ISHT Lecture 3 2020-11-12

Theorem 1: If X Kan complex the map

$$\eta : X \to \operatorname{Sing}|X|$$

is a homotopy equivalence.

$$\sigma \in X([n]) \longmapsto \underset{\underset{|\Delta^n| \times \{\sigma\}}{\shortparallel}}{|\Delta^n|} \to |X| = \coprod_{m \geq 0} |\Delta^m| \times X([m]) \Big/ \sim$$

Theorem: S simplicial set, $T \subseteq S$ subset, X Kan complex

$f : |S| \to |X|$ continuous map s.t. $f|_{|T|} = |g|$ $g : T \to X$ map of sSet

$\Rightarrow \exists f', H : f \sim f'$ relative to $|T|$, $g' : S \to X$ simplicial map s.t.

- $g'|_T = g$
- $|g'| = f'$

Proof of thm 1: $\eta : X \to \operatorname{Sing}|X|$, consider $|\eta|$

$$\mathrm{id}_{|X|} = |\mathrm{id}_X|$$

$$|X| \xrightarrow{|\eta|} |\operatorname{Sing}|X|| \xrightarrow{\varepsilon_{|X|}} |X|$$

$\varepsilon_Y : |\operatorname{Sing} Y| \to Y$ $\quad (t \in |\Delta^n|, \sigma : |\Delta^n| \to Y) \longmapsto \sigma t$

Moreover, η is the inclusion of a simplicial subset.

So $\varepsilon_{|X|}$ satisfies the hypotheses of the simplicial approximation theorem:

$\exists g : \operatorname{Sing}|X| \to X$, $H : \underbrace{|g|} \sim \underbrace{\varepsilon_{|X|}}$ rel. to $\underbrace{|X|}$ (*)

s.t. $g \circ \eta = \mathrm{id}_X$.

$H : \underbrace{|\operatorname{Sing}|X|| \times [0,1]}_{|\operatorname{Sing}|X| \times \Delta^1|} \to |X|$ $\quad [0,1] \;\text{"="}\; |\Delta^1|$

by adjunction
$\iff \tilde{H} : \operatorname{Sing}|X| \times \Delta^1 \to \operatorname{Sing}|X|$.

(*) $\tilde{H}|_{X \times \Delta^1} = X \times \Delta^1 \to X \xrightarrow{\eta} \operatorname{Sing}|X|$

$\tilde{H}|_{\operatorname{Sing}|X| \times \{0\}} = \operatorname{Sing}|X| \xrightarrow{g} X \xrightarrow{\eta} \operatorname{Sing}|X|$.

- $\tilde{H}|_{\operatorname{Sing}|X| \times \{1\}} = \operatorname{Sing}|X| \xrightarrow{\mathrm{id}} \operatorname{Sing}|X|$. $\square$

§2 Homotopy coherent diagrams

Def: $\underline{\mathcal{C}}$ is a simplicial category if it is a category enriched in simplicial sets. Concretely this is the datum of

- A set $\operatorname{ob}\underline{\mathcal{C}}$ of objects
- $\forall x,y \in \operatorname{ob}\underline{\mathcal{C}}$ a simplicial set $\operatorname{Map}_{\underline{\mathcal{C}}}(x,y)$
- $\forall x,y,z$ a composition map $\operatorname{Map}_{\underline{\mathcal{C}}}(x,y) \times \operatorname{Map}_{\underline{\mathcal{C}}}(y,z) \to \operatorname{Map}_{\underline{\mathcal{C}}}(x,z)$
- $\forall x$ $\quad id \in \operatorname{Map}_{\underline{\mathcal{C}}}(x,x)([0]) = \Delta^0 \to \operatorname{Map}_{\underline{\mathcal{C}}}(x,x)$

\+ compatibilities (ass. & unitality of the composition).

We'll say $\underline{\mathcal{C}}$ is a Kan-enriched category if it is a simplicial category where $\operatorname{Map}_{\underline{\mathcal{C}}}(x,y)$ is a Kan complex $\forall x,y \in \operatorname{ob}\underline{\mathcal{C}}$.

