Thm (stability): In Sp, if we have a square

$$\begin{array}{ccc} X & \to & Y \\ \downarrow & & \downarrow \\ Z & \to & T \end{array}$$

then it is a pullback square iff it is a pushout square.

We will call such squares exact squares.

Def: Let $\mathcal{C}$ be an ∞-category. An exact sequence is a square

$$\begin{array}{ccc} X' & \to & X \\ \downarrow & & \downarrow \\ 0 & \to & X'' \end{array}$$

that is both pullback and pushout.

Ex: If $\mathcal{C} = Ab$ (more generally any abelian category), an exact sequence is exactly a short exact sequence.

Def: E a spectrum, its nth homotopy group is the group $(n \in \mathbb{Z})$

$$\pi_n E = [\Sigma^n \mathbb{S}, E] = \pi_{m+n} E_m \quad m \geq -n \qquad [\,,\,] = \pi_0 \mathrm{Map}_{Sp}(-,-)$$

$$[\Sigma^n \mathbb{S}, E] = [\Sigma^\infty S^n, \Sigma^\infty E] = \pi_0 \mathrm{Map}_{Space_*}(S^{n+m}, \Omega^\infty \Sigma^m E) = E_m$$

$$E_m = \Omega E_{m+1} \Rightarrow \pi_{m+n} E_m = \pi_{m+n} \Omega E_{m+1} = \pi_{m+1+n} E_{m+1}.$$

$\mathbb{S} = \Sigma^\infty S^0$, $\Sigma^n \mathbb{S} = \Sigma^\infty S^n$ $(n \geq 0)$ $(\Sigma^{-n} = \Omega^n$ for $n<0)$

Remark: $\pi_n E = \pi_2 E_{n+2} \in Ab$.

Ex: $\pi_n HA = \begin{cases} 0 & n \neq 0 \\ A & n = 0 \end{cases} = \pi_{m+n} K(A,m)$

Ex: $X \in Space_*$, $\pi_n \Sigma^\infty X = \begin{cases} 0 & n < 0 \\ \pi_n^s X = \pi_n QX & n \geq 0 \end{cases}$ ($\pi_* \Sigma^\infty X$ stable homotopy)

Lemma (Whitehead theorem): Let $f: E \to F$ be a map of spectra. Then f is an equivalence iff any of $\pi_n f$ is an iso $\forall n \in \mathbb{Z}$.

Proof: $f \Rightarrow ((f_n)_{n \in \mathbb{Z}}, f_n \sim \Omega f_{n+1})$

$\Leftarrow$: It's enough to show f_n is an eq. $\forall n$.

$f_n: \pi_i(E_n, *) \to \pi_i(F_n, *)$ is an iso $\forall i$, $\Rightarrow f_n$ is an eq. after restricting to the conn. component of the basepoint (classical Whitehead).

$\Rightarrow \Omega f_n: \Omega E_n \to \Omega F_n$ is an eq. $\Rightarrow f_{n-1}$ is an eq. $\forall n \in \mathbb{Z}$. But that's what we wanted □

Remark: Let $E' \to E \to E''$ exact sequence of spectra. Then $\exists$ a les of homotopy groups

$$\cdots \to \pi_n E' \to \pi_n E \to \pi_n E'' \to \pi_{n-1} E' \to \cdots$$

That's because if $E' \to E \to E''$ exact sequence $\Rightarrow \Omega^\infty \Sigma^{-n} E' \to \Omega^\infty \Sigma^{-n} E \to \Omega^\infty \Sigma^{-n} E''$ fiber seq. of pointed spaces (Σ^n eq., so preserves exact seq., Ω^∞ right adjoint so preserves pullback squares). Then this comes from the LES of a fibration in homotopy.

