LEVINE’S CHOW’S MOVING LEMMA

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The main result of this note is proven in [4], using results from [2]. What is here is essentially a simplified version of the proof (at the expense of some generality). A very useful reference for the classical moving lemma is [6].

1. Dimension of $B$-schemes of finite type

Let $f : X \to B$ be an irreducible $B$-scheme of finite type, where $B$ is a nice enough scheme (regular, universally catenary, Noetherian). We can define
\[
\dim X = \dim X_{\xi} + \dim \tilde{\xi}
\]
where $\xi$ is the generic point of $X$. This definition is set up in order to make true that the dimension of the generic fiber is equal to the dimension of the total space minus the dimension of the base. This definition is well behaved if the base has a well-behaved notion of dimension of points ([7, Tag 02JW]). Note that if $Y \subseteq X$ locally closed, the codimension of $Y$ in $X$ using Krull dimension is the same as the codimension using this new notion of dimension.

We say that a finite type $B$-scheme $C$ is of pure relative dimension (oprд) if the dimension of all the fibers is constant, that is for every $b \in B$
\[
\dim C_b = \dim C - \dim B.
\]

Note that every irreducible flat finite type $B$-scheme is oprд ([7, Tag 00ON]). When $B$ is regular Noetherian scheme of dimension one, this is equivalent to being dominant onto a connected component of $B$ ([7, Tag 00QK]).

2. Higher Chow groups

Recall the definition of Bloch’s higher Chow groups: if $X$ is a regular noetherian scheme we can denote for $p, q \geq 0$ with $X^{(p,q)}$ the set of irreducible closed subsets $Z \subseteq X \times \Delta^p$ such that for every $S \subseteq \{0, \ldots, p\}$
\[
\text{codim}(Z \cap X \times \Delta^S) = q.
\]
Bloch’s cycle complex is now the simplicial abelian group
\[
z^q(X) = Z(X^{(\bullet,q)}).
\]

Now let us suppose that $C = \{C_1, \ldots, C_m\}$ is a finite family of irreducible locally closed subsets of $X$ and $\{d_1, \ldots, d_m\}$ are positive integers. We can form the set $X_{C,d}^{(p,q)}$ of cycles that meet the $C_i$’s $d_i$-properly as the set of those $Z \in X^{(p,q)}$ such that for every $S \subseteq \{0, \ldots, p\}$ and every $i = 1, \ldots, m$
\[
\text{codim}(Z \cap C_i \times \Delta^S) \geq q - d_i.
\]
Similarly we have the simplicial subgroup $z^q_{C,d}(X) \subseteq z^q(X)$. 
Theorem 1 (Chow’s moving lemma ([4] Th. 2.6.2)). Let $X$ be a “nice enough” affine smooth $B$-scheme and $C = \{C_1, \ldots, C_m\}$ a finite family of irreducible locally closed oprd subsets. Then the inclusion

$$z^q_{C,d}(X) \subseteq z^q(X)$$

is a weak equivalence.

We will make precise in the following what it means to be “nice enough”. For example when $B$ is a field, every affine scheme will be “nice enough”. We will show that every smooth scheme has a Nisnevich cover by “nice enough” schemes.

The following lemma is a very important reduction, often allowing us to reduce from the case of a general $B$ to the case when $B = \text{Spec } k$ for $k$ a field.

Lemma 1.1. Let $C = \{C_1, \ldots, C_m\}$ be a collection of locally closed oprd subsets of $X$. Then $W \in X^{(p,q)}$ is in $X^q_{C,d}$ if and only if for every $b \in B$, $W_b$ is in $X^{(p, q - \text{codim } b)}$.

Proof. Suppose that for every $b \in B$, $W_b$ is in $X^{(p, q - \text{codim } b)}$. Then we need to prove that for every $C_i$ and every $S$

$$\dim(W \cap C_i \times \Delta^S) \leq \dim(C_i) + \#S - q + d_i.$$  

let $w \in W$ be a generic point of $W \cap C_i \times \Delta^S$. It is above some $b \in B$, so that

$$\dim w \leq \dim(C_i)_b + \#S - q + \text{codim } b + d_i + \dim b = \dim(C_i) + \#S - q + d_i.$$  

Since this is true for all the irreducible components of $W \cap C_i \times \Delta^S$, the thesis is proven. The viceversa is analogous. \qed

3. The case of affine space

3.1. Weak homotopy invariance. In order to prove the moving lemma for affine space we will need to be able to talk about $\mathbb{A}^1$-homotopies of maps. This would be very easy if we had $\mathbb{A}^1$-homotopy invariance for the higher Chow groups, but unfortunately we do not have it yet (and in fact you need the moving lemma to prove it). So we will have to prove a weaker version of homotopy invariance, that is enough for our purposes.