- The elements of $\operatorname{ob}\underline{\mathcal{C}}$ will be called objects of $\mathcal{C}$
- The 0-simplices of $\operatorname{Map}_{\underline{\mathcal{C}}}(x,y)([0])$ will be called maps or morphisms $(f : x \to y)$
- The 1-simplices of $\operatorname{Map}_{\underline{\mathcal{C}}}(x,y)([1])$ will be called homotopies $H : f \sim g$.

Ex:
- $\underline{\mathrm{Top}}$ the category whose objects are (cg) topological spaces $\quad$ "homotopy types"
  $\operatorname{Map}_{\underline{\mathrm{Top}}}(X,Y) = $ Kan complex of maps $\quad [n] \mapsto \{X \times |\Delta^n| \to Y\}$
- $\underline{\mathrm{Kan}}$ the simpl. cat. whose objects are Kan complexes & $\quad$ "models homotopy types"
  $\operatorname{Map}_{\underline{\mathrm{Kan}}}(X,Y) = \operatorname{Map}(X,Y) \quad [n] \mapsto \{X \times \Delta^n \to Y\}$
- $\underline{\mathrm{Man}}$ objects are smooth manifolds
  $\operatorname{Map}_{\underline{\mathrm{Man}}}(M,N) = \operatorname{Emb}(M,N)$
- $\mathrm{Ch}(R)$ R ring $\quad$ objects are chain complexes
  $\underline{\mathrm{Hom}}_{\mathrm{Ch}(R)}(C_\bullet, D_\bullet) = $ Kan complex whose 0-simplices are $C_\bullet \to D_\bullet$ chain maps
  1-simplices are chain homotopies $C_\bullet[1] \to D_\bullet$ etc..

Recall: A coherent square was a square of topological spaces

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "g'"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd} \quad + \text{ a homotopy } H : g'f \sim f'g$$

From now on, we fix a Kan-enriched category $\underline{\mathcal{C}}$, I category, we want to guess a reasonable notion of "I-shaped coherent diagram".

① $I = \overset{\text{"}[0]}{*}$, a coherent diagram should be just an object $x_0 \in \operatorname{ob}\underline{\mathcal{C}}$.

② $I = [1] = \{0 \to 1\}$, a coherent diagram should be just two objects x_0, x_1 and an arrow $f_{01} \in \operatorname{Map}(x,y)$

③ $I = [2] = \left\{ \begin{tikzcd} 0 \arrow[r] \arrow[dr] & 1 \arrow[d] \\ & 2 \end{tikzcd} \right\}$ a coherent diagram should be
- three objects x_0, x_1, x_2
- three arrows $f_{01} : x_0 \to x_1$, $f_{12} : x_1 \to x_2$, $f_{02} : x_0 \to x_2$
- $f_{012} : f_{02} \sim f_{12} \circ f_{01}$.

④ $I = [3] = \left\{ \begin{tikzcd} 0 \arrow[r] \arrow[dr] \arrow[drr] & 1 \arrow[d] \arrow[dr] & \\ & 2 \arrow[r] & 3 \end{tikzcd} \right\}$
- $x_0, x_1, x_2, x_3 \in \operatorname{ob}\underline{\mathcal{C}}$
- $f_{ij} : x_i \to x_j$ arrow $\forall i < j$.
- $f_{ijk} : f_{ik} \sim f_{jk} \circ f_{ij} \quad \forall i < j < k$

Let's take a look at $\operatorname{Map}(x_0, x_3)$

$$\begin{tikzcd} f_{03} \arrow[r, "f_{013}"] \arrow[d, "f_{023}"'] & f_{13} \circ f_{01} \arrow[d, "f_{123} \circ f_{01}"] \\ f_{23} \circ f_{02} \arrow[r, "f_{23} \circ f_{012}"'] & f_{23} \circ f_{12} \circ f_{01} \end{tikzcd}$$

