ISHT Lecture 4 — 2020-11-16

§ Basics of ∞-category theory

Today: No proofs (refer to Prof. Cisinski's class or references in the notes) $\qquad \bigcup_{i\neq j} \partial_i \Delta^n \subseteq \Delta^n$

Def: $\mathcal{C}$ simplicial set is an ∞-category if $\forall n \geq 2, \forall 0 < i < n, \forall f_0 : \Lambda^n_i \to \mathcal{C}$ $\exists f : \Delta^n \to \mathcal{C}$, $f|_{\Lambda^n_i} = f_0$.

$n = 2, i = 1$: $\forall$ f_{01}, f_{12} $\rightsquigarrow$ $\exists\, f_{02}$ filling the triangle — "we can compose arrows"

Ex: · $\mathcal{C}$ be an ordinary category $\Rightarrow N\mathcal{C}$ is an ∞-category
· X Kan complex $\Rightarrow X$ is an ∞-category
· $\mathcal{C}$ Kan enriched cat, $N^{\Delta}\mathcal{C}$ is an ∞-category

We will call 0-simplices of $\mathcal{C}$ "objects" and 1-simplices "arrows".

Lemma: $\mathcal{C}$ be a simplicial set. TFAE
① $\mathcal{C}$ is an ∞-category
② $\forall$ diagram of the form

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \mathrm{Hom}(\Delta^2, \mathcal{C}) \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & \mathrm{Hom}(\Lambda^2_1, \mathcal{C}) \end{array}$$

("simplicial set of pairs of composable arrows w/ a choice of composition"; "simplicial set of pairs of composable arrows")

there exists a lift
③ $\forall$ diagram of the form

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{Hom}(\Delta^2, \mathcal{C}) \\ \downarrow & \overset{\exists}{\nearrow} & \downarrow \\ B & \longrightarrow & \mathrm{Hom}(\Lambda^2_1, \mathcal{C}) \end{array}$$

$\exists$ a lift

Corollary: $\mathcal{C}$ ∞-category, S simplicial set $\Rightarrow \mathrm{Hom}(S, \mathcal{C})$ is an ∞-category, which we write $\mathrm{Fun}(S, \mathcal{C})$. (functor category, ∞-category of diagrams)

Proof:

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{Hom}(\Lambda^2_1, \mathrm{Hom}(S,\mathcal{C})) \\ \downarrow & \overset{\exists}{\nearrow} & \downarrow \\ B & \longrightarrow & \mathrm{Hom}(\Delta^2, \mathrm{Hom}(S,\mathcal{C})) \end{array} \iff \begin{array}{ccc} A\times S & \longrightarrow & \mathrm{Hom}(\Lambda^2_1, \mathcal{C}) \\ \downarrow & \nearrow & \downarrow \\ B\times S & \longrightarrow & \mathrm{Hom}(\Delta^2, \mathcal{C}) \end{array}$$

& the lift exists because $\mathcal{C}$ is an ∞-category. □

Def: $\mathcal{C}$ ∞-category, $x, y \in \mathrm{ob}\,\mathcal{C}$ $\qquad$ ($\mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}\times\mathcal{C}$ comes from precomposition w/ $\partial\Delta^1 \hookrightarrow \Delta^1$)

$$\mathrm{Map}_{\mathcal{C}}(x,y) := \mathrm{Fun}(\Delta^1, \mathcal{C}) \times_{\mathcal{C}\times\mathcal{C}} \{(x,y)\}$$

FACT: $\mathrm{Map}_{\mathcal{C}}(x,y)$ is a Kan complex.

Def "composition":

$$\mathrm{Map}_{\mathcal{C}}(x,y) \times \mathrm{Map}_{\mathcal{C}}(y,z) = \mathrm{Fun}(\Lambda^2_1, \mathcal{C}) \times_{\mathcal{C}\times\mathcal{C}\times\mathcal{C}} \{(x,y,z)\}$$

$$\emptyset \longrightarrow \mathrm{Fun}(\Delta^2, \mathcal{C}) \times_{\mathcal{C}\times\mathcal{C}\times\mathcal{C}} \{(x,y,z)\} \xrightarrow{\partial_1} \mathrm{Fun}(\Delta^1, \mathcal{C}) \times_{\mathcal{C}\times\mathcal{C}} \{(x,z)\} = \mathrm{Map}_{\mathcal{C}}(x,z)$$