$[\pi_n \Omega^\infty \Sigma^{-n} E = \pi_{n-m} E]$

Def: E, F: The mapping spectrum $\mathrm{map}(E,F) \in Sp$ is the spectrum

$$(\{\mathrm{Map}_{Sp}(E, \Sigma^n F)\}_{n \in \mathbb{Z}}, \ \&: \mathrm{Map}_{Sp}(E, \Sigma^n F) \simeq \mathrm{Map}_{Sp}(E, \Omega\Sigma\Sigma^n F) = \Omega \mathrm{Map}_{Sp}(E, \Sigma^{n+1} F))$$

Remark: $E \in Sp$, $X \in Space_*$

$$E^n X := [X, E_n] = \pi_0 \mathrm{Map}_{Space_*}(X, E_n) \simeq \pi_0 \mathrm{Map}_{Sp}(\Sigma^\infty X, \Sigma^n E)$$

$$\simeq \pi_0 \mathrm{map}(\Sigma^\infty X, E)_n = \pi_{-n} \mathrm{map}(\Sigma^\infty X, E)$$

[The technicality proves only that they are the same as pointed sets, but Eckmann-Hilton shows they are the same as abelian groups]

Def: $E, F \in Sp$

$$E^n F := \pi_{-n} \mathrm{map}(F, E)$$

Lemma: $F' \to F \to F''$ exact seq of spectra, we have a les

$$\cdots \to E^{n-1} F' \to E^n F'' \to E^n F \to E^n F' \to E^{n+1} F'' \to \cdots$$

Proof: $\mathrm{map}(-, E): Sp^{op} \to Sp$ is a right adjoint (will follow in a second) so it preserves exact squares $\Rightarrow$ exact sequences (follows from stability)

$$\mathrm{map}(F'', E) \to \mathrm{map}(F, E) \to \mathrm{map}(F', E) \quad \text{exact seq.}$$

Then you take the les in π_*. □

Def: $E, F \in Sp$, the $\otimes$-product (smash product $E \wedge F$)

$$E \otimes F := \underset{n,m \in \mathbb{Z}}{\mathrm{colim}}\ \Omega^{n+m} \Sigma^\infty (E_n \wedge F_m)$$

adjoint to $\Sigma\Sigma^\infty(E_0 \wedge F_0) \simeq \Sigma^\infty(\Sigma E_0 \wedge F_0) \to \Sigma^\infty(E_1 \wedge F_0)$; $\Sigma E_0 \simeq \Sigma\Omega E_1 \to E_1$

$$\begin{array}{ccccc} \Sigma^\infty E_0 \wedge F_0 & \to & \Omega\Sigma^\infty(E_1 \wedge F_0) & \to & \cdots \\ \downarrow & & \downarrow & & \\ \Omega\Sigma^\infty(E_0 \wedge F_1) & \to & \Omega^2 \Sigma^\infty(E_1 \wedge F_1) & \to & \cdots \\ \vdots & & \vdots & & \end{array}$$

Prop: $E, F, K \in Sp$, there's a natural eq.

$$\mathrm{Map}_{Sp}(E \otimes F, K) \simeq \mathrm{Map}_{Sp}(E, \mathrm{map}(F, K))$$

[In Ab: $\mathrm{Hom}(A \otimes B, C) \simeq \mathrm{Hom}(A, \mathrm{Hom}(B, C))$]

Proof:

$$\begin{aligned}
\mathrm{Map}_{Sp}(E \otimes F, K) &\simeq \mathrm{Map}_{Sp}(\underset{n,m}{\mathrm{colim}}\ \Omega^{n+m}\Sigma^\infty(E_n \wedge F_m), K) \\
&\simeq \lim_{n,m} \mathrm{Map}_{Sp}(\Omega^{n+m}\Sigma^\infty(E_n \wedge F_m), K) \\
&\simeq \lim_{n,m} \mathrm{Map}_{Sp}(\Sigma^\infty(E_n \wedge F_m), \Sigma^{n+m} K) \\
&\simeq \lim_{n,m} \mathrm{Map}_{Space_*}(E_n \wedge F_m, \Omega^\infty \Sigma^{n+m} K) \qquad (\Sigma^n K)_m \\
&\simeq \lim_{n,m} \mathrm{Map}_{Space_*}(E_n, \mathrm{Map}_{Space_*}(F_m, \Omega^\infty\Sigma^{n+m} K)) \\
&\simeq \lim_n \mathrm{Map}_{Space_*}(E_n, \underbrace{\lim_m \mathrm{Map}_{Space_*}(F_m, (\Sigma^n K)_m)}_{\mathrm{Map}_{Sp}}) \\
&\simeq \lim_n \mathrm{Map}_{Space_*}(E_n, \underbrace{\mathrm{Map}_{Sp}(F, \Sigma^n K)}_{\mathrm{map}(F,K)_n}) \\
&\simeq \mathrm{Map}_{Sp}(E, \mathrm{map}(F, K)) \quad . \ \square
\end{aligned}$$

