§1 Simplicial homotopy theory

Def: Δ = category of finite non-empty totally ordered sets, and non-decreasing maps.

The typical object is of the form

$[n] = \{0 < 1 < \cdots < n\}$

$[0] = \{0\}$, $[1] = \{0 < 1\}$, $[2] = \{0 < 1 < 2\}$

$s : [n] \to [0]$ $\quad s_i = 0 \quad \forall i$ (degeneracy)

$\partial_i : [n-1] \to [n]$, $0 \leq i \leq n$ $\quad \partial_i(j) = \begin{cases} j & j < i \\ j+1 & j \geq i \end{cases}$ (face maps)

$s_i : [n] \to [n-1]$, $0 \leq i \leq n$ $\quad s_i(j) = \begin{cases} j & j \leq i \\ j-1 & j \geq i+1 \end{cases}$ (degeneracy maps)

We have a functor $\Delta \to \mathrm{Top}$

$[n] \longmapsto |\Delta^n| = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \ \sum t_i = 1\}$

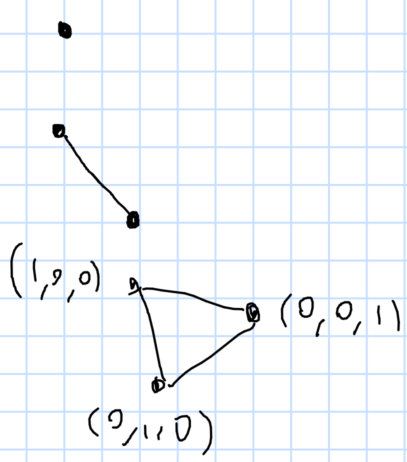
$S \longmapsto |\Delta^S| = \{(t_s)_{s \in S} \in \mathbb{R}^S \mid t_s \geq 0, \ \sum t_s = 1\}$

$f : [n] \to [m]$ $\quad f(t_0, \dots, t_n) = \left(\sum_{i \in f^{-1} 0} t_i, \sum_{i \in f^{-1} 1} t_i, \dots, \sum_{i \in f^{-1} m} t_i \right)$

$|\Delta^0| =$

$|\Delta^1| =$

$|\Delta^2| =$



$|\Delta^n|$ = convex envelope of the basis vectors.

$\partial_i : |\Delta^{n-1}| \to |\Delta^n|$ exactly the inclusions of the faces

$s : |\Delta^n| \to |\Delta^0| = *$

Def: A *simplicial set* is a functor

$X : \Delta^{op} \longrightarrow \mathrm{Set}$

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$$X : \Delta^{op} \longrightarrow Set$$

Ex: X topological space, $\operatorname{Sing} X$ is the simplicial set
$$[n] \longmapsto \operatorname{Hom}_{Top}(|\Delta^n|, X)$$

Slogan: $\operatorname{Sing} X$ knows all about the homotopy type of X

Inspired by this example we'll often refer to $X([0])$ as the set of points in X, $X([1])$ as the set of paths in X (more generally $X([n])$ is the set of n-simplices)

Ex: $C_*(X)_n$ = free abelian group on $X([n])$ & face maps given by $\sum (-1)^i \partial_i$

Ex: Suppose $\mathcal{C}$ is a category, then we define its *nerve* as the simplicial set
$$N\mathcal{C} : [n] \longmapsto \operatorname{Fun}([n], \mathcal{C})$$
$N\mathcal{C}_0$ = set of objects in $\mathcal{C}$
$N\mathcal{C}_1$ = set of arrows in $\mathcal{C}$
$N\mathcal{C}_2$ = set of pairs of composable arrows in $\mathcal{C}$ etc..

Ex: $\Delta^n := N([n])$
$$\Delta^n : [m] \longmapsto \operatorname{Hom}([m],[n])$$
simplicial analog of $|\Delta^n|$.