(with $\exists$ a lift of the first map along $\mathrm{Fun}(\Delta^2,\mathcal{C})\times\{(x,y,z)\} \to \mathrm{Map}_{\mathcal{C}}(x,y)\times\mathrm{Map}_{\mathcal{C}}(y,z)$)

Def: $\mathrm{id}_x = sx \in \mathrm{Fun}(\Delta^1, \mathcal{C})$

$$\begin{array}{ccccc} \emptyset & \longrightarrow & \mathrm{Fun}(\Delta^2,\mathcal{C}) \times_{\mathcal{C}\times\mathcal{C}\times\mathcal{C}} \{(x,y,z)\} & \hookrightarrow & \mathrm{Fun}(\Delta^2,\mathcal{C}) \xrightarrow{\partial_1} \mathrm{Fun}(\Delta^1,\mathcal{C}) \\ \downarrow & & \downarrow & \ulcorner & \downarrow \\ \mathrm{Map}_{\mathcal{C}}(x,y)\times\mathrm{Map}_{\mathcal{C}}(y,z) & \xrightarrow{\sim} & \mathrm{Fun}(\Lambda^2_1,\mathcal{C}) \times_{\mathcal{C}\times\mathcal{C}\times\mathcal{C}} \{(x,y,z)\} & \hookrightarrow & \mathrm{Fun}(\Lambda^2_1, \mathcal{C}) \end{array}$$

FACT: Two arrows $f, g : x \to y$ are connected by a path in $\mathrm{Map}_{\mathcal{C}}(x,y)$ iff they are homotopic rel ∂ in the old sense: i.e. $\exists$ a 2-simplex with boundary f, g, id_y (square with sides f, g, id_x, id_y).

FACT: h is homotopic to $g \circ f$ (for some choice of $\circ$) if and only if $\exists$ a 2-simplex with edges f, g, h.

FACT: $\mathrm{Map}_{N^{\Delta}\mathcal{C}}(x,y) \simeq \mathrm{Map}_{\mathcal{C}}(x,y)$ $\quad \forall$ Kan enriched cat. $\mathcal{C}$. ("higher")

Def: Let $\mathcal{C}$ be an ∞-category, then the homotopy category $h\mathcal{C}$ is the (ordinary) category whose objects are the 0-simplices of $\mathcal{C}$ and s.t.

$$\mathrm{Hom}_{h\mathcal{C}}(x,y) = \pi_0 \mathrm{Map}_{\mathcal{C}}(x,y) = \text{homotopy eq. classes of morphisms } x \to y$$

Composition is given by applying π_0 to any of the composition maps

$$\mathrm{Map}_{\mathcal{C}}(x,y) \times \mathrm{Map}_{\mathcal{C}}(y,z) \longrightarrow \mathrm{Map}_{\mathcal{C}}(x,z)$$

Ex: Let $\mathrm{Space} = N^{\Delta}\underline{\mathrm{Kan}}$ (∞-category of spaces), then $h\mathrm{Space}$ is the homotopy category of spaces.

Def: $\mathcal{C}$ ∞-cat., $f : x \to y$ arrow in $\mathcal{C}$ is an equivalence if its class in $h\mathcal{C}$ is an isomorphism, or eq. $\exists g : y \to x$ s.t. $gf \sim \mathrm{id}_x$, $fg \sim \mathrm{id}_y$.

FACT: f is an equivalence iff the map $\Delta^1 \xrightarrow{\{f\}} \mathcal{C}$ can be extended to $N(0 \rightleftarrows 1)$.
Note, that this gives a g & an infinite amount of coherences.

FACT: f is an equivalence iff $\forall \alpha : \Lambda^n_0 \to \mathcal{C}$ s.t. $\alpha|_{[0,1]} = f$ $\exists \bar{\alpha} : \Delta^n \to \mathcal{C}$ extension.

Ex: $n = 2$: α = (horn with edges f, g) $\Rightarrow \bar{\alpha}$ = (triangle with edges f, g, gf^{-1})

Corollary: $\mathcal{C}$ ∞-category is a Kan complex iff all arrows are equivalences.
That's why Kan complexes are sometimes called ∞-groupoids.

Ex: X Kan complex $hX = \Pi_1 X$

Def: $\mathcal{C}$ ∞-category, $x \in \mathrm{ob}\,\mathcal{C}$ object
· x is initial if $\forall y \in \mathrm{ob}\,\mathcal{C}$ $\quad \mathrm{Map}_{\mathcal{C}}(x,y) \sim *$.
· x is terminal if $\forall y \in \mathrm{ob}\,\mathcal{C}$ $\quad \mathrm{Map}_{\mathcal{C}}(y,x) \sim *$.