Let $(X \times \Delta^1)^{(p,q)}_{C,d,h}$ be the space of those $Z \subseteq X \times \Delta^1 \times \Delta^p$ closed irreducible subsets such that for every nondegenerate simplex $\sigma$ of $\Delta^p \times \Delta^1$

- The codimension of $Z \cap (X \times \sigma)$ in $X \times \sigma$ is $q$;
- For every $i = 1, \ldots, m$ the codimension of $Z \cap (C_i \times \sigma)$ is greater or equal than $q - d_i$.

As usual we form $z^q_{C,d,h}(X \times \Delta^1)$ the corresponding complex. We have maps

$$i_0^*, i_1^* : z^q_{C,d,h}(X \times \Delta^1) \to z^q_{C,d}(X)$$

given by sending $[W]$ to $[W \cap \Delta^p \times \{j\}]$ for $j = 0, 1$.

Lemma 1.2 ([4] Lm. 3.2.4). There is a homotopy between $i_0^*$ and $i_1^*$ natural in $X$ with respect to flat maps.
Proof. The idea is to construct a simplicial homotopy
\[ z^q_{C,d,h}(X \times \Delta^1) \to \text{Map}(\Delta^1, z^q_{C,d}(X)). \]
So for every nondegenerate simplex \( \sigma = \Delta^r \) of \( \Delta^p \times \Delta^1 \) we need to give a map
\[ z^q_{C,d,h}(X \times \Delta^1, p) \to z^q_{C,d}(X, r) \]
which is natural in \( \sigma \) and \( p \). But this is simply the map sending
\[ [W] \mapsto [W \cap X \times \sigma]. \]
It is clear that this is a homotopy between \( i_0^* \) and \( i_1^* \) and that this homotopy is natural in \( X \). \( \square \)

3.2. Translations. Let us fix a smooth \( B \)-scheme \( Y \) (in fact \( Y \) is not even required to be smooth, up to changing our notation to work with dimension instead of codimension). Let \( C = \{C_1, \ldots, C_m\} \) irreducible locally closed opsd subsets of \( \mathbb{A}^n_B \) and \( d_1, \ldots, d_m \geq 0 \) positive integers. We want to prove the following statement

Theorem 2 (Moving lemma for affine space ([4 Pr. 3.3.4]). Let \( B \) be a semilocal PID. Then the inclusion
\[ z^q_{C,d}(\mathbb{A}^n_B) \to z^q(\mathbb{A}^n_B) \]
is an equivalence for all smooth \( B \)-schemes \( Y \).

The idea behind this theorem is that if \( W \in (\mathbb{A}^n_Y)^{(p,q)} \), it intersects all \( C_i \)'s properly after a generic translation. Let us introduce the generic translation. Let
\[ B' = \text{Spec} A(x_1, \ldots, x_n) \]
(where \( A(x_1, \ldots, x_n) \) is the localization of \( A[x_1, \ldots, x_n] \) at all primitive polynomials, that is those polynomials that do not become zero at any point of \( B = \text{Spec} A \)) and let us consider the map
\[ \phi : \mathbb{A}^n_B \times \Delta^1 \to \mathbb{A}^n_B \]
given by the map of rings
\[ A[t_1, \ldots, t_n] \to A(x_1, \ldots, x_n)[t_1, \ldots, t_n, s] \]
sending
\[ t_i \mapsto t_i + sx_i. \]
We’ll also denote by \( \phi_0, \phi_1 \) the restrictions of \( \phi \) to \( \mathbb{A}^n_B \times \{0\} \) and \( \mathbb{A}^n_B \times \{1\} \) respectively. It is clear that \( \phi_0 \) is just the pullback along the map \( B' \to B \) and that \( \phi_1 \) is our “generic translation”. Let’s see how our cycles behave under them. For brevity we will write \( Y' \) for \( Y \times_B B' \).

Lemma 2.1. Let \( W \in (\mathbb{A}^n_Y)^{(p,q)} \). Then
- We have \( \phi_1^{-1} W \in (\mathbb{A}^n_Y)^{(p,q)} \); 
- Suppose that \( W \in (\mathbb{A}^n_Y)^{(p,q)} \). Then \( \phi_1^{-1} W \in (\mathbb{A}^n_Y \times \Delta^1)^{(p,q)} \).

\footnote{We are committing a slight abuse of notation by writing \( z^q_{C,d} \) where it should be \( z^q_{C \times \mathbb{A}^n_d} \)}
**Proof.** By lemma [1.1] it is enough to prove this when $B$ is a field. In fact, since the $C_i$’s are equidimensional, $W \in (\mathbb{A}^n_{Y'})^{(p,q)}$ if and only if $W_b \in (\mathbb{A}^n_{Y_b})^{(p,q-1)}$ for every closed point $b \in B$ and $W_q \in (\mathbb{A}^n_{Y_q})^{(p,q)}$ for every generic point $\eta \in B$.