This comes w/ some important simplicial subset
$$\partial\Delta^n = \text{"union of } \partial_i\Delta^n \ \forall i\text{"} : [m] \longmapsto \{ f \in \operatorname{Hom}([m],[n]) \mid f \text{ is not surjective} \}$$

Ex: $\partial\Delta^0 = \emptyset$, $\partial\Delta^1 = \Delta^0 \amalg \Delta^0$

$\forall\, 0 \leq i \leq n$
$$\Lambda^n_i = \text{"union of } \partial_j\Delta^n \ \forall i \neq j\text{"} : [m] \longmapsto \{ f \in \operatorname{Hom}([m],[n]) \mid \operatorname{Im} f \not\supseteq \{0, \dots, \hat{i}, \dots, n\} \}$$

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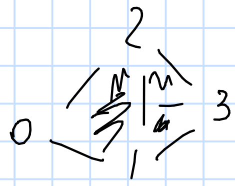
Ex: $\Lambda^2_1 =$
$$\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] & & 2 \end{tikzcd}$$
$\Lambda^2_0 =$
$$\begin{tikzcd} & 1 \\ 0 \arrow[ur] \arrow[r] & 2 \end{tikzcd}$$
$\Lambda^3_0 =$ 

Prop: $\mathrm{Sing} : \mathrm{Top} \to \mathrm{sSet}$ has a left adjoint

$|-| : \mathrm{sSet} \to \mathrm{Top}$ (geometric realization)

$$\mathrm{Hom}_{\mathrm{sSet}}(X, \mathrm{Sing}\, Y) = \mathrm{Hom}_{\mathrm{Top}}(|X|, Y)$$

$$|X| = \coprod_{n \ge 0} X([n]) \times |\Delta^n| \Big/ \sim$$

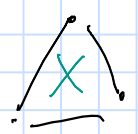
$\forall f : [n] \to [m]$

$x \in X([m])$, $t \in |\Delta^n|$

$(f^* x, t) \sim (x, ft)$

Ex: $|\Delta^n| = |\Delta^n|$ long & boring

. $|-|$ preserves colimits : $|\partial\Delta^n| =$ union of the proper faces in $|\Delta^n|$

$|\partial\Delta^2| =$ 

Proof: What is a map $X \to \mathrm{Sing}\, Y$?

$x \in X([m]) \longmapsto \sigma_x : |\Delta^m| \to Y$

$\forall f : [n] \to [m]$, $\forall x \in X([m])$

$$\begin{tikzcd} {|\Delta^m|} \arrow[r, "\sigma_x"] & Y \\ {|\Delta^n|} \arrow[u, "f"] \arrow[ur, "\sigma_{f^*x}"'] \arrow[ur, phantom, "\circlearrowleft"] & \end{tikzcd}$$

$(\Leftrightarrow)$ $\coprod_{m \ge 0} X([m]) \times |\Delta^m| \to Y$ that respects the equivalence relation I wrote. $\Leftrightarrow$ $|X| \to Y$. $\square$

Let's examine $|\Delta^1 \times \Delta^1|$

What are the simplices of $\Delta^1 \times \Delta^1$? They are pairs of maps $f:[n] \to [1]$, $g:[n] \to [1]$

$f: [00 0 \cdots 1 \cdots 1]$

$g: [001 \cdots 1]$

$$\begin{tikzcd}
(0,0) \arrow[r, "{([00],[01])}"] \arrow[d] \arrow[dr, "{([01],[01])}"] & (0,1) \arrow[d, "{([01],[11])}"] \\
(1,0) \arrow[r] & (1,1)
\end{tikzcd}$$

<u>Claim:</u> $\Delta^1 \times \Delta^1 = \Delta^2 \cup_{\Delta^1} \Delta^2$, $\Delta^1 = ([01],[01])$

The two 2-simplices are $([001],[011])$ and $([011],[001])$

for example $([001],[001]) = s_0([01],[01])$.