Ex: · $\mathcal{C} = N\mathcal{D}$ category: $x \in \mathrm{ob}\,\mathcal{C}$ is initial iff it is initial in $\mathcal{D}$
· X Kan complex has an initial object iff it is contractible.

Def: Let S be a simplicial set, the right cone $S^{\triangleright}$ is the simplicial set

$$[n] \longmapsto \{ (f, \sigma) \mid f : \Delta^n \to \Delta^1,\ \sigma : f^{-1}_0 \to S \}$$

Ex: $S = N\mathcal{C} \Rightarrow S^{\triangleright} = N(\mathcal{C}^{\triangleright})$ where $\mathcal{C}^{\triangleright}$ has objects $\mathrm{ob}\,\mathcal{C} \amalg \{\infty\}$

$$\mathrm{Hom}_{\mathcal{C}^{\triangleright}}(x,y) = \begin{cases} \mathrm{Hom}_{\mathcal{C}}(x,y) & \text{if } x, y \in \mathrm{ob}\,\mathcal{C} \\ * & y = \infty \\ \emptyset & x = \infty,\ y \neq \infty \end{cases}$$

$S^{\triangleright}$: S with arrows from every simplex to a new point $\cdot\infty$; $\quad i \in \{0, \dots, n\}$, $\Delta^{\{0,\dots,i\}} \to S$

Δ^1: $\cdot \longrightarrow \cdot$

Ex: $S = \Lambda^2_0$ = (two arrows out of one vertex) $\Rightarrow S^{\triangleright} \simeq \Delta^1 \times \Delta^1$ = (square) (homotopy)

Def: Let $\mathcal{C}$ be an ∞-category, $p : S \to \mathcal{C}$ a diagram. A colimit of p is a diagram $\bar{p} : S^{\triangleright} \to \mathcal{C}$ s.t. $\bar{p}|_S = p$ & it is an initial object of

$$\mathrm{Fun}(S^{\triangleright}, \mathcal{C}) \times_{\mathrm{Fun}(S,\mathcal{C})} \{p\} \quad \text{(category of such diagrams)}$$

Ex: When $S = \Lambda^2_0$, $\mathcal{C} = \mathrm{Space}$ this notion of colimit coincides w/ the notion of homotopy pushout we have seen last semester.

Remark: There's a dually defined notion of left cone $S^{\triangleleft}$ & of limit.

$S = \Lambda^2_1$ = (two composable arrows) $\qquad \mathrm{Fun}(S, \mathcal{C})$ "=" $\mathrm{Fun}(\Delta^2, \mathcal{C})$

Def: $\mathcal{C}, \mathcal{D}$ ∞-cat., $F, G : \mathcal{C} \to \mathcal{D}$ functors. A natural transformation $F \Rightarrow G$ is just a point in $\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, G)$.
The space of nat. transformations is $\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, G)$

Def: $\mathcal{C}, \mathcal{D}$ ∞-cat. $\quad \mathrm{Map}(\mathcal{C}, \mathcal{D}) :=$ maximal Kan subcomplex of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$

Ex: $\mathcal{C}$ ∞-cat, its maximal Kan subcomplex is just $\mathcal{C} \times_{N(h\mathcal{C})} N(h\mathcal{C}^{\simeq}) =: \mathcal{C}^{\simeq}$, $\iota\mathcal{C}$ $\quad$ ($h\mathcal{C}^{\simeq}$: subcat. of $h\mathcal{C}$ w/ all objects & all isomorphisms)

Def: $\mathrm{Cat}_{\infty} = N^{\Delta}(q\mathrm{Cat})$ where $q\mathrm{Cat}$ is the simplicial category whose objects are ∞-cat. & $\mathrm{Map}_{q\mathrm{Cat}}(\mathcal{C}, \mathcal{D}) = \mathrm{Map}(\mathcal{C}, \mathcal{D})$.

Remark: An eq. $F : \mathcal{C} \to \mathcal{D}$ of ∞-categories is just an eq. in Cat_{∞}, i.e. $\exists G : \mathcal{D} \to \mathcal{C}$ $\quad GF \simeq \mathrm{id}_{\mathcal{C}}$, $FG \simeq \mathrm{id}_{\mathcal{D}}$.

Def: $\mathcal{C}$ ∞-cat, a zero object $0 \in \mathrm{ob}\,\mathcal{C}$ is an object that is both initial & terminal.
If $\mathcal{C}$ has a 0 object, it's called pointed.

Ex: · $\mathcal{C}$ ptd ∞-cat (ex $\mathrm{Space}_* = N^{\Delta}(\mathrm{Kan}_*)$), the suspension Σx of $x \in \mathrm{ob}\,\mathcal{C}$ is the pushout

$$\begin{array}{ccc} x & \longrightarrow & 0 \\ \downarrow & \lrcorner & \downarrow \\ 0 & \longrightarrow & \Sigma x \end{array}$$

· As before, the loopspace Ωx is the pullback

$$\begin{array}{ccc} \Omega x & \longrightarrow & 0 \\ \downarrow & \ulcorner & \downarrow \\ 0 & \longrightarrow & x \end{array}$$

Ex:

$$\mathrm{Map}_{\mathcal{C}}(\Sigma x, y) \simeq \mathrm{Map}_{\mathrm{Fun}(\Lambda^2_0, \mathcal{C})}\left(\begin{array}{ccc} x & \to & 0 \\ \downarrow & & \\ 0 & & \end{array}, \begin{array}{ccc} y & \xrightarrow{\mathrm{id}_y} & y \\ & & \downarrow \mathrm{id}_y \\ & & y \end{array} \right) \simeq$$

$$\simeq \mathrm{Map}_{\mathcal{C}}(0,y) \times_{\mathrm{Map}_{\mathcal{C}}(x,y)} \mathrm{Map}_{\mathcal{C}}(x,y) \times_{\mathrm{Map}_{\mathcal{C}}(x,y)} \mathrm{Map}(0,y) \simeq \mathrm{Map}_{\mathrm{Fun}(\Lambda^2_2, \mathcal{C})}\left(x = x,\ 0 \to y \right)$$

(with $\mathrm{Map}_{\mathcal{C}}(0,y) \simeq *$)

$$\simeq \mathrm{Map}_{\mathcal{C}}(x, \Omega y)$$

Σ, Ω are functors w/ a natural eq. $\mathrm{Map}_{\mathcal{C}}(\Sigma -, -) \simeq \mathrm{Map}_{\mathcal{C}}(-, \Omega -)$ we say such a pair is an adjunction $\Sigma \dashv \Omega$.

Ex (geometric realization) A geometric realization is just a (homotopy) colimit of a diagram $X : N\Delta^{op} \to \mathcal{C}$

Let $X : \Delta^{op} \to \mathrm{Top}$ be a simplicial space, we can consider

$$\bar{X} : N\Delta^{op} \to N\mathrm{Top} \xrightarrow{N\mathrm{Sing}} N^{\Delta}\underline{\mathrm{Kan}} = \mathrm{Space}$$

$$\mathrm{colim}\,\bar{X} = (\mathrm{Sing}) \coprod_{n \geq 0} X([n]) \times |\Delta^n| \Big/ \sim \qquad \forall f : [n] \to [m],\ (f^* x, t) \sim (x, f_* t) \in X([m]) \times |\Delta^m|$$

(with $f^*x, t \in X([n]) \times |\Delta^n|$)

Thm (Bousfield–Kan formula):
$X : S \to \mathcal{C}$ diagram. Then there's an equivalence between $\mathrm{colim}\,X$ and the geometric realization of the simplicial object

$$[n] \longmapsto \coprod_{\sigma \in S([n])} X(\sigma 0)$$

($\sigma 0$: last vertex of σ $\in S([0])$)

Ex:

$$\begin{array}{ccc} x & \to & y \\ \downarrow & \lrcorner & \downarrow \\ z & \to & t \end{array}$$

pushout diagram, t is eq. to the geom. realization

$$[n] \longmapsto y \amalg \underbrace{x \amalg \dots \amalg x}_{n\text{-times}} \amalg z$$

Recall: In 1-categories t would be given by $\mathrm{coeq}(y \amalg x \amalg z \rightrightarrows y \amalg z)$
In ∞-cat. our simplicial diagram is of the form

$$\left(\cdots\ y \amalg x \amalg x \amalg x \amalg z \;\substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow}\; y \amalg x \amalg x \amalg z \;\substack{\rightarrow\\\leftarrow\\\rightarrow}\; y \amalg x \amalg z \;\substack{\rightarrow\\\leftarrow\\\rightarrow}\; y \amalg z \right)$$

Ex: $N\,\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \cong \mathrm{Fun}(N\mathcal{C}, N\mathcal{D})$ $\quad \mathcal{C}, \mathcal{D}$ categories