Let us denote $\Delta^1_b = \Delta^1 \setminus \{0\}$ (morally we think of it as the interval $[0,1]$). For any nondegenerate simplex $\sigma$ of $\Delta^p \times \Delta^1$ we denote $\sigma_0 = \sigma \cap \Delta^p \times \{0\}$ and $\sigma_+ = \sigma \cap \Delta^p \times \Delta^1_b$. We will prove that for any locally closed $C \subseteq \mathbb{A}^n_B$,

$$\text{codim}_{C \times \mathbb{Y}' \times \mathbb{A}_+} (\phi^{-1} W \cap C \times \mathbb{Y}' \times \mathbb{A}_+) = q$$

This implies everything we might want. Let us consider the map

$$(W \cap \mathbb{A}_+^n \times \mathbb{A}_+) \times C \to \mathbb{A}^n_C$$

sending

$$(w, c) \mapsto \frac{w_1 - c}{s}$$

(recall that $W \subseteq \mathbb{A}^n_B \times_B \mathbb{Y} \times \Delta^p \times \Delta^1$, in the above formula $w_1$ is the first component and $s$ is the last). The generic fiber of this map is precisely $\phi^{-1} W \cap (C \times \mathbb{Y}' \times \mathbb{A}_+)$, so it is either empty or it has dimension

$$\dim(W \cap \mathbb{A}_+^n \times \mathbb{A}_+) + \dim C - n = (n + \dim \mathbb{Y} + \dim \sigma - q) + \dim C - n =$$

$$= \dim((C \times \mathbb{Y}') \times \mathbb{A}_+) - q.$$  

\[\square\]

Ok great, so we have a well-defined map

$$\phi^* : z_h(\mathbb{A}_+^n \times \Delta^1)/z_{C,d}(\mathbb{A}_+^n \times \Delta^1) \to z(\mathbb{A}_+^n)/z_{C,d}(\mathbb{A}_+^n)$$

such that $i^*_h \phi^*$ is the pullback along $B' \to B$ and $i^*_h \phi^* = \phi^*_1$ is the zero map. But $i^*_h$ and $\phi^*$ are homotopic, so we have that $\phi^*_0$ and $\phi^*_1$ are homotopic. That is the pullback along $B' \to B$ is nullhomotopic.

Recall that our goal is to show that the cofiber

$$z^q(\mathbb{A}_+^n)/z_{C,d}(\mathbb{A}_+^n)$$

is contractible, and so far we have showed that the map

$$z^q(\mathbb{A}_+^n)/z_{C,d}(\mathbb{A}_+^n) \to z^q(\mathbb{A}_+^n)/z_{C,d}(\mathbb{A}_+^n)$$

is nullhomotopic. To conclude our proof it suffices to show that the pullback along $B' \to B$ is injective on homotopy groups. This is the first (and only) place where we will use the hypothesis that $B$ is a semilocal PID. Since we will need this fact more generally in the future let us collect it in a separate proposition.

**Proposition 2.1.** Let $B' = \text{Spec } A(x_1, \ldots, x_n)$ a purely transcendental extension and let $X$ be a smooth scheme, $C = \{C_1, \ldots, C_m\}$ a finite collection of locally closed oprd subsets and $d = \{d_1, \ldots, d_m\}$, $d' = \{d'_1, \ldots, d'_m\}$ collections of positive integers with $d'_i \leq d_i$. Then the map

$$z^q_{C,d}(X)/z^q_{C,d'}(X) \to z^q_{C,d}(X_{B'})/z^q_{C,d'}(X_{B'})$$

is an injection on homotopy groups.

**Proof.** Let us first assume that all residue fields of $B$ are infinite. Then let us choose $Z \in z^q_{C,d}(X, p)$ such that all of its boundaries lie in $z^q_{C,d'}(X, p - 1)$ and such that its image is nullhomotopic after basechange along $B' \to B$. This means that there is $Y \in z^q_{C,d'}(X_{B'}, p + 1)$ such that

$$z^q_{C,d}(X, p) = z^q_{C,d'}(X_{B'}, p + 1)$$

where $z^q$ is the order $q$ component.
Now $Y$ and $W$ are cycles of $X \times \Delta^* \times \mathbb{A}^n_B$ lying above the generic section $B'$. Let us take their closure (that is the closure of all of their components) in $X \times \Delta^* \times \mathbb{A}^n_B$.