Every higher simplex factors through one of those 2-simplices

$|\Delta^1 \times \Delta^1|$ = 2-dimensional square.

We can write this as $|\Delta^1 \times \Delta^1| \to |\Delta^1| \times |\Delta^1|$ is a homeomorphism

<u>Proposition:</u> X, Y simplicial sets. The map

$$|X \times Y| \to |X| \times |Y|$$

is a homeomorphism.

<u>Proof:</u> Reduction to $X = \Delta^m$, $Y = \Delta^n$.

Let's assume first $Y = \Delta^n$. Let us consider the poset of simplicial subsets of X A s.t.

$$|A \times \Delta^n| \to |A| \times |\Delta^n|$$

is a homeomorphism. By Zorn's lemma this has a maximal element A [we're using that top spaces are cg to show $- \times |\Delta^n|$ commutes w/ colimits]

Let A such a maximal element. If $A = X$ ✓

If $A \neq X$ we can find a simplex $\sigma \in X([m])$ not in A of minimal dimension.

σ has to be nondegenerate (otherwise it wouldn't be minimal) $\Rightarrow$ we can take $A' = A \cup_{\partial \Delta^m} \Delta^m$ but then using the fact that $|- \times \Delta^n|$, $|-| \times |\Delta^n|$ commute w/ products we get a contradiction.

Then to prove it for a gen Y, you fix X & consider the poset of $B \subseteq Y$ s.t.
$|X \times B| \xrightarrow{\sim} |X| \times |B|$. $\square$ (Gabriel - Zisman, thm 3.1)

FACT: If Z is a cg top space

$- \times^{g} Z$ has a left adj $\mathrm{Map}(Z, -)^{cg}$ $\Rightarrow$ $- \times^{g} Z$ commutes w/ colimits

$(A \cup_B C) \times Z = (A \times Z) \cup_{B \times Z} (C \times Z)$.

The gen. statement would be $|X \times Y| \xrightarrow{\sim} (|X| \times |Y|)^{cg} = |X| \times^{g} |Y|$

The next goal is to define $\pi_n(X, x)$, but for doing this we need to ask something more of X.
For example $\pi_0 X$. Ideally you want to take $X([0])$ & quotient by the relation $x \sim y$
if there's a path from x to y, i.e. $\exists \gamma \in X([1])$ $\partial_1 \gamma = x$, $\partial_0 \gamma = y$.
Problem: $\sim$ is not an equivalence relation in general!
Ex: $X = \Delta^1$ $0 \sim 1$ but $1 \nsim 0$.
Let's try to prove transitivity $\gamma : x \to y$, $\delta : y \to z$

$$\begin{tikzcd} \Delta^1 \cup_{\Delta^0} \Delta^1 = \Lambda^2_1 \arrow[r] \arrow[d] & X \\ \Delta^2 \arrow[ur, dashed, "want"'] & \end{tikzcd} \qquad \begin{tikzcd} & y \arrow[dr] & \\ x \arrow[ur] \arrow[rr, dashed, phantom, "\shortparallel" description] \arrow[rr, dashed] & & z \end{tikzcd}$$

But this is not possible in general.

Def: $X \in sSet$ is a Kan complex if $\forall n \geq 0$, $\forall 0 \leq i \leq n$ $\forall$
$$f : \Lambda^n_i \to X$$
$\exists$ an extension $\tilde{f} : \Delta^n \to X$.

Def: $X \in sSet$ is a <u>Kan complex</u> if $\forall n \geq 0$, $\forall 0 \leq i \leq n$ $\forall$

$$f : \Lambda^n_i \longrightarrow X$$

$\exists$ an extension $\tilde{f} : \Delta^n \longrightarrow X$.

Ex: Let Y be a topological space, then $\mathrm{Sing}\, Y$ is a Kan complex.

