§ Cohomology theories and Brown representability

$\mathrm{Space} = \infty$-cat of spaces $(= N^\Delta \mathrm{Kan})$

$\mathrm{Space}_* = \infty$-cat of pointed spaces $(= N^\Delta \mathrm{Kan}_*)$

$h\mathrm{Space}_*$ = homotopy category of ptd CW complexes

FACT: Space_* can be generated under colimits by S^n ("every (ptd) space has a cell structure")

$X = \operatorname{colim}_n X^{(n)}$ ← n-skeleton

$$\begin{tikzcd} \bigvee S^n \arrow[r] \arrow[d] & X^{(n)} \arrow[d] \\ \bigvee D^n = * \arrow[r] & X^{(n+1)} \end{tikzcd}$$

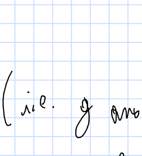
Def: A cofiber sequence $X' \xrightarrow{f} X \xrightarrow{g} X''$ is just a pushout diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ * \arrow[r] & X'' \end{tikzcd}$$

Concretely f, g maps, $H: gf \sim *$ which is "universal" (i.e. g and H identify X'' w/ the mapping cone of f). We will often write the mapping cone of f as $\mathrm{cof}\, f$ or X/X' or Cf.

Def: A cohomology theory is a pair (E^*, ∂) where

$E^*: h\mathrm{Space}_*^{op} \longrightarrow \mathrm{grAb}$ ← graded abelian groups



$= \Sigma X$

and ∂ is a natural isom $\partial: E^* \simeq E^{*+1} \circ \Sigma$ s.t.

① For any collection of pointed spaces $\{X_\alpha\}_{\alpha \in A}$ the map

$$E^*\Big(\bigvee_{\alpha \in A} X_\alpha\Big) \xrightarrow{\sim} \prod_{\alpha \in A} E^*(X_\alpha)$$

← product in graded abelian groups

is an isomorphism $(A = \emptyset \Rightarrow E^*(*) = 0)$

② $\forall\, X' \xrightarrow{f} X \xrightarrow{g} X''$ cofiber sequence the sequence

$$E^*(X'') \xrightarrow{g^*} E^*(X) \to E^*(X')$$

is exact in the middle.

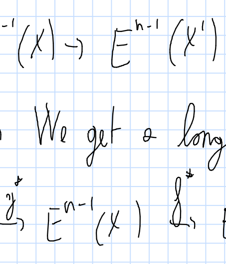
Ex: Let M be an abelian group, then $X \mapsto \tilde{H}^*(X; M)$, equipped w/ the suspension isom. is a cohomology theory.

Recall: $X' \xrightarrow{f} X \xrightarrow{g} X''$ cofiber seq. $\Rightarrow X \xrightarrow{g} X'' \xrightarrow{d} \Sigma X$ cofiber sequence

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & X'' \arrow[r] & ? \end{tikzcd}$$

The big square is also a pushout square $\Rightarrow\ ? = \Sigma X'$.

We can iterate this: $X'' \xrightarrow{d} \Sigma X \xrightarrow{\Sigma f} \Sigma X'$, $\Sigma X' \xrightarrow{\Sigma f} \Sigma X \xrightarrow{\Sigma g} \Sigma X''$, ...



$\sim$ $= \Sigma X' = X''/X$

If (E^*, ∂) is a cohomology theory, we know

$$E^n(X'') \xrightarrow{g^*} E^n(X) \xrightarrow{f^*} E^n(X') \quad \text{exact seq.} \qquad X' \to X \to X''$$

$$E^{n-1}(X') \overset{\partial}{\simeq} E^n(\Sigma X') \xrightarrow{d^*} E^n(X'') \xrightarrow{g^*} E^n(X) \qquad X \to X'' \to \Sigma X'$$

$$E^{n-1}(X) \to E^{n-1}(X') \to E^n(X'') \qquad X'' \to \Sigma X' \to \Sigma X$$

$\Rightarrow$ We get a long exact sequence

$$\cdots \xrightarrow{g^*} E^{n-1}(X) \xrightarrow{f^*} E^{n-1}(X') \xrightarrow{\partial^*} E^n(X'') \xrightarrow{g^*} E^n(X) \xrightarrow{f^*} E^n(X') \xrightarrow{\partial^*} E^{n+1}(X'') \xrightarrow{g^*} \cdots$$

Notation: $\forall\, A \subseteq X \quad E^n(X, A) := E^n(X/A)$

So for the cofiber seq. $A \subseteq X \to X/A$, the les above is the les of a pair.

