§1 Filtered colimits

Def: S simplicial set is *filtered* if $\forall K$ finite simplicial set (i.e. finitely many nondegenerate simplices) and every map $K \to S$, we can extend it to $K^{\triangleright} \to S$.

Exercise: P poset, $(N)P$ is filtered if $\forall p, p' \in P$ $\exists q \in P$ $q \geq p$ & $q \geq p'$.

Example: $(\mathbb{N}, \leq)$ and $(\mathbb{Z}, \leq)$ are filtered.

Ex: $\mathcal{C}$ ∞-cat. w/ all finite colimits (e.g. $\mathcal{C} = (N)\mathrm{Fin}$), then $\mathcal{C}$ is filtered.

Property:
① The forgetful functor $\mathrm{Space}_* \to \mathrm{Space}$ preserves filtered colimits
② Filtered colimits in Space commute w/ pullbacks and finite products (HTT. 5.3.3.3)
③ $\pi_0 : \mathrm{Space} \to \mathrm{Set}$ commutes w/ all colimits, because it's the left adjoint to the inclusion $\mathrm{Set} \hookrightarrow \mathrm{Space}$ $\Rightarrow$

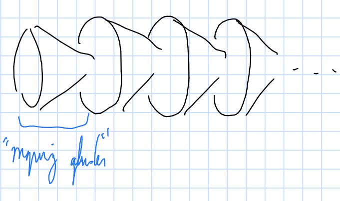
$$\begin{tikzcd} \pi_n : \mathrm{Space}_* \arrow[r] \arrow[d, "\Omega^n"', "②"] & \mathrm{Set} \\ \mathrm{Space}_* \arrow[r, "①"'] & \mathrm{Space} \arrow[u, "\pi_0"'] \end{tikzcd}$$

commutes w/ filtered colimits

colimits indexed by $\mathbb{N}$ can be represented by the mapping telescope in Top

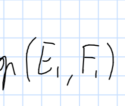
$X : \mathbb{N} \to \mathrm{Top}$ $\qquad X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots$

$$\operatorname{colim} X = \coprod_{i \geq 0} X_i \times [0,1] \Big/ (x,1) \sim (f_i x, 0)$$



$X : I \to \mathrm{Space}_* \xrightarrow{U} \mathrm{Space}$ $\qquad * \to UX$

$$\begin{tikzcd} \operatorname{colim} * \arrow[r] \arrow[d] & \operatorname{colim} UX \arrow[d] \\ * \arrow[r] & \operatorname{colim} X \end{tikzcd}$$



$$\operatorname{colim} X = \operatorname{colim} UX \Big/ \underbrace{\operatorname{colim} *}_{\text{already contractible if } I \text{ filtered}}$$

Recall: Sp : objects are $\{(E_n)_{n \in \mathbb{Z}}, \delta_n : E_n \xrightarrow{\sim} \Omega E_{n+1}\}$

$$\mathrm{Map}_{\mathrm{Sp}}(E,F) = \lim_n \left(\mathrm{Map}(E_0, F_0) \xleftarrow{\Omega} \mathrm{Map}(E_1, F_1) \xleftarrow{\Omega} \dots \right)$$

$\Omega^\infty : \mathrm{Sp} \to \mathrm{Space}_*$ $\quad E \mapsto \Omega^\infty E := E_0$ and it has a left adjoint

$\Sigma^\infty : \mathrm{Space}_* \to \mathrm{Sp}$ $\quad X \mapsto \Sigma^\infty X := (\{Q\Sigma^n X\}_{n \geq 0}, \delta_n : Q\Sigma^n X \xrightarrow{\sim} \Omega Q \Sigma^{n+1} X)$

$Q : \mathrm{Space}_* \to \mathrm{Space}_*$ $\quad QX = \operatorname{colim}_n \Omega^n \Sigma^n X$ $\qquad \pi_i QX = \pi_i^s X$.

$$\mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty X, E) \simeq \mathrm{Map}_{\mathrm{Space}_*}(X, \Omega^\infty E).$$

Proposition: $E : I \to \mathrm{Sp}$ any diagram, $\lim_I E$ exists and it's given by the spectrum
$$\left(\{\lim_I E(i)_n\}, \delta_n : \lim_I E(i)_n \xrightarrow{\sim} \Omega \lim_I E(i)_{n+1}\right) =: \lim_I E$$

Proof: First we need to check that δ_n is an equivalence, but this follows because Ω commutes w/ limits.
$$\Omega \lim_I E(i)_{n+1} \xleftarrow{\simeq} \lim_I \Omega E(i)_{n+1} \xleftarrow[\sim]{\delta_n(i)} \lim_I E(i)_n : \delta_n.$$
Then
$$\mathrm{Map}_{\mathrm{Sp}}(F, \lim_I E) = \lim_n \mathrm{Map}_{\mathrm{Space}_*}(F_n, \lim_I E(i)_n) \simeq$$
$$= \lim_n \lim_I \mathrm{Map}_{\mathrm{Space}_*}(F_n, E(i)_n) \simeq \lim_I \lim_n \mathrm{Map}_{\mathrm{Space}_*}(F_n, E(i)_n)$$
$$= \lim_I \mathrm{Map}_{\mathrm{Sp}}(F, E(i)). \quad \square$$

Prop: Let $E : I \to \mathrm{Sp}$ be a diagram w/ I filtered. Then $\operatorname{colim}_I E$ exists in Sp and it's given by
$$\left(\{\operatorname{colim}_I E(i)_n\}, \delta_n : \operatorname{colim}_I E(i)_n \xrightarrow{\sim} \Omega \operatorname{colim}_I E(i)_{n+1}\right)$$

Proof: Same as above but we're using that Ω commutes w/ filtered colimits. $\square$

Corollary: ΩE is given by the spectrum
$$\left(\{E_{n-1}\}_{n \in \mathbb{Z}}, \delta_{n-1} : E_{n-1} \xrightarrow{\sim} \Omega E_n\right)$$
i.e. it's given by "shifting"

Proof: $\Omega E = (\{\Omega E_n\}, \delta_n : \Omega E_n \xrightarrow{\sim} \Omega^2 E_{n+1})$, but this is given by the formula above. $\square$

$$\left(\begin{tikzcd} \Omega c \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & * \arrow[d] \\ * \arrow[r] & c \end{tikzcd}\right)$$

$\Rightarrow \Omega : \mathrm{Sp} \to \mathrm{Sp}$ is an equivalence w/ inverse
$$E \mapsto \left(\{E_{n+1}\}_{n \in \mathbb{Z}}, \delta_{n+1} : E_{n+1} \xrightarrow{\sim} \Omega E_{n+2}\right)$$
But we know that if Ω has a left adjoint, this is given by Σ
$\Rightarrow \Sigma E$ exists & it's given by the above formula.

Notation: Sometimes we'll write $\Omega^n E$ as $\Sigma^{-n} E$, due to this remark above.

Def: A *prespectrum* is a sequence E_n of pointed spaces and maps
$$\delta_n : \Sigma E_n \to E_{n+1} \qquad (E_n \to \Omega E_{n+1})$$
To a prespectrum we can associate a spectrum by taking the colim
$$E := \operatorname{colim}(\Sigma^\infty E_0 \to \Omega \Sigma^\infty E_1 \to \Omega^2 \Sigma^\infty E_2 \to \dots)$$
adjoint to $\Sigma \Sigma^\infty E_0 = \Sigma^\infty \Sigma E_0 \to \Sigma^\infty E_1$

Ex: If I take the prespectrum $\{\Sigma^n X\}_{n \geq 0}, \Sigma\Sigma^n X \xrightarrow{id} \Sigma^{n+1} X$, the associated spectrum is $\Sigma^\infty X$.

Lemma (standard presentation): Let E be a spectrum. Then the natural map
$$\operatorname{colim}_n \Omega^n \Sigma^\infty E_n \xrightarrow{\sim} E,$$
given by the maps $\Sigma^\infty E_n \to \Sigma^n E$ adjoint to $E_n \xrightarrow{\sim} \Omega^\infty \Sigma^n E$, is an equivalence

Proof:
$$\mathrm{Map}_{\mathrm{Sp}}(\operatorname{colim}_n \Omega^n \Sigma^\infty E_n, F) =$$
Ω, Σ inverse equivalences
$$= \lim_n \mathrm{Map}_{\mathrm{Sp}}(\Omega^n \Sigma^\infty E_n, F) = \lim_n \mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty E_n, \Sigma^n F)$$
$\Sigma^\infty \dashv \Omega^\infty$
$$= \lim_n \mathrm{Map}_{\mathrm{Space}_*}(E_n, \Omega^\infty \Sigma^n F) \simeq \lim_n \mathrm{Map}_{\mathrm{Space}_*}(E_n, F_n) =: \mathrm{Map}_{\mathrm{Sp}}(E,F). \square$$

Corollary: Sp has all colimits

Proof: $E : I \to \mathrm{Sp}$, $\{\operatorname{colim}_I E(i)_n\}_n$, $\delta_n : \Sigma \operatorname{colim}_I E(i)_n \to \operatorname{colim}_I E(i)_{n+1}$

