapsto 0$, $1 \mapsto 2$ $\qquad f(0,1) = \{(0,1),(1,2)\}$

$M \in Mon_{cl}(\mathcal{C}) \qquad T \longmapsto M^{Gap(T)}$

$\Delta^{op} \to \mathcal{C} \qquad [f : T \to S] \longmapsto M^{Gap(S)} \to M^{Gap(T)}$

$$M^{Gap(S)} \cong \prod_{(t,t') \in Gap(T)} M^{f(t,t')} \xrightarrow{\mu_{f(t,t')}} \prod_{(t,t') \in Gap(T)} M \qquad \left(Gap(S) = \coprod f(t,t')\right)$$

This gives a functor $Mon_{cl}(\mathcal{C}) \to Mon(\mathcal{C})$.

$Mon(\mathcal{C}) \to Mon_{cl}(\mathcal{C}) \qquad g_0 : [1] \to [0]$

$$[M : \Delta^{op} \to \mathcal{C}] \longmapsto \Big(M([1]),\; \eta : * \simeq M([0]) \xrightarrow{M(g_0)} M([1]),$$
$$\mu : M([1]) \times M([1]) \xrightarrow[(g_1,g_2)^{-1}]{\sim} M([2]) \xrightarrow{M(h)} M([1]) \Big) \qquad \square$$

$h : [1] \to [2]$, $0 \mapsto 0$, $1 \mapsto 2$

<u>Remark:</u> $\pi_0 : Space \to Set$ preserves products $\Rightarrow$ M E_1-space $\Rightarrow$ $\pi_0 M$ monoid in sets.

We say M is <u>group-like</u> or an <u>E_1-group</u> if $\pi_0 M$ is a group.

<u>Exercise:</u> M E_1-space is an E_1-group iff the map

$$M([1]) \times M([1]) \to M([1]) \times M([1]) \qquad (Mg_1, Mh) : M([2]) \to M([1]) \times M([1])$$
$$(x,y) \longmapsto (x, xy)$$

is an equivalence.

<u>Example:</u> Let X be a pointed space, ΩX has a canonical E_1-group structure given by concatenation of loops.

$\Delta^{op} \longrightarrow Space$

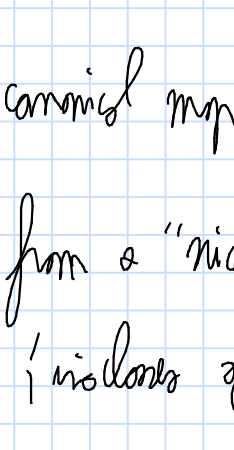
$T \longmapsto \lim_{T^\triangleright} X_T \qquad T^\triangleright = t \to *,\; t' \to * \qquad X_T = (* \to X, \dots)$

$T = [0] \quad \lim(* \to X) = *$

$T = [1] \quad \lim \begin{pmatrix} & * \\ & \downarrow \\ * & \to X \end{pmatrix} \simeq \Omega X$

$T = [2] \quad \lim \begin{pmatrix} & & * \\ & * & \downarrow \\ * & \to & X \end{pmatrix} \simeq \lim \begin{pmatrix} & Map_*(*,X) \\ & \downarrow \\ Map_*(*,X) & \to Map_*(S^0,X) \end{pmatrix}$

$$\simeq Map_*\left(colim \begin{pmatrix} S^0 & \to * \\ \downarrow & \\ * & \end{pmatrix}, X \right) \simeq Map_*(S^1,X) \times Map_*(S^1,X) \simeq \Omega X \times \Omega X$$



$\simeq S^1 \vee S^1$

$S^1 \to S^1 \vee S^1$

$\pi_0 \Omega X = \pi_1 X$ and we know this is a group.

$\Omega : Space_* \to Mon(Space)^{gp}$ ← subcat. of groups

Def: Let $M \in Mon(Space)$, its <u>classifying space</u> BM is just the colimit of M seen as a functor

$$BM := colim_{\Delta^{op}} M$$

It is pointed by the canonical map $M([0]) \simeq * \to colim_{\Delta^{op}} M = BM$

<u>Remark:</u> When M comes from a "nice" topological group G there's a bijection

$[X, BG] \leftrightarrow \{$iso classes of principal G-bundles on $X\}$

The proof follows the same strategy as exercise 2 in the last exercise sheet.

$B : Mon(Space) \to Space_*$. WTS it's the left adjoint to Ω.