Goal: For any cohomology theory (E^*, ∂) $\exists$ a seq. of pointed spaces Y_n s.t.

$$E^n(X) = [X, Y_n]_*.$$

Theorem (Brown representability theorem): Let $h\mathrm{Space}_*^{\geq 0} \subseteq h\mathrm{Space}_*$ be the subcategory of connected pointed spaces. Then a functor

$$F: (h\mathrm{Space}_*^{\geq 0})^{op} \to \mathrm{Set}$$

is representable (i.e. $\exists Z \quad [X, Z]_* \simeq FX$) if and only if it has the following two properties:

① $\forall$ collection $\{X_\alpha\}_{\alpha \in A}$ of pointed connected spaces, the map

$$F\Big(\bigvee_{\alpha \in A} X_\alpha\Big) \to \prod_{\alpha \in A} F(X_\alpha)$$

is a bijection

② $\forall$ pushout square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] & Y' \end{tikzcd}$$

the map $F(Y') \to F(X') \times_{F(X)} F(Y)$ is surjective.

Remarks: Both pointedness & connectivity in the hypothesis are necessary.
For example $\exists$ a functor
$$F: h\mathrm{Space}_*^{op} \to \mathrm{Set}$$
satisfying ① & ② but not representable

Corollary: Let (E^*, ∂) be a cohomology theory. Then there exist a (unique) collection of pointed spaces $\{E_n\}_{n \in \mathbb{Z}}$ and equivalences $[E_n \in \mathrm{Space}_* (= N^\Delta \mathrm{Kan}_*)]$

$$\delta_n: E_n \xrightarrow{\sim} \Omega E_{n+1}$$

s.t. $\exists$ natural isomorphisms $E^n(X) \simeq [X, E_n]_*$ making the following diagram commute

$$\begin{tikzcd} E^n(X) \arrow[r, "\partial"] \arrow[d, "\wr"] & E^{n+1}(\Sigma X) \arrow[d, "\wr"] \\ {[X, E_n]_*} \arrow[r, "\delta_n", "\sim"'] & {[X, \Omega E_{n+1}]_* \simeq [\Sigma X, E_{n+1}]} \end{tikzcd}$$

Proof: That such data determines a cohomology theory is left as an exercise.
The first step is to prove $\forall n$ $E^n|_{h\mathrm{Space}_*^{\geq 0}}$ is representable, i.e. that it satisfies the hypotheses of the Brown representability theorem.
Hypothesis ① follows from the definition of a cohomology theory. The key point is hypothesis ②

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\text{Ⓐ}"] & Y \arrow[d, "g'"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

WTS $E^n(Y') \to E^n(Y) \times_{E^n(X)} E^n(X')$ is onto.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d] \\ * \arrow[r] & \mathrm{cof}\, f' \end{tikzcd}$$

Remark: The pushout square Ⓐ induces an equivalence $\underset{Y/X}{\mathrm{cof}\, f} \simeq \underset{Y'/X'}{\mathrm{cof}\, f'}$

$$\begin{tikzcd} \cdots \arrow[r] & E^n(Y'/X') \arrow[r, "\partial'"] \arrow[d, "\wr"] & E^n(Y') \arrow[r, "f'^*"] \arrow[d, "g'^*"] & E^n(X') \arrow[r, "\partial'"] \arrow[d, "g^*"] & E^{n+1}(Y'/X') \arrow[r] \arrow[d, "\wr"] & \cdots \\ \cdots \arrow[r] & E^n(Y/X) \arrow[r, "\partial"] & E^n(Y) \arrow[r, "f^*"] & E^n(X) \arrow[r, "\partial"] & E^{n+1}(Y'/X') \arrow[r] & \cdots \end{tikzcd}$$

$\alpha \in E^n(X')$, $\beta \in E^n(Y)$

$$g^*\alpha = f^*\beta$$

$$\partial(g^*\alpha) = \partial f^*\beta = 0 \Rightarrow \partial'\alpha = 0 \Rightarrow \exists \tilde{\gamma} \quad \alpha = f'^*\tilde{\gamma}$$

$$\beta_0 = \beta - (g')^*\tilde{\gamma} \qquad f^*\beta_0 = f^*\beta - f^*(g')^*\tilde{\gamma} = g^*\alpha - g^* f'^*\tilde{\gamma} = 0$$

$$\beta_0 = \partial\varepsilon \Rightarrow \gamma = \tilde{\gamma} + \partial'\varepsilon. \text{ This works. } \checkmark$$

$\Rightarrow \exists \tilde{Z}_n$ s.t. $\forall X$ ptd connected space $E^n X \simeq [X, \tilde{Z}_n]_*$.

Now we can use the following observation: if X is any ptd space $\Rightarrow \Sigma X$ connected.

$$E^n X \overset{\partial}{\simeq} E^{n+1}\Sigma X \simeq [\Sigma X, \tilde{Z}_n]_* \simeq [X, \Omega\tilde{Z}_n]_*$$

$E_n := \Omega\tilde{Z}_n$.