For $b \in B$ closed point we have that $Y_b$ is a cycle of $X \times \mathbb{A}^n_{k(b)}$ such that the generic fiber satisfies a finite number of codimension constraint. So there is an open subset $U_b \subseteq \mathbb{A}^n_{k(b)}$ such that for every point $x_b \in U_b$ the fiber $Y_{x_b}$ satisfies the same codimension constraints and the same for $W_{x_b}$.

Since the field $k(b)$ is infinite we can choose such points $x_b$ such that they are rational. Moreover, since $B$ is a semilocal PID we can apply the Chinese remainder theorem and get a section $x: B \rightarrow \mathbb{A}^n_B$ such that its value at a closed point $b \in B$ is the point $x_b$. But then the pullbacks $Y_x$ and $W_x$ provide a nullhomotopy of $Z$.

To do the case of finite residue fields we need a little more. Using the next lemma we can find $B''$, $B'''$ finite étale covers of $B$ of coprime degree such that $Z$ goes to zero after pullback along $B'' \rightarrow B$ and $B''' \rightarrow B$. But since the composition of pushforward and pullback gives multiplication by the degree we have that $[B'': B]/Z = [B''': B]/Z = 0$. Since the degrees are coprime $Z = 0$.

**Lemma 2.2 (theorem).** Let $A$ be a semilocal PID, $U \subseteq \mathbb{A}^n_A$ be an open subscheme, faithfully flat over $A$. Then there are $A'$, $A''$ finite étale extensions of $A$ of coprime degree such that there are sections $\text{Spec } A', \text{Spec } A'' \rightarrow U$.

**Proof.** Let $b \in \text{Spec } A$ be a maximal ideal of $A$. Then there exists a positive integer $r_b > 0$ such that for every separable extension $F/k(b)$ of degree greater than $r_b$ there are $F$-rational points in $U_b$. We can prove this by induction on $n$ (the dimension of the ambient space). If $n = 1$ we can just pick $r_b$ equal to the number of points in the complement of $U_b$. Suppose now it is true for all open subsets of $\mathbb{A}^n_{k(b)}$. Then by induction we can find $r_1$ such that for every extension of degree greater than $r_1$ there is a rational line $L$ not contained in the complement of $U_b$. So we can take $r_b = 2r_1 + 1$.

Since $A$ is a semilocal PID we can find $l', l''$ different primes greater than all of the $\mathfrak{m}_i$’s and $f_b, f'_b \in k(b)[T]$ irreducible separable monic polynomial of degree $l, l'$. Then using the Chinese remainder theorem we can find $f, f'$ monic separable irreducible polynomial such that they reduce to $f_b, f'_b \mod b$. Then we can let $A' = A[T]/(f(T))$ and $A'' = A[T]/(f''(T))$. □

**4. Homotopy invariance**

Armed with the weak moving lemma we can now prove homotopy invariance.

**Theorem 3 (lemma).** Let $B$ the spectrum of a semilocal PID. Then for every $X$ smooth $B$-scheme the map

$$p^*: z^q(X \times \mathbb{A}^1) \rightarrow z^q(X)$$

is an equivalence.

In order to prove it we will need first to prove a different version of lemma 1.2

**Lemma 3.1 (Lm. 3.3.6).** Let $i_0, i_1: X \rightarrow X \times \mathbb{A}^1$ be the inclusions at 0 and 1 respectively. Then the maps

$$i^*_0, i^*_1: z^q_{X \times \{0\}, X \times \{1\}}(X \times \mathbb{A}^1) \rightarrow z^q(X)$$
are induce the same map on homotopy groups.

Proof. Since $\pi^*$ is injective on homotopy groups by proposition \ref{prop:homotopy_groups}, it is enough to show that $\pi^* i_0^* = i_0^* \pi^*$ and $\pi^* i_1^* = i_1^* \pi^*$ are homotopic. Let $B' = \text{Spec} \ A[x]$ and let $\phi : A^{1,1} \times \Delta^1 \to A^{1,1}$ be the map corresponding to the map of rings $A[t] \to A(x)[t,s] \quad t \mapsto t + sx$.

Let $C = \{\{0\}, \{1\}\}$ as a collection of subsets of $A^{1,1}$. By the reasoning after lemma \ref{lem:homotopy_groups} the maps $\phi^*_1, \pi^* : z^q(X \times A^1) \to z^q(X_{B'} \times A^1)$ are homotopic. So it is enough to show that $i_0^* \phi^*_1$ and $i_1^* \phi^*_1$ are homotopic. But by proceeding as in the proof of lemma \ref{lem:homotopy_groups} we know that $\phi^*_1$ lands in $z^q_A(X_{B'}, A^1)$, so the thesis follows from lemma \ref{lem:homotopy_groups}.

Armed with this lemma the proof of theorