ISHT Lecture 6 2020-11-23

Thm (Brown representability): Let $h\,\mathrm{Space}_*^{\geq 0}$ be the homotopy category of pointed connected spaces, then a functor

$$F : (h\,\mathrm{Space}_*^{\geq 0})^{op} \longrightarrow \mathrm{Set}$$

is representable if and only if it satisfies the following two conditions

① $\forall \{X_\alpha\}_{\alpha \in A}$ collection of pd. conn. spaces

$$F\Big(\bigvee_{\alpha\in A} X_\alpha\Big) \xrightarrow{\sim} \prod_{\alpha\in A} F(X_\alpha)$$

② $\forall$ pushout square in $\mathrm{Space}_*^{\geq 0}$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d, "j"'] \\ X' \arrow[r] & Y' \end{tikzcd}$$

the map $F(Y') \longrightarrow F(X') \times_{F(X)} F(Y)$ is surjective.

Proof: $F = [-, Z]_*$ let us show it satisfies ① & ②.

① $$\Big[\bigvee_{\alpha\in A} X_\alpha, Z\Big]_* \xrightarrow{\sim} \prod_{\alpha \in A} [X_\alpha, Z]$$

because $\bigvee$ is the coproduct (π_0 respects product)

② $$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "j"'] \arrow[dr, phantom, "\overset{H}{\Swarrow}"] & X' \arrow[d] \\ Y \arrow[r, "J"] & Y' \end{tikzcd}$$

$$[Y', Z]_* = \pi_0 \mathrm{Map}_*(Y', Z) \simeq \pi_0\Big(\mathrm{Map}_*(X', Z) \times_{\mathrm{Map}_*(X,Z)} \mathrm{Map}(Y, Z)\Big)$$

$$= \text{“homotopy classes of triples } (f : X' \to Z,\ g : Y \to Z,\ k : fi \sim gj)\text{”}$$

$$[X', Z]_* \times_{[X,Z]_*} [Y, Z]_* = \text{pairs } \big([f], [g], \underbrace{[fi] = [gj]}_{\text{a } k \text{ exists}}\big)$$

$\Rightarrow$ the map in ② is surjective.

Other direction: We have $F : (h\,\mathrm{Space}_*^{\geq 0})^{op} \to \mathrm{Set}$ satisfying ① & ②

We want Z, $\xi \in F(Z)$ inducing $[-, Z]_* \xrightarrow{\sim} F(-)$.

Claim: $\forall X$ connected pd space, $\forall \xi \in FX$ $\exists f_X : X \to Z_X$, $\tilde{\xi} \in F(Z_X)$ s.t.

- $f_X^* \tilde{\xi} = \xi$
- $\forall m \geq 1$ $$\begin{matrix} g & \longmapsto & g^*\tilde{\xi} \\ [S^m, Z_X]_* & \xrightarrow{\sim} & F(S^m) \end{matrix}$$ is a bijection.

$Z = Z_*$ (i.e. we apply the claim for $X = *$, $FX \overset{\text{by ①}}{=} * \ni \xi$), $\tilde{\xi} \in FZ$

Want to show $\forall X$

$$\begin{matrix} [X, Z]_* & \xrightarrow{\sim} & FX \\ \rotatebox{90}{$\in$}\ g & \longmapsto & g^*\tilde{\xi} \end{matrix}$$

We know this for when $X = S^m$, $m \geq 1$. Let us show that it is surjective

Let's take $\eta \in FX$, let's consider $X' = X \vee Z$

$$F(X') \overset{①}{=} F(X) \times F(Z) \ni (\eta, \tilde{\xi})$$

$\Rightarrow \exists Z'$ $g' : X' \to Z'$, $\tilde{\xi}' \in F(Z')$ $(g')^* \tilde{\xi}' = (\eta, \tilde{\xi})$

and $\tilde{\xi}' : [S^m, Z']_* \xrightarrow{\sim} F(S^m)$

We have a map $Z \hookrightarrow X' = X \vee Z \longrightarrow Z'$ pulling $\tilde{\xi}'$ to $\tilde{\xi}$. Therefore we have a comm. diagram

$$\begin{tikzcd} \pi_m Z = {[S^m, Z]_*} \arrow[rr] \arrow[dr, "\wr"'] & & {[S^m, Z')_*} = \pi_m Z' \arrow[dl, "\wr"] \\ & F(S^m) & \end{tikzcd}$$