Now $E^{n+1}\Sigma X = [X, \Omega E_{n+1}]_*$ & so the natural iso $E^n \simeq E^{n+1}\Sigma$ comes from $\delta_n: E_n \simeq \Omega E_{n+1}$ by Yoneda. □

Def: A spectrum is the datum of a collection of pointed spaces $\{E_n\}_{n \in \mathbb{Z}}$ and equivalences

$$\delta_n: E_n \xrightarrow{\sim} \Omega E_{n+1} \qquad E = (\{E_n\}_{n \in \mathbb{Z}}, \delta_n)$$

Sometimes we're going to define E_n for $n \geq N$ $(E_i := \Omega^{N-i} E_N \quad \forall i < N)$

From a spectrum we want to get a cohomology theory.

IDEA: $E^n(X) := [X, E_n]_* = \pi_0 \mathrm{Map}_*(X, E_n) \in \mathrm{Set}$

We certainly have a suspension isom $E^n \simeq E^{n+1} \circ \Sigma$ using δ_n.

$$E^n(X) = \pi_0 \mathrm{Map}_*(X, E_n) \simeq \pi_0 \mathrm{Map}_*(X, \Omega^2 E_{n+2}) \simeq \pi_0 \Omega^2 \mathrm{Map}_*(X, E_{n+2})$$
$$\overset{!}{=} \pi_2 \mathrm{Map}(X, E_{n+2})$$ and this gives an abelian group structure to $E^n(X)$.

The axiom ① ("product axiom") is just the universal property of coproducts.
Axiom ② is exactness for Puppe seq: $\forall T \in \mathrm{Space}_*$

$$[X'', T]_* \to [X, T]_* \to [X', T]_*$$

is an exact sequence of pointed sets.

The Eckmann-Hilton argument shows that $\pi_n \Omega X \simeq \pi_{n+1} X$ is an isom of groups for $n \geq 1$.

Def: $\mathrm{Sp} = \infty$-cat. of spectra $= \lim(\mathrm{Space}_* \xleftarrow{\Omega} \mathrm{Space}_* \xleftarrow{\Omega} \cdots)$

concretely: objects are $\{E_n, \delta_n: E_n \simeq \Omega E_{n+1}\}_{n \geq 0}$, maps

$$\mathrm{Map}_{\mathrm{Sp}}(E, F) = \lim_n\big(\leftarrow \mathrm{Map}_{\mathrm{Space}_*}(E_n, F_n) \xleftarrow{\Omega} \mathrm{Map}_{\mathrm{Space}_*}(E_{n+1}, F_{n+1}) \xleftarrow{\Omega} \cdots\big)$$

i.e. a map is $f_n: E_n \to F_n \ \forall n$ +

$$\begin{tikzcd} E_n \arrow[r, "f_n"] \arrow[d] \arrow[dr, phantom, "\swarrow H_n"] & F_n \arrow[d] \\ \Omega E_{n+1} \arrow[r, "\Omega f_{n+1}"'] & \Omega F_{n+1} \end{tikzcd}$$

$\forall n$.

$f \sim f'$

$K_n: f_n \sim f_n'$

$$\begin{tikzcd} E_n \wedge [0,1]_+ \arrow[r, "K_n"] \arrow[d] \arrow[dr, phantom, "\swarrow \tilde{H}_n"] & F_n \arrow[d] \\ \Omega(E_{n+1} \wedge [0,1]_+) \arrow[r, "\Omega K_{n+1}"'] & \Omega F_{n+1} \end{tikzcd}$$

$\tilde{H}_n$ extending H_n, H_n'

m-simplex in $\mathrm{Map}_{\mathrm{Sp}}(E, F)$

$f_n: E_n \times \Delta^m \to F_n$ +

$$\begin{tikzcd} E_n \wedge |\Delta^m|_+ \arrow[r, "f_n"] \arrow[d] \arrow[dr, phantom, "\swarrow H_n"] & F_n \arrow[d] \\ \Omega(E_{n+1} \wedge |\Delta^m|_+) \arrow[r, "\Omega f_{n+1}"] & F_{n+1} \end{tikzcd}$$

Because an m-simplex in $\mathrm{Map}_{\mathrm{Sp}}(E, F)$ is an m-simplex $\sigma_n \in \mathrm{Map}_{\mathrm{Space}_*}(E_n, F_n)$ + a path $\Omega\sigma_{n+1} \simeq \sigma_n$ in $\mathrm{Map}(\Delta^m, \mathrm{Map}_{\mathrm{Space}_*}(E_n, F_n))$.

If we have a tower of spaces $\cdots \leftarrow X_n \xleftarrow{p_n} X_{n+1} \leftarrow \cdots$

An m-simplex in $\lim_n X_n$ is $\{\sigma_n \in X_n([m]),\ \gamma_m: \sigma_n \sim p_n\sigma_{m+1} \text{ in } \mathrm{Hom}(\Delta^m, X_n)\}$