$M \in Mon(Space)$, $\exists\, M \to \Omega BM$

$$\begin{array}{ccc} M([1]) & \xrightarrow{M(d_0)} & M([0]) \simeq * \\ \downarrow{\scriptstyle M(d_1)} & & \downarrow \\ M([0]) & \longrightarrow & colim_{\Delta^{op}} M = BM \\ \| & & \\ * & & \end{array} \Rightarrow \text{We get a map } M([1]) \to \Omega BM.$$

$(\Delta^{op})^\triangleright \longrightarrow Space \qquad \forall T \quad T^\triangleright \xrightarrow{D} \Delta$

$[n] \longmapsto M([n]) \qquad t \mapsto [0]$

$\infty \longmapsto BM \qquad \infty \mapsto T$

$t \to \infty \mapsto [0] \xrightarrow{t} T$

$(T^{op \triangleleft})^\triangleright \xrightarrow{D^{op}} (\Delta^{op})^\triangleright \xrightarrow{M} Space$

$M(T) \xrightarrow{M([0] \xrightarrow{t} T)} M([0]), \dots, M([0]) \to BM \qquad M(T) \to (\Omega BM)(T)$

So this gives a map $\boxed{\eta : M \to \Omega BM}$ in $Mon(Space)$.

$\varepsilon : B\Omega X \to X$

$$\| $$
$$colim\left(\lim \begin{pmatrix} & * \\ * & \to X \end{pmatrix}\right) \to colim\left(\lim \begin{pmatrix} & X \\ & \downarrow id \\ X & \xrightarrow{id} X \end{pmatrix}\right) \simeq X$$

$\Omega X \longrightarrow X$

$\gamma : S^1 \to X \longmapsto \gamma(-1)$

<u>Lemma:</u> ε, η satisfy the triangular identities for an adjunction

$BM \xrightarrow{\eta_{BM}} B\Omega BM \xrightarrow{B\varepsilon_M} BM$ (composite $= id$) $\qquad \Omega X \to \Omega B \Omega X \to \Omega X$

<u>Proof:</u> $BM = colim_{[n] \in \Delta^{op}} M([n]) \to colim_{[n] \in \Delta^{op}} \lim \begin{pmatrix} & * \\ * & \to BM \end{pmatrix} \to colim_{[n] \in \Delta^{op}} \lim \begin{pmatrix} & BM \\ & \| \\ BM & = BM \end{pmatrix}$

Unwrapping this is

$colim_{[n] \in \Delta^{op}} M([n]) \to colim_{[n] \in \Delta^{op}} BM$ & this is the identity by def. of colim

$BM \to |\Delta^{op}| \times BM$ ($|\Delta^{op}| = *$) $\qquad \square$

[Boardman-Vogt, May, Segal]

<u>Thm (Recognition thm for loopspaces):</u> Let M be an E_1-space, $X \in Space_*$

① $\eta : M \to \Omega BM$ is an equivalence iff M is an E_1-group

② BM is connected

③ $\varepsilon : B\Omega X \to X$ is equivalent to the inclusion of the connected component of the basepoint. In particular ε is an equivalence iff X connected

$\Rightarrow$ The adjunction $B \dashv \Omega$ restricts to an eq. $Space_*^{\geq 0} \simeq Mon(Space)^{gp}$.

<u>Corollary:</u> $M \mapsto \Omega BM$ is the left adjoint to the inclusion $Mon(Space)^{gp} \subseteq Mon(Space)$

It's called <u>group completion</u> $M^{gp} := \Omega BM$.

Def: $F, G : I \to \mathcal{C}$ two diagrams, $\alpha : F \Rightarrow G$ nat. transformation. We say α is <u>cartesian</u> if $\forall$ arrow $i \to j$ in I the square

$$\begin{array}{ccc} F_i & \to & F_j \\ \downarrow{\scriptstyle \alpha_i} & & \downarrow{\scriptstyle \alpha_j} \\ G_i & \to & G_j \end{array}$$

is a pullback square.

<u>Theorem (Descent property for colimits):</u> Let I be a small cat. $F, G : I^\triangleright \to Space$ two diagrams $\alpha : F \Rightarrow G$ nat. transformation s.t. $\alpha|_I$ is cartesian. Suppose G is a colimit diagram. Then α is cartesian iff F is a colimit diagram

In particular $colim\, F \times_{colim\, G} G_i \simeq F_i$.

