ISHT Lecture 2 2020-11-09

Last time: · defined simplicial sets & Kan complexes

· Y topological space, $\mathrm{Sing}\,Y$ Kan complex $[n] \mapsto \mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, Y)$

Sing is part of an adjunction $|-| \dashv \mathrm{Sing}$

$$|X| = \coprod_{n \geq 0} |\Delta^n| \times X([n]) \Big/ \sim$$

Remark 1: $|X|$ is always a CW complex w/ $X^{(m)} = \mathrm{Im}\left(\coprod_{0 \leq n \leq m} |\Delta^n| \times X([n]) \longrightarrow |X| \right)$

and m-cells in bijection w/ the non-degenerate simplices

Remark 2: · $\varepsilon : |\mathrm{Sing}\,Y| \longrightarrow Y$

$$\coprod_n |\Delta^n| \times \mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, Y) \Big/ \sim \longrightarrow Y$$

$$(t, f) \longmapsto ft$$

· $\eta : X \longrightarrow \mathrm{Sing}\,|X|$

$$\sigma \in X([n]) \longmapsto |\Delta^n| \longrightarrow |X| = \coprod |\Delta^n| \times X([n]) \Big/ \sim$$

inclusion of $|\Delta^n| \times \{\sigma\}$ ($\{\sigma\} \subset X([n])$, $\sigma \in X([n])$)

Goal for today: · ε is a weak equivalence

· η is a homotopy eq. if X is a Kan complex

Upshot: $\mathrm{Sing}\,Y$ knows everything about the w. homotopy type of Y

Homotopy groups: $\sigma, \tau \in X([n])$ we say that $\sigma \sim \tau$ (σ is homotopic to τ rel to the boundary)

if $\exists\, \eta \in X([n+1])$ $\partial_n \eta = \sigma$, $\partial_{n+1} \eta = \tau$, $\partial_i \eta = s_{n-1} \partial_i \sigma = s_{n-1} \partial_i \tau$

$$\pi_n(X, x) = \{ \sigma \in X([n]) \mid \partial_i \sigma = x \ \forall i \} \big/ \sim .$$

$\pi_n(X,x) := \{ \sigma \in X([n]) \mid \partial_i \sigma = x \ \forall i \} / \sim$.

$\alpha, \beta \in X([n])$ $\quad \alpha|_{\partial\Delta^n} = \beta|_{\partial\Delta^n} = x$

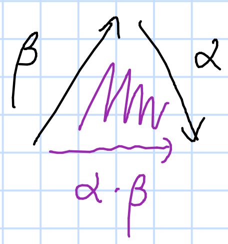
$\alpha \cdot \beta$ in the following manner: we construct

$\eta_0 : \Lambda^{n+1}_n \to X$ s.t.

- $\partial_{n-1} \eta_0 = \alpha$
- $\partial_{n+1} \eta_0 = \beta$
- $\partial_i \eta_0 = x \qquad 0 \leq i < n-1$

Case $n=1$

Λ^2_1

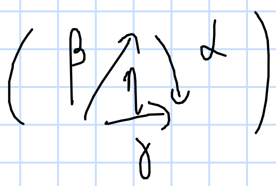


Since X is a Kan complex, $\exists \eta \in X([n+1])$ $\eta|_{\Lambda^{n+1}_n} = \eta_0$. We define $\alpha \cdot \beta = \partial_n \eta$

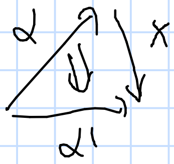
Proposition: $[\alpha \cdot \beta] \in \pi_n(X,x)$ depends only on $[\alpha], [\beta] \in \pi_n(X,x)$ and it induces a group operation on $\pi_n(X,x)$. $\quad n \geq 1$

Proof: We'll do only the $n=1$ case

Define (α,β,γ) to be a composition pair if $\exists \eta \in X([2])$ $\partial_0 \eta = \alpha$, $\partial_1 \eta = \gamma$, $\partial_2 \eta = \beta$



. Remark: $\alpha \sim \alpha'$ if and only if (x, α, α') is a composition pair



, Remark (α, x, α) is always a composition pair $\quad s_0 \alpha \in X([2])$

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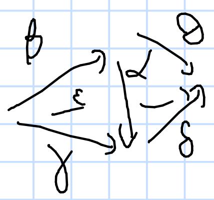
Key idea: (α,β,γ), $(\gamma,\delta,\varepsilon)$, (α,δ,θ) composition pairs $\Rightarrow$ $(\theta,\beta,\varepsilon)$ is a comp. pair