$\Rightarrow$ by Whitehead $Z \to Z'$ is also an equivalence $\Rightarrow$ the map

$$X \hookrightarrow X' = X \vee Z \longrightarrow Z' \simeq Z$$

is the g we wanted (i.e. $g^*\tilde{\xi} = (i_1^*)(g'^*)(\tilde{\xi}') = \eta$)

Injectivity: $f, g : X \to Z$ $f^*\tilde{\xi} = g^*\tilde{\xi} = \eta$

$$\begin{tikzcd} X \vee X \arrow[r, "f \vee g"] \arrow[d, "\nabla"] & Z \arrow[d, "\partial"] \\ X \arrow[r, "b"'] & Y \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

$\Rightarrow \psi \in F(Y) \overset{②}{\longrightarrow\!\!\!\!\!\to} FX \times_{F(X)\times F(X)} FZ \ni (\eta, \tilde{\xi})$. Apply Claim to (Y, ψ)

$\partial^*\psi = \tilde{\xi}$, $b^*\psi = \eta \Rightarrow \exists\ h : Y \longrightarrow Z'$, $\psi' \in FZ'$ s.t. $h^*\psi' = \psi$,

$\forall m \geq 0$ $\psi' : [S^m, Z']_* \xrightarrow{\sim} F(S^m)$.

WTS $Z \xrightarrow{\partial} Y \xrightarrow{h} Z'$ is an equivalence $(h\partial)^*\psi' = \tilde{\xi}$, so we have a com. diag

$$\begin{tikzcd} \pi_m Z = {[S^m, Z]_*} \arrow[rr, "\sim"] \arrow[dr, "\wr"', "\tilde{\xi}"] & & \pi_m Z' = {[S^m, Z']_*} \arrow[dl, "\wr", "\psi'"'] \\ & FS^m & \end{tikzcd}$$

$\Rightarrow$ But, by definition $\partial f \sim \partial g \Rightarrow \overbrace{h\partial}^{\text{equivalence}} f \sim \overbrace{h\partial}^{} g \Rightarrow f \sim g$.

Claim: $\forall X$ connected pd space, $\forall \xi \in FX$ $\exists f_X : X \to Z_X$, $\tilde{\xi} \in F(Z_X)$ s.t.

- $f_X^* \tilde{\xi} = \xi$
- $\forall m \geq 1$ $$\begin{matrix} g & \longmapsto & g^*\tilde{\xi} \\ \tilde{\xi} : [S^m, Z_X]_* & \xrightarrow{\sim} & F(S^m) \end{matrix}$$ is a bijection.

First step: build (Z_0, ξ_0) s.t. ξ_0 is surjective.

$$Z_0 = X \vee \bigvee_{\gamma \in FS^m,\, m \geq 1} S^m \overset{①}{\Rightarrow} F(Z) = F(X) \times \prod_\gamma F(S^m)$$

$\xi_0 = (\xi, \{\gamma\}_\gamma)$: $\qquad f_0 : X \to Z_0$ (incl. of the summand)

$$\xi_0 : [S^n, Z_0]_* \longrightarrow F(S^n)$$

onto $\forall m \geq 1$.

To fix it, we will construct a sequence of spaces

$$Z_0 \xrightarrow{z_0} Z_1 \xrightarrow{z_1} Z_2 \longrightarrow \cdots$$

$$X \longrightarrow *$$

$$0 : * = F(*) \longrightarrow FX$$

$\xi_i \in F(Z_i)$ s.t. $z_i^* \xi_{i+1} = \xi_i$

Suppose we constructed Z_i, let $K_i(-) = \ker\left(\xi_i : [-, Z_i]_* \to F(-)\right)$

$$\begin{tikzcd}
\bigvee_{\gamma \in K_i(S^m),\, m \geq 1} S^m \arrow[r] \arrow[d] & * \arrow[d] \\
Z_i \arrow[r, "z_{i+1}"] & Z_{i+1}
\end{tikzcd}
\qquad
\begin{tikzcd}
F(Z_{i+1}) \arrow[r] \arrow[d] & \{0\} \arrow[d] \\
F(Z_i) \arrow[r] & \prod_\gamma F(S^m) \\
\xi_i \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & 0
\end{tikzcd}$$

$$F(Z_{i+1}) \overset{②}{\twoheadrightarrow} F(Z_i) \underset{F(\bigvee S^m)}{\times} F(*) \ni (\xi_i, 0)$$

$$\begin{tikzcd}
F(Z_{i+1}) \\
\xi_{i+1} \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

$$\begin{tikzcd}
\xi \arrow[d, phantom, sloped, "\in"] & \xi_0 \arrow[d, phantom, sloped, "\in"] & \xi_1 \arrow[d, phantom, sloped, "\in"] & \xi_2 \arrow[d, phantom, sloped, "\in"] & \\
F(X) & F(Z_0) \arrow[l] & F(Z_1) \arrow[l, "z_0^*"'] & F(Z_2) \arrow[l, "z_1^*"'] & \cdots \arrow[l]
\end{tikzcd}$$