This is not a spectrum: it is just a prespectrum. But we can consider the spectrum presented by it:
$$\operatorname{colim}_I E = \operatorname{colim}_n \Omega^n \Sigma^\infty \operatorname{colim}_I E(i)_n$$
$$\mathrm{Map}_{\mathrm{Sp}}(\operatorname{colim}_I E, F) = \lim_n \mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty \operatorname{colim}_I E(i)_n, \Sigma^n F)$$
$$= \lim_n \lim_I \mathrm{Map}_{\mathrm{Space}_*}(E(i)_n, \Omega^\infty \Sigma^n F) = \lim_I \mathrm{Map}_{\mathrm{Sp}}(E(i), F). \quad \square$$

Prop: Let us consider a square in Sp
$$(\star) \quad \begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_1 \arrow[d] \\ X_2 \arrow[r] & X_{12} \end{tikzcd}$$
Then $(\star)$ is a pushout square iff $(\star)$ is a pullback square. [Stability]

Corollary: Let E, F two spectra. Then the natural map
$$E \amalg F \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} E \times F$$
is an equivalence (Sp is semiadditive). We will write $E \oplus F$ for this.

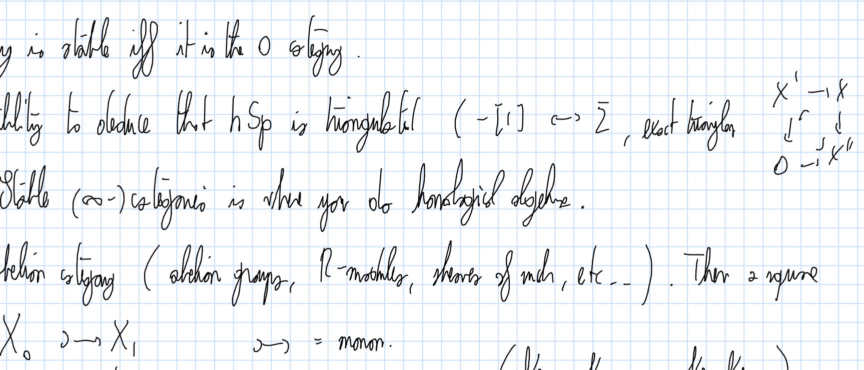
Proof: Let us consider the squares
$$\begin{tikzcd} 0 \arrow[r] \arrow[d, "\wr"'] & E \arrow[d, "\wr"] \\ 0 \arrow[r] & E \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} 0 \arrow[r, "\sim"] \arrow[d] & 0 \arrow[d] \\ F \arrow[r, "\sim"] & F \end{tikzcd}$$
These are pullback squares $\Rightarrow$ their product is a pullback square
$$\begin{tikzcd} 0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & E \arrow[d] \\ F \arrow[r] & E \times F \end{tikzcd} \qquad E \amalg_0 F = E \amalg F \to E \times F.$$
But by stability, it is a pushout square $\square$

Proof of stability: pushout $\Rightarrow$ pullback

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d] & 0 \arrow[d] & \\ X_2 \arrow[r] \arrow[d] & X_{12} \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Z \arrow[r] \arrow[d] & \Sigma X_0 \arrow[r] & \Sigma X_2 \arrow[d] \\ & 0 \arrow[r] & \Sigma X_1 \arrow[r] & \Sigma X_{12} \end{tikzcd}$$

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "\wr"'] & X_1 \times_{X_{12}} X_2 \arrow[d] \\ \Omega\Sigma X_0 \arrow[r] & \Omega\Sigma X_1 \times_{\Omega\Sigma X_{12}} \Omega\Sigma X_2 \end{tikzcd}$$

has both a left & a right inverse $\Rightarrow$ it's an eq.
$\Rightarrow$ all arrows are equivalences.



$\square$

Ex: A 1-category is stable iff it is the 0 category.

Ex: Use stability to deduce that $h\mathrm{Sp}$ is triangulated ($-[1] \leftrightarrow \Sigma$, exact triangles $\begin{tikzcd} X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d] \\ 0 \arrow[r] & X'' \end{tikzcd}$)

Slogan: Stable (∞-)categories is where you do homological algebra.

Remark: $\mathcal{A}$ abelian category (abelian groups, R-modules, sheaves of mod, etc...). Then a square
$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d, two heads] & X_1 \arrow[d, two heads] \\ X_2 \arrow[r, hook] & X_{12} \end{tikzcd} \qquad \begin{array}{l} \hookrightarrow = \text{monom.} \\ \twoheadrightarrow \text{ epim.} \end{array} \qquad (\ker \operatorname{coker} = \operatorname{coker} \ker)$$
is pullback iff it is pushout.