$s f_{01}$ $\quad$ $s f_{23}$

$$\begin{tikzcd} \partial(\Delta^1 \times \Delta^1) \arrow[r] \arrow[d] & \operatorname{Map}(x_0, x_3) \\ \Delta^1 \times \Delta^1 \arrow[ur, "f_{0123}"'] & \end{tikzcd}$$

part of the diagram

$\Delta^1 \times \Delta^1 = $ poset of paths from 0 to 3 under refinement
$= N\{A \subseteq \{0,\dots,3\} \mid 0,3 \in A\}$

$$\begin{tikzcd} \{0,3\} \arrow[r] \arrow[d] & \{0,1,3\} \arrow[d] \\ \{0,2,3\} \arrow[r] & \{0,1,2,3\} \end{tikzcd}$$

- $I = [n] = \{0 < 1 < \dots < n\}$
  - $x_0, \dots, x_n \in \operatorname{ob}\underline{\mathcal{C}}$
  - $\forall i < j$ $\quad P_{ij} = N\{A \subseteq \{i, i+1, \dots, j-1, j\} \mid i,j \in A\}$ $\quad$ "poset of paths from i to j"
    $|P_{ij}| = [0,1]^{j-i-1} = [1]^{j-i-1}$
    $F_{ij} : P_{ij} \to \operatorname{Map}_{\underline{\mathcal{C}}}(x_i, x_j)$
  - $\forall i < j < k$

$$\begin{tikzcd} P_{ij} \times P_{jk} \arrow[r, "\cup"] \arrow[d, "F_{ij} \times F_{jk}"'] \arrow[dr, phantom, "\circlearrowleft"] & P_{ik} \arrow[d, "F_{ik}"] \\ \operatorname{Map}(x_i, x_j) \times \operatorname{Map}(x_j, x_k) \arrow[r, "\circ"] & \operatorname{Map}(x_i, x_k) \end{tikzcd} \qquad (A, B) \mapsto A \cup B$$

  - $F_{ii} : \overset{P_{ii}}{\overset{\shortparallel}{*}} \to \operatorname{Map}(x_i, x_i)$ is id_{x_i}.

$\mathfrak{C}[n] = $ simplicial category whose objects are $0, \dots, n$
$\operatorname{Map}_{\mathfrak{C}[n]}(i,j) = P_{ij} \qquad (i > j \;\; = \emptyset)$
$\circ$ $\operatorname{Map}_{\mathfrak{C}[n]}(i,j) \times \operatorname{Map}_{\mathfrak{C}[n]}(j,k) \xrightarrow{\circ} \operatorname{Map}_{\mathfrak{C}[n]}(i,k)$ is given by $\cup$.
$\operatorname{Map}_{\mathfrak{C}[n]}(i,i) = \{\overset{id_i}{\{i\}}\}$

Then a homotopy coherent diagram $[n] \to \underline{\mathcal{C}}$ is the same thing as a simplicial functor
Cordier $\mathfrak{C}[n] \to \underline{\mathcal{C}}$.

Def: Let $\underline{\mathcal{C}}$ be a Kan-enriched category, the coherent nerve $N^\Delta\underline{\mathcal{C}}$ is the simplicial set
$$[n] \longmapsto \operatorname{Fun}_\Delta(\mathfrak{C}[n], \underline{\mathcal{C}})$$

$f : [n] \to [m]$ $\qquad$ $\mathfrak{C}[n] \to \mathfrak{C}[m]$
$i \mapsto fi$
$P_{ij} \to P_{fi\, fj}$
$A \mapsto fA$

Ex: $\mathcal{C}$ ordinary category. You can see it as a Kan enriched cat. $\operatorname{Map}_{\mathcal{C}}(x,y) = \operatorname{Hom}_{\mathcal{C}}(x,y)$
$N^\Delta\mathcal{C} = N\mathcal{C}$

Def: I category, $\underline{\mathcal{C}}$ Kan-enriched category, an I-shaped (homotopy) coherent diagram is a map of simplicial sets
$$NI \to N^\Delta\underline{\mathcal{C}}$$

Ex: $I = [1] \times [1] = \begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$ $\qquad NI = \Delta^1 \times \Delta^1 = \Delta^2 \cup_{\Delta^1} \Delta^2$ $\quad \begin{tikzcd} 00 \arrow[r] \arrow[d] \arrow[dr] & 01 \arrow[d] \\ 10 \arrow[r] & 11 \end{tikzcd}$

so a map $NI \to N^\Delta\underline{\mathcal{C}}$ is just two functors
$\mathfrak{C}[2] \to \underline{\mathcal{C}}$, $\mathfrak{C}[2] \to \underline{\mathcal{C}}$ coinciding on $\mathfrak{C}[1] \subseteq \mathfrak{C}[2]$ $\quad$ $0 \xrightarrow{d_2} 2$

i.e.
$$\begin{tikzcd} x_{00} \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, "z"] & x_{01} \arrow[d, "g'"] \\ x_{10} \arrow[r, "f'"'] & x_{11} \end{tikzcd} \qquad H, \; K$$

In the topological case you can think of this as picking $z = H(\cdot, 1/2)$

$N^\Delta\underline{\mathcal{C}}$ knows all the interesting stuff about $\underline{\mathcal{C}}$

Exercise: $\mathrm{Cat} \to \mathrm{sSet}$, $\mathcal{C} \mapsto N\mathcal{C}$ is fully faithful, i.e. $\operatorname{Fun}(\mathcal{C}, \mathcal{D}) = \operatorname{Hom}_{\mathrm{sSet}}(N\mathcal{C}, N\mathcal{D})$

Remark: N^Δ has a left adjoint $\mathfrak{C}[-] : \mathrm{sSet} \to \mathrm{Cat}_\Delta$ and if $\underline{\mathcal{C}}$ is Kan enriched
$$\mathfrak{C}[N^\Delta\underline{\mathcal{C}}] \to \underline{\mathcal{C}}$$
is a Dwyer-Kan equivalence, i.e. it is a homotopy eq. on mapping spaces and "essentially surjective"

Simplicial sets of the form $N^\Delta\underline{\mathcal{C}}$ have a special property: for example if you have an arrow $f : x \to y$, $g : y \to z$, you can find a 2-simplex σ

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "z"'] & \arrow[u, Rightarrow, "\sigma"] & z \end{tikzcd}$$

i.e. $\forall$ map $F : \Lambda^2_1 \to N^\Delta\underline{\mathcal{C}}$, you can find an extension $\bar{F} : \Delta^2 \to N^\Delta\underline{\mathcal{C}}$.
(quasicat, weak Kan compl.)

Def: A simplicial set $\mathcal{C}$ is an ∞-category if $\forall n \geq 2$, $0 < i < n$ every map
$$F : \Lambda^n_i \to \mathcal{C} \text{ extends to some } \bar{F} : \Delta^n \to \mathcal{C}$$

Exercise: $N^\Delta\underline{\mathcal{C}}$ is an ∞-category $\forall$ Kan-enriched category $\underline{\mathcal{C}}$

Remark: If $\mathcal{C}$ is an ∞-cat., $\mathcal{C} \to N^\Delta\mathfrak{C}[\mathcal{C}]$ is an equivalence in the appropriate sense.

Examples: $\mathrm{Space} := N^\Delta\mathrm{Kan}$ (∞-cat. of spaces or animas)

Def: An object in an ∞-cat is a 0-simplex $\qquad \operatorname{ob} N^\Delta\underline{\mathcal{C}} = \operatorname{ob}\underline{\mathcal{C}}$
A morphism in an ∞-cat is a 1-simplex $\qquad \operatorname{Mor} N^\Delta\underline{\mathcal{C}} = \operatorname{Mor}\underline{\mathcal{C}}$

$$\mathrm{Cat}_{\mathrm{Kan}}[\text{Dwyer-Kan eq.}^{-1}] \simeq q\mathrm{Cat}[\text{eq.}^{-1}]$$

$$I \to N^\Delta\underline{\mathcal{C}} \quad \Longleftrightarrow \quad \mathfrak{C}[I] \to \underline{\mathcal{C}}$$