(morally $\delta(\alpha\beta) = (\delta\alpha)\beta$)

$\alpha \sim \alpha' \qquad (x,\alpha,\alpha')$
$\alpha\beta = \gamma \qquad (\alpha,\beta,\gamma) \quad \rightsquigarrow \quad (x,\gamma,\gamma')$ comp. pair $\Rightarrow \gamma \sim \gamma'$.
$\alpha'\beta = \gamma' \qquad (\alpha',\beta,\gamma')$



$\zeta_0 : \Lambda^3_2 \to X$

$\partial_0 \zeta_0 = (\alpha, \delta, \theta)$
$\partial_1 \zeta_0 = (\delta, \gamma, \varepsilon)$
$\partial_3 \zeta_0 = (\alpha, \beta, \gamma)$

$\Rightarrow$ extend it to $\zeta : \Delta^3 \to X$

$\partial_2 \zeta = (\theta, \beta, \varepsilon)$. $\square$

$\underline{Ex}$: This group operation is the classical group operation on $\pi_n(\mathrm{Sing}\, Y, x) = \pi_n(Y, x)$ for Y topological space.

$\underline{Remark}$: If $f: X \to X'$ map of Kan complexes, we get functorial maps

$$\begin{aligned} f_* : \pi_n(X, x) &\longrightarrow \pi_n(X', fx) \\ [\alpha] &\longmapsto [f\alpha] \end{aligned}$$

$\underline{Next\ goal}$: How does this interact w/ homotopies of maps.

$\underline{Def}$: S, T simplicial sets, want to define a simplicial set $\mathrm{Hom}(S, T)$ s.t.

$$[n] \longmapsto \mathrm{Hom}_{sSet}(S \times \Delta^n, T)$$

0-simplices: maps $f: S \to T$
1-simplices: "homotopies" $f: S \times \Delta^1 \to T$
etc.

$\underline{Q}$: When is $\mathrm{Hom}(S,T)$ a Kan complex?

$\underline{Lemma}$: Let X be a simplicial set. TFAE

① X is a Kan complex

② $\forall n \geq 0$, $\forall$ diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \mathrm{Hom}(\Delta^1, X) \arrow[d, "ev_0"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

$\exists$ a lift

$$\begin{aligned} ev_0 : \mathrm{Hom}(\Delta^1, X) &\longrightarrow X \\ [\Delta^m \times \Delta^1 \to X] &\longmapsto [\Delta^m \times \{0\} \to X] \end{aligned}$$

③ $\forall A \subseteq B$ simplicial subsets & $\forall$ diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathrm{Hom}(\Delta^1, X) \arrow[d, "ev_0"] \\ B \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

$\exists$ a lift.

$\underline{Proposition}$: S any simplicial set, X Kan complex $\Rightarrow \mathrm{Hom}(S, X)$ Kan complex (which will be denoted $\mathrm{Map}(S, X)$)

$\underline{Proof}$: Verify condition ③ in the lemma.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathrm{Hom}(\Delta^1, \mathrm{Hom}(S, X)) \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & \mathrm{Hom}(S, X) \end{tikzcd}$$

Proposition: S any simplicial set, X Kan complex $\Rightarrow$ $\mathrm{Hom}(S,X)$ Kan complex
(which will be denoted $\mathrm{Map}(S,X)$)

Proof: Verify condition ③ in the lemma.

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & \mathrm{Hom}(\Delta^1, \mathrm{Hom}(S,X)) = \mathrm{Hom}(S, \mathrm{Hom}(\Delta^1, X)) \arrow[d, "ev_0"] \\ B \arrow[r] \arrow[ur, dashed] & \mathrm{Hom}(S,X) \end{tikzcd} \iff \begin{tikzcd} A\times S \arrow[r] \arrow[d, hook] & \mathrm{Hom}(\Delta^1, X) \arrow[d, "ev_0"] \\ B\times S \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

but we can do that because X is a Kan complex & by the lemma. □

$f,g : X \to Y$ maps of Kan complexes, we say they are homotopic if $f \sim g$ as points of $\mathrm{Map}(X,Y)$.
i.e. $\exists\, H : X\times\Delta^1 \to Y$ $\quad H|_{X\times\{0\}} = f$, $H|_{X\times\{1\}} = g$.