Proof: a map $\Lambda^n_i \xrightarrow{f} \mathrm{Sing}\, Y$ is the same thing as a map $|\Lambda^n_i| \xrightarrow{|f|} Y$, & our goal is to extend it to $|\Delta^n| \xrightarrow{|\tilde{f}|} Y$. But the inclusion $|\Lambda^n_i| \subseteq |\Delta^n|$ has a retraction (for example sending the barycenter of the i-th face to the i-th vertex & extending linearly:

$$\begin{tikzcd} |\Lambda^n_i| \arrow[d, hook] \arrow[dr, equal] & \\ |\Delta^n| \arrow[r, "r"'] & |\Lambda^n_i| \end{tikzcd}$$

$$|\tilde{f}| = |f| \circ r. \qquad \square$$

Remark: If X is a Kan complex, the relation $\sim$ on X_0 we defined above is an equivalence relation.

Ex:

$$\begin{tikzcd} & x \arrow[d] \\ y \arrow[r, equal, "i_s^* y"'] & y \end{tikzcd} = \begin{tikzcd} \Lambda^2_0 \arrow[r] \arrow[d, hook] & X \\ \Delta^2 \arrow[ur] & \end{tikzcd} \qquad (\text{symmetry})$$

Ex: X, Y topological spaces [as ugly as you like]

$\mathrm{Map}(X,Y)$ is the simplicial set

$$[n] \longmapsto \mathrm{Hom}_{Top}(X \times |\Delta^n|, Y)$$

pts are continuous maps
paths are homotopies.

$\mathrm{Map}(X,Y)$ is a Kan complex (<u>exercise</u>)

$\pi_0 \mathrm{Map}(X,Y) = [X,Y]$ (homotopy classes of maps)

Ex: M, N smooth manifolds

$\mathrm{Emb}(M,N) \subseteq \mathrm{Map}(M,N)$ simplicial subset whose n-simplices are maps

$f: M \times |\Delta^n| \to N$ s.t. $\forall t \in |\Delta^n|$ $f|_{M\times\{t\}}$ is a smooth embedding

so pts of $\mathrm{Emb}(M,N)$ are embeddings & paths are isotopies.

In the smooth category this is still Sing of something (but it's very hard to define)

In the PL category it's not known if this is Sing of something.

Ex: $\mathrm{Emb}(M,N)$ is a Kan complex.

Def: $\pi_0 X = X([0])/\sim$

Def: $\sigma, \tau \in X([n])$ are homotopic relative to the ∂ if $\sigma|_{\partial\Delta^n} = \tau|_{\partial\Delta^n}$

and $\exists\, \psi \in X([n+1])$ w/

- $\partial_n \psi = \sigma$, $\partial_{n+1}\psi = \tau$
- $\forall\, 0 \le i \le n-1$ $\partial_i \psi = s_{n-1}\partial_i \sigma = s_{n-1}\partial_i\tau$

Ex: $n=1$ $x \xrightarrow{\sigma} y$, $x \xrightarrow{\tau} y$

$$\begin{tikzcd} & y \arrow[d, equal, "sy"] \\ x \arrow[ur, "\sigma"] \arrow[r, "\tau"'] \arrow[ur, phantom, "\psi", shift right=3ex] & y \end{tikzcd}$$

Exercise: $X = \mathrm{Sing}\, Y$, $\sigma, \tau : |\Delta^n| \to Y$ are homotopic rel to ∂ iff they are homotopic rel to $|\partial\Delta^n| \subseteq |\Delta^n|$ in the classical sense.

Proposition: Being homotopic rel ∂ is an equivalence relation on $X([n])$ if X is a Kan complex

Def: X Kan complex, $x \in X([0])$

$$\pi_n(X,x) := \{\sigma \in X([n]) \mid \sigma|_{\partial\Delta^n} = x\}/\sim . \quad \text{(n-th homotopy group)}$$

Ex: Δ^n is not a Kan complex, in general $\mathcal{C}$ category $N\mathcal{C}$ is a Kan complex iff $\mathcal{C}$ is a groupoid.