$$\mathrm{Map}_{Sp}(E \otimes F, K) \simeq \mathrm{Map}_{Sp}(E, \mathrm{map}(F, K))$$

Corollary: $F = \mathbb{S}$ $\quad \mathrm{map}(\mathbb{S}, K) = K \qquad \mathrm{Map}(\mathbb{S}, \Sigma^n K) \simeq \Omega^\infty \Sigma^n K = K_n$

$\Rightarrow \mathrm{Map}_{Sp}(E \otimes \mathbb{S}, K) \simeq \mathrm{Map}_{Sp}(E, K)$

$\Rightarrow E \otimes \mathbb{S} \simeq E$.

Corollary: $(E \otimes F) \otimes G \simeq \underset{n,m,l}{\mathrm{colim}}\ \Omega^{n+m+l} \Sigma^\infty (E_n \wedge F_m \wedge G_l) \simeq E \otimes (F \otimes G)$

Proof: $\otimes$ commutes w/ colimits in each variable since it's a left adjoint, and then plug in the standard presentations of E, F, G plus the next corollary. □

$$(\mathrm{colim}\ \Omega^n \Sigma^\infty E_n \otimes \mathrm{colim}\ \Omega^m \Sigma^\infty F_m) \otimes \mathrm{colim}\ \Omega^l \Sigma^\infty G_l \simeq \underset{n,m,l}{\mathrm{colim}}\ \Omega^{n+m+l}((\Sigma^\infty E_n \otimes \Sigma^\infty F_m) \otimes \Sigma^\infty G_l)$$

Corollary: $\Sigma^\infty X \otimes \Sigma^\infty Y \simeq \Sigma^\infty (X \wedge Y)$

Proof:

$$\mathrm{Map}_{Sp}(\Sigma^\infty X \otimes \Sigma^\infty Y, E) \simeq \mathrm{Map}_{Sp}(\Sigma^\infty X, \mathrm{map}(\Sigma^\infty Y, E)) \quad (\otimes \dashv \mathrm{map})$$

$$\overset{\Sigma^\infty \dashv \Omega^\infty}{\simeq} \mathrm{Map}_{Space_*}(X, \Omega^\infty \mathrm{map}(\Sigma^\infty Y, E)) \simeq \mathrm{Map}_{Space_*}(X, \mathrm{Map}_{Sp}(\Sigma^\infty Y, E)) \quad \text{(def of map)}$$

$$\overset{\Sigma^\infty \dashv \Omega^\infty}{\simeq} \mathrm{Map}_{Space_*}(X, \mathrm{Map}_{Space_*}(Y, \Omega^\infty E)) \simeq \mathrm{Map}_{Space_*}(X \wedge Y, \Omega^\infty E) \quad (\wedge \dashv \mathrm{Map}_*) \ . \ \square$$

Corollary: $\mathrm{Map}_{Sp}(E, \mathrm{map}(F, G)) \simeq \mathrm{Map}_{Sp}(F, \mathrm{map}(E, G))$ $\quad [\mathrm{map}(-,G) \vdash \mathrm{map}(-,G)^{op}]$, $\ Sp^{op} \to Sp$, $\ Sp \to Sp^{op}$

$$\| \qquad\qquad \|$$

$$\mathrm{Map}_{Sp}(E \otimes F, G) \simeq \mathrm{Map}_{Sp}(F \otimes E, G)$$

Def: $E \in Sp$, $X \in Space_*$

$$E_n X := \pi_n (E \otimes \Sigma^\infty X) \ \ (= \underset{m}{\mathrm{colim}}\ \pi_{n+m}(E_m \wedge X))$$

(plug in the std presentation for E)

Exercise: $E = HA$, $E_n(-)$ satisfies the Eilenberg-Steenrod axioms $\Rightarrow$ it coincides w/ $\tilde{H}_n(-;A)$ on CW complexes.

Def: $E \in Sp$ connective if $\pi_n E = 0$ $\forall n < 0$.

Ex: HA, $\Sigma^\infty X$ are connective

Thm (Recognition theorem): There's an equivalence between connective spectra and E_∞-groups in Space.

Let $T \in \Delta$ nonempty finite totally ordered set $\quad (T = [n] = (0 < 1 < \cdots < n))$

A gap in T is just a pair (t, t') of elements in T s.t. $t < t'$ & $\nexists s$ $t < s < t'$

Ex: $T = 0 < 1 < 2 < \cdots < n \qquad \mathrm{Gap}(T)$ = set of gaps

$\mathrm{Gap}(T) = \{(0,1), (1,2), (2,3), (3,4), \ldots, (n-1,n)\}$

$(t,t') \in \mathrm{Gap}(T)$: $[1] \xrightarrow{g_{(t,t')}} T \qquad 0 \mapsto t, \ 1 \mapsto t'$; $\ [1] = \{0 < 1\}$

$T = [n] \qquad g_i = g_{(i-1,i)} : [1] \to [n] \qquad 1 \leq i \leq n.$

Def: Let $\mathcal{C}$ be an ∞-cat w/ finite products. Then an associative monoid in $\mathcal{C}$ is a functor

$$M: \Delta^{op} \to \mathcal{C}$$

s.t. $\forall [n] \in \Delta$

$$\prod g_i^*: M([n]) \xrightarrow{\sim} \prod_{i=1}^n M([1])$$

is an equivalence. Associative monoids in Space are called E_1-spaces.

Ex: $\mathcal{C} = Set$ (any 1-category), M_0 an associative monoid in the "classical sense".

$m: M_0 \times M_0 \to M_0$, $1: * \to M_0$, satisfying associativity & unitality.

$$M(T) = \{m: \mathrm{Gap}(T) \to M_0\}$$

$f: T \to S$

$$M(S) \to M(T)$$

$$(\alpha: \mathrm{Gap}(S) \to M) \mapsto \mathrm{Gap}(T) \to M$$

$$(t,t') \mapsto \prod_{\substack{(s,s') \in \mathrm{Gap}(S) \\ ft \leq s < s' \leq ft'}} \alpha(s,s') \quad \text{(mult. in the monoid)}$$

$[0] \mapsto *$

$[1] \mapsto M_0$ $\ (0,1)$

$[2] \mapsto M_0 \times M_0$ $\ (0,1) \ (1,2)$

$g: [1] \to [0]$

$g^*: * \to M_0 \iff g^* = 1$

$g: [1] \to [2]$, $0 \mapsto 0$, $1 \mapsto 2$

$g_1: [1] \to [2]$, $0 \mapsto 0$, $1 \mapsto 1$ $\qquad g: [1] \to [2]$

$pr_1: M_0 \times M_1 \to M_1 \qquad pr_2$

$g^*: M_0 \times M_0 \to M_0 \quad (m_1, m_2) \mapsto m_1 m_2$

$$\begin{array}{ccc} M([1]) & \xrightarrow{\sim} & M([1]) \times_{M([0])} \\ \downarrow & & \downarrow \\ M([2]) & \xrightarrow{(g_1^*, g_2^*)} & M([1]) \times M([1]) \\ & g^* \searrow \quad \swarrow m & \\ & M([1]) & \end{array}$$

$[2] \to [1]$: $0 \mapsto 0$, $1 \mapsto 1$, $2 \mapsto 1$

$m \cdot 1 = m$