Proof of the lemma:

① X Kan complex , ② $\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \mathrm{Hom}(\Delta^1, X) \arrow[d, "ev_0"] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "\exists"] & X \end{tikzcd}$, ③ $\begin{tikzcd} A \arrow[r] \arrow[d, hook] & \mathrm{Hom}(\Delta^1, X) \arrow[d, "ev_0"] \\ B \arrow[r] \arrow[ur, dashed, "\exists"] & X \end{tikzcd}$

③ $\Rightarrow$ ② ✓

③ $\Rightarrow$ ①

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & X \\ \Delta^n \arrow[ur, dashed, "?"] \end{tikzcd}$$

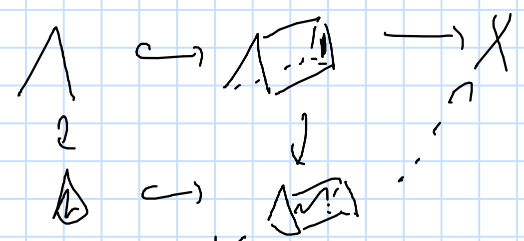
Trick: Consider the following map

$$r : \Delta^n \times \Delta^1 = N([n]\times[1]) \longrightarrow \Delta^n$$

which $r(j,0) = \begin{cases} j & j \neq i+1 \\ i & j = i+1 \end{cases}$

$$\begin{array}{cccccccc} 0 & 1 & \cdots & i & i+1 & \cdots & n \\ \downarrow & \downarrow & & \downarrow & \swarrow & & \downarrow \\ 0 & 1 & \cdots & i & i+1 & \cdots & n \end{array}$$

$r(j,1) = j$



Point $r(\Lambda^n_i \times \Delta^1) \subseteq \Lambda^n_i$
$r(\Delta^n \times \{0\}) \subseteq \Lambda^n_i$

$$\begin{tikzcd} \Lambda^n_i\times\{1\} \arrow[r, hook] \arrow[d] \arrow[rr, bend left, "id"] & \Lambda^n_i\times\Delta^1 \cup \Delta^n\times\{0\} \arrow[r, "r"] \arrow[d, "i"] & \Lambda^n_i \arrow[r] & X \\ \Delta^n\times\{1\} \arrow[r] & \Delta^n\times\Delta^1 \arrow[urr, dashed, "\exists ?"'] & & \end{tikzcd}$$

$$\begin{tikzcd} \Lambda^n_i\times\Delta^1 \cup \Delta^n\times\{0\} \arrow[r] \arrow[d] & X \\ \Delta^n\times\Delta^1 \arrow[ur, dashed, "?"] \end{tikzcd} \iff \begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & \mathrm{Hom}(\Delta^1, X) \arrow[d, "ev_0"] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "?"] & X \end{tikzcd}$$

✓

① X Kan complex , ② $\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \mathrm{Hom}(\Delta^1,X) \arrow[d, "ev_0"] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "\exists"] & X \end{tikzcd}$, ③ $\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] & \mathrm{Hom}(\Delta^1,X) \arrow[d, "ev_0"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "\exists"] & X \end{tikzcd}$

② $\Rightarrow$ ③ consider the poset of pairs $(A \subseteq C \subseteq B,\ h: C \to \mathrm{Hom}(\Delta^1,X))$

By Zorn we find a maximal element $\quad h|_A = f,\ ev_0 h = g|_C$

& we use ② to show that if the maximal element does not have all simplices, it's not maximal.

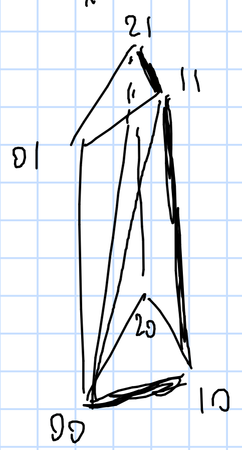
① $\Rightarrow$ ②

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \mathrm{Hom}(\Delta^1,X) \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "?"] & X \end{tikzcd} \qquad X \text{ Kan complex}$$

$$\Updownarrow$$

$$\begin{tikzcd} \partial\Delta^n\times\Delta^1 \cup \Delta^n\times\{0\} \arrow[r] \arrow[d, hook] & X \\ \Delta^n\times\Delta^1 \arrow[ur, dashed, "?"'] & \end{tikzcd}$$