$(\, Z = \operatorname{colim} Z_i \,)$

$$\begin{tikzcd}
\bigvee_{i \geq 0} Z_i \arrow[r] \arrow[d] & \bigvee_{i \geq 0} Z_{2i} \arrow[d] \\
\bigvee_{i \geq 0} Z_{2i+1} \arrow[r] & \operatorname{colim} Z_i =: Z_X
\end{tikzcd}$$

$Z_i \to Z_{i+1} \to \cdots \to Z_{2i}$

$Z_i \to \cdots \to Z_{2i} \to Z_{2i+1}$

$$F(Z_X) \overset{②}{\twoheadrightarrow} \prod F(Z_{2i}) \underset{\prod F(Z_i)}{\times} \prod F(Z_{2i+1}) \ni (\xi_{2i}, \xi_{2i+1})$$

$$\begin{tikzcd}
F(Z_X) \\
\tilde{\xi} \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

$$\tilde{\xi} : [S^m, Z_X]_* \longrightarrow F(Z_X)$$

we know it is surjective $\qquad \begin{tikzcd} {[S^m, Z_0]_*} \arrow[r] \arrow[rr, two heads, bend right] & {[S^m, Z_X]_*} \arrow[r] & F(Z_X) \end{tikzcd}$

$g, g' : S^m \to Z_X$ s.t. $g^* \tilde{\xi} = (g')^* \tilde{\xi}$. $\qquad F(S^m \vee S^m) = F(S^m) \times F(S^m)$

WTS g, g' factor through some Z_i

Simplify $g, g' \in \pi_m Z_X \Rightarrow$ take $g \cdot (g')^{-1} = g''$ one can show $(g'')^* \tilde{\xi} \overset{①}{=} 0$.

Up to replacing g, g' w/ g'' I can assume $g' = *$.

Claim: $\pi_m Z_X \simeq \operatorname{colim}_i \pi_m Z_i$ $\qquad$ [Z_X is a mapping telescope]

Easy way to show this is $\tilde{H}_m Z_X \simeq \operatorname{colim} \tilde{H}_m Z_i$ + relative Hurewicz.

$\Rightarrow g : S^m \to Z_i \Rightarrow g^* \xi_i = 0 \Rightarrow z_i \circ g \sim 0$. □

Corollary: If A is an abelian group, there is a natural isomorphism

$$\tilde{H}^n(X, A) \simeq [X, K(A,n)]_*$$

where $K(A,n)$ is the Eilenberg-MacLane space, i.e. a pointed space w/ the property

$$\pi_i K(A,n) = \begin{cases} 0 & i \neq n \\ A & i = n \end{cases}$$

Proof: We know $\tilde{H}^n$ satisfies the hypothesis of Brown representability $\Rightarrow \exists$ a space $K(A,n)$ representing it.

$$\pi_i K(A,n) = [S^i, K(A,n)]_* \simeq \tilde{H}^n(S^i, A) = \begin{cases} 0 & i \neq n \\ A & i = n \end{cases}$$

Let K be another space w/ this property. Then by Hurewicz

$\tilde{H}^n(K; \mathbb{Z}) \simeq A \Rightarrow (UCT) \quad \tilde{H}^n(K, A) = \operatorname{Hom}(A, A)$

$\tilde{H}^i(K; \mathbb{Z}) \simeq 0 \quad i < n \qquad \qquad \tilde{H}^n(K,A) = [K, K(A,n)]_*$

We take $f : K \to K(A,n)$ corresponding to $\mathrm{id}_A \in \operatorname{Hom}(A,A)$ and this gives an iso on homotopy groups (Hurewicz: iso on $\tilde{H}^n \Rightarrow$ iso on π_n). □

Ex: $S^1 = K(\mathbb{Z}, 1) \Rightarrow \tilde{H}^1(X, \mathbb{Z}) \simeq [X, S^1]_*$.