<u>Remark:</u> Via categorical reformulations, this property is eq. to

$$Space_{/colim\, G} \simeq \lim_I Space_{/G_i}$$

["colimits have the van Kampen property"]. Then this follows immediately from the straightening/unst. thm.

$$Fun(colim_I G, Space) \overset{?}{\simeq} \lim_I Fun(G_i, Space)$$

<u>Lemma:</u> $Y : \Delta^{op} \to \mathcal{C}$. Then there's an equivalence

$$Y([0]) \simeq colim_{[n]} Y([n+1]) \qquad ([n+1] = [n] \cup \{\infty\})$$

<u>Proof:</u> $\Delta_{+\infty}$ = cat. of finite non-empty totally ord. set. & monotone maps preserving the maximum

$\cap$

Δ

<u>Rmk:</u> $\Delta_{+\infty} \subseteq \Delta$ has a left adjoint $T \mapsto T \cup \{+\infty\}$

$\Rightarrow$ This left adjoint is cofinal $\qquad X_1 \rightrightarrows X_0 \to X_{-1} \qquad A \to B \to C$

$$colim_{T \in \Delta^{op}} Y(T \cup \{+\infty\}) \simeq colim_{T \in \Delta^{op}_{+\infty}} Y(T) \simeq Y([0]) \qquad \square$$

($\Delta^{op}_{+\infty}$ has a term. obj $[0]$)

Prop: $M \in Mon(Space)$, $\eta : M \to \Omega BM$ is an eq. iff M is an E_1-group.

<u>Proof:</u> If η is an eq. $\pi_0 \eta : \pi_0 M \simeq \pi_1 BM$, so M is a group.

To do the other direction, note that η can be checked to be an eq. on underlying spaces. So we need to check

$$\begin{array}{ccc} M([1]) & \xrightarrow{M(d_0)} & M([0]) \\ \downarrow{\scriptstyle M(d_1)} & & \downarrow \\ M([0]) & \to & BM \end{array} \quad \text{is cartesian.}$$

To do so, let us introduce an auxiliary simplicial space

$PM : [n] \mapsto M([n+1] = [n] \cup \{\infty\})$

$\alpha : PM \to M$ induced by the inclusion $[n] \hookrightarrow [n] \cup \{\infty\}$

<u>Claim:</u> If M is a group, α is cartesian.

With this claim we're done: then by descent

$$\begin{array}{ccc} M[1] = PM([0]) & \xrightarrow{d_0} & colim\, PM \simeq M[0] \\ d_1 \downarrow & \lrcorner & \downarrow \\ M([0]) & \to & colim\, M = BM \end{array}$$

(extra degeneracy)

Let $f : [n] \to [m]$ map in Δ

$$\begin{array}{ccc} PM([m]) & \to & PM([n]) \\ \| & & \| \\ M([m+1]) & \to & M([n+1]) \\ \downarrow & & \downarrow \\ M([m]) & \to & M([n]) \end{array} \qquad \begin{array}{ccc} (m_1, \dots, m_{m+1}) & \longmapsto & (\dots, \prod_{i \geq fn} m_i) = (m'_1, \dots, m'_{n+1}) \\ \downarrow & & \downarrow \\ (m_1, \dots, m_m) & \to & (m'_1, \dots, m'_n) \end{array}$$

$[n] = [n-1] \cup \{n\}$

$[m] = [fn-1] \cup \{fn, \dots, m\}$

$$\begin{array}{ccc} M([fn-1]) \times M(\{fn, \dots, m, m+1\}) & \xrightarrow{(\sim, \sim)} & M([n-1]) \times M(n, n+1) \\ \downarrow{\scriptstyle (id, -)} & & \downarrow{\scriptstyle (id, -)} \\ M([fn-1]) \times M(\{fn, \dots, m\}) & \xrightarrow{(\sim, -)} & M([n-1]) \times M(n) \end{array}$$

I can decompose this square as the product of two squares, so by induction I can assume $n = 0$

$f : [0] \to [m] \iff i \in \{0, \dots, m\}$

$$\begin{array}{ccc} M([m+1]) & \longrightarrow & M([1]) \\ \downarrow & \lrcorner & \downarrow \\ M([m]) & \longrightarrow & M([0]) = * \end{array} \quad \text{\& I want to say this is cartesian}$$

$$M([m+1]) \longrightarrow M([m]) \times M([1])$$
$$\| \qquad\qquad \|$$
$$M([1])^{m+1} \qquad M([1])^m$$

$(x_1, \dots, x_{m+1}) \longmapsto (x_1, \dots, x_m, x_1 \cdots x_{m+1})$

As in the exercise, by induction on m we see this is an eq. if M is a group.

$M([2]) \xrightarrow{\sim} M([1]) \times M([1]) \qquad \square$