Let $\sigma_i: \Delta^{n+1} \to \Delta^n\times\Delta^1$ corresponding $(0,0)<(1,0)<\dots(i,0)<(i,1)<\dots(n,1)$



$n=2$

$i=1$

$$\begin{tikzcd} (0,1) \arrow[r, dash] & (1,1) \arrow[r, dash] & (2,1) \arrow[r, dash] & \dots \arrow[r, dash] & (n,1) \\ (0,0) \arrow[r, dash] \arrow[u] & (1,0) \arrow[r, dash] \arrow[u] & (2,0) \arrow[r, dash] \arrow[u] & \dots \arrow[r, dash] & (n,0) \arrow[u] \end{tikzcd}$$

$$\bigcup \sigma_i = \Delta^n\times\Delta^1$$

$B_i = \partial\Delta^n\times\Delta^1 \cup \Delta^n\times\{0\} \cup \sigma_0 \cup \dots \cup \sigma_i \quad \subseteq \Delta^n\times\Delta^1$

Remarks: $B_{-1} = \partial\Delta^n\times\Delta^1 \cup \Delta^n\times\{0\}$

$B_n = \Delta^n\times\Delta^1$

$B_i = B_{i-1} \cup_{\Lambda^{n+1}_i} \Delta^{n+1}_{\sigma_i}$

$$\begin{tikzcd} \Lambda^{n+1}_i \arrow[r] \arrow[d] & B_{i-1} \arrow[d] \\ \Delta^{n+1} \arrow[r] & B_i \end{tikzcd}$$

Every simplex in $\underset{N([n]\times[1])}{\overset{\shortparallel}{\Delta^n\times\Delta^1}}$ is contained in some σ_i

$f:[m] \to [n]\times[1]$

$$\begin{tikzcd} B_{-1} \arrow[r] \arrow[d] & X \\ B_0 \arrow[d] & \\ \vdots \arrow[d] & \\ B_n \arrow[uuur, dashed] & \end{tikzcd} \qquad \text{It's enough} \qquad \begin{tikzcd} \Lambda^{n+1}_i \arrow[r] \arrow[d] & B_{i-1} \arrow[r] \arrow[d] & X \\ \Delta^n \arrow[r] \arrow[ur, dashed] & B_i \arrow[ur, dashed] & \end{tikzcd}$$

which we can. $\square$

<u>Lemma</u>: X Kan complex, $\gamma : \Delta^1 \to X$ path

$ev_0 : \mathrm{Hom}(\Delta^1, X) \to X$

inducing $\pi_n(\mathrm{Hom}(\Delta^1, X), \gamma) \to \pi_n(X, \underset{\partial_1\gamma}{\underbrace{\gamma 0}})$ & this is an isom $\forall n$

<u>Construction</u>: $\gamma_* : \pi_n(X, \gamma 0) \xrightarrow{ev_0^{-1}} \pi_n(\mathrm{Hom}(\Delta^1, X), \gamma) \xrightarrow{ev_1} \pi_n(X, \gamma 1)$

<u>Ex</u>: $H : X \times \Delta^1 \to Y$ homotopy $H|_{X \times \{0\}} = f$, $H|_{X \times \{1\}} = g$, $\gamma = H|_{\{x\} \times \Delta^1}$.

① Then the following diagram commutes

$$\begin{tikzcd} & \pi_n(Y, fx) \arrow[dd, "\gamma_*"] \\ \pi_n(X, x) \arrow[ur, "f_*"] \arrow[dr, "g_*"'] \arrow[r, phantom, "\circlearrowleft"] & \\ & \pi_n(Y, gx) \end{tikzcd}$$

② f homotopy equivalence $\Rightarrow$ f_* is an isomorphism.