Ex: By applying the corollary to $X_+ = X \sqcup \{*\}$ $\quad H^n(X; A) \simeq \tilde{H}^n(X_+; A) \simeq [X_+, K(A,n)]_*$

$$\simeq [X, K(A,n)]$$

One can show $[X, S^1] \longleftrightarrow \{$ isomorphism classes of line bundles $\}$

Examples of spectra

Remember a spectrum $(\{E_n\}_{n \in \mathbb{Z}}, E_n \xrightarrow{\sim} \Omega E_{n+1})$

$$HA = \left(\{K(A,n)\}_{n \geq 0},\ K(A,n) \xrightarrow{\sim} \underbrace{\Omega K(A,n+1)}_{\text{has } \pi_n = A,\ 0 \text{ other degrees}}\right)$$

i.e. it is the spectrum corresponding to $\tilde{H}^*(-; A)$.

In fact this gives a fully faithful functor $H : (N)Ab \to Sp$.

Def: $0 = H0 = (\{*\}, * \xrightarrow{\sim} \Omega *)$

Def: $\Omega^\infty : Sp \to Space_* \qquad (E_n, E_n \xrightarrow{\sim} \Omega E_{n+1}) \longmapsto E_0$ (0-th space)

Ex: $\Omega^\infty HA = A$ pointed by the zero element.

Ω^∞ has a left adjoint $\Sigma^\infty : Space_* \to Sp$

If $X \in Space_*$ we can build $\qquad \Sigma X \to \Omega\Sigma(\Sigma X)$

$$QX = \operatorname{colim}\left(X \to \Omega\Sigma X \xrightarrow{\Omega(\mathrm{unit}_{\Sigma X})} \Omega^2 \Sigma^2 X \xrightarrow{\Omega^2 \mathrm{unit}_{\Sigma^2 X}} \cdots\right) \left(= \Omega^\infty \Sigma^\infty X\right) \in Space_*$$

$$\pi_n QX \simeq \pi_n^s X = \operatorname{colim}_m \pi_{n+m} \Sigma^m X = \operatorname{colim}_m \pi_n(\Omega^m \Sigma^m X)$$

$$\simeq \pi_n\left(\operatorname{colim}_m \Omega^m \Sigma^m X\right) = \pi_n QX.$$

(*) Ω commutes w/ filtered colimits in $Space_*$

Remark: $\Omega Q \Sigma X \simeq \Omega\left(\operatorname{colim}_m \Omega^m \Sigma^m \Sigma X\right) \overset{(*)}{\simeq} \operatorname{colim}_m \Omega^{m+1} \Sigma^{m+1} X \simeq QX$.

$\Rightarrow$ We can build a spectrum

$$\Sigma^\infty X = \{ Q\Sigma^n X \,,\; Q\Sigma^n X \xrightarrow{\sim} \Omega Q \Sigma^{n+1} X \}$$

__Lemma:__ Let $X \in \mathrm{Space}_*$, $E \in \mathrm{Sp}$. There's a natural equivalence

$$\mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty X, E) \simeq \mathrm{Map}_{\mathrm{Space}_*}(X, \Omega^\infty E)$$

__Proof:__ $\mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty X, E) \simeq \lim\limits_n \mathrm{Map}_{\mathrm{Space}_*}((\Sigma^\infty X)_n, E_n)$

$$\simeq \lim_n \mathrm{Map}_{\mathrm{Space}_*}(Q\Sigma^n X, E_n) \simeq \lim \mathrm{Map}_{\mathrm{Space}_*}(\operatorname{colim}_m \Omega^n \Sigma^m X, E_n)$$

$$\simeq \lim_{n,m} \mathrm{Map}_{\mathrm{Space}_*}(\Omega^n \Sigma^{n+m} X, E_n)$$

$$\begin{tikzcd}
\mathrm{Map}_*(X, E_0) \simeq \mathrm{Map}(X, \Omega^\infty E) & & \\
\mathrm{Map}_*(\Omega\Sigma X, E_0) \arrow[u] & \mathrm{Map}_*(\Sigma X, E_1) \arrow[l, "\Omega"'] \arrow[ul, "\sim"'] & \\
\mathrm{Map}_*(\Omega^2\Sigma^2 X, E_0) \arrow[u] & \mathrm{Map}_*(\Omega\Sigma^2 X, E_1) \arrow[l, "\Omega"'] \arrow[u] & \mathrm{Map}_*(\Sigma^2 X, E_2) \arrow[l] \arrow[ul, "\sim"'] \\
\vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

We can take the limit over the constant diagonal diagram. $\square$

$X = S^0$, $\Sigma^\infty S^0 = \mathbb{S}$, the corr. cohomology theory stable cohomotopy

$$\pi^{s,m}(Y) = \operatorname{colim}_n [\Sigma^n Y, S^{n+m}]_* \quad (= [Y, QS^0]_*)$$

Y finite CW