<u>Lemma</u>: X Kan complex, $\gamma : \Delta^1 \to X$ path

$ev_0 : \mathrm{Hom}(\Delta^1, X) \to X$

inducing $\pi_n(\mathrm{Hom}(\Delta^1, X), \gamma) \to \pi_n(X, \underset{\partial_1\gamma}{\underbrace{\gamma 0}})$ & this is an isom $\forall n$

<u>Proof</u>: There's a map $\delta : X \to \mathrm{Hom}(\Delta^1, X)$

$\Delta^n \to X \mapsto \Delta^n \times \Delta^1 \xrightarrow{pr_1} \Delta^n \to X$

$ev_0 \delta = id \Rightarrow (ev_0)_* \delta_* = id \Rightarrow ev_0$ is surjective. The trick is only to verify injectivity

$\alpha, \beta : \Delta^n \to \mathrm{Hom}(\Delta^1, X)$, $ev_0 \alpha \sim ev_0 \beta$. $\exists \eta : \Delta^{n+1} \to X$ satisfying some conditions

Proof: There's a map $\delta : X \to \mathrm{Hom}(\Delta^1, X)$

$$\Delta^n \to X \longmapsto \Delta^n \times \Delta^1 \xrightarrow{pr_1} \Delta^n \to X$$

$ev_0\, \delta = id \Rightarrow (ev_0)_* \delta_* = id \Rightarrow ev_0$ is surjective. The trick is only to verify injectivity

$\alpha, \beta : \Delta^n \to \mathrm{Hom}(\Delta^1, X)$, $ev_0 \alpha \sim ev_0 \beta$. $\exists\, \eta : \Delta^{n+1} \to X$ satisfying some conditions

$$\begin{tikzcd} \mathrm{Hom}(\Delta^1, X) \arrow[r] & X \\ \partial\Delta^{n+1} \arrow[u, "\eta_0"] \arrow[r] & \Delta^{n+1} \arrow[u, "\eta"'] \arrow[ul, dashed, "\bar{\eta}"'] \end{tikzcd}$$

$\partial_n \eta = ev_0 \alpha$, $\partial_{n+1} \eta = ev_0 \beta$, $\partial_i \eta = \gamma_0$

Let $\eta_0 : \partial\Delta^{n+1} \to X$ the homotopy that a witness to $\alpha \sim \beta$ should have, i.e.

$\partial_n \eta_0 = \alpha$, $\partial_{n+1} \eta_0 = \beta$, $\partial_i \eta_0 = \gamma$.

By the lemma $\exists \bar{\eta}$ which witnesses $\alpha \sim \beta$. $\square$

We have seen: a <u>homotopy equivalence between Kan complexes induces an isom on homotopy groups</u>.

<u>Theorem</u>: Let X be a Kan complex, then

$$\eta : X \longrightarrow \mathrm{Sing}\,|X|$$

is a homotopy equivalence

<u>Corollary 1</u>: Let Y be a topological space, then

$$\varepsilon : |\mathrm{Sing}\,Y| \longrightarrow Y$$

is a weak equivalence

<u>Proof</u>: Look at the com diagram

$$\begin{tikzcd} \mathrm{Sing}\,|\mathrm{Sing}\,Y| \arrow[r, "\mathrm{Sing}\,\varepsilon"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Sing}\,Y \\ \mathrm{Sing}\,Y \arrow[u, "\eta_{\mathrm{Sing}\,Y}"] \arrow[ur, equal] & \end{tikzcd}$$

$\eta_{\mathrm{Sing}\,Y}$ is a homotopy eq. by the theorem $\Rightarrow$ $\mathrm{Sing}\,\varepsilon$ is a homotopy equivalence $\Rightarrow$ $\mathrm{Sing}\,\varepsilon$ is an iso on π_n.

But $\pi_n(\mathrm{Sing}\,Z) = \pi_n Z$ ✓ □

<u>Corollary</u>: Suppose $f : X \to Y$ is a weak eq. of Kan complexes (i.e. $\forall x \in X$ $f_* : \pi_n(X,x) \xrightarrow{\cong} \pi_n(Y, fx)$)

$\Rightarrow$ f is a homotopy eq.

<u>Proof</u>: Consider the com diagram

$$\begin{tikzcd} \mathrm{Sing}\,|X| \arrow[r, "w.e."] & \mathrm{Sing}\,|Y| \\ X \arrow[u, "h.e."] \arrow[r, "w.e."] & Y \arrow[u, "h.e."] \end{tikzcd}$$

but $|X| \xrightarrow{|f|} |Y|$ is a w.e. of CW-complexes $\Rightarrow$ it is a homotopy eq.

$\Rightarrow$ $\mathrm{Sing}\,|f|$ is a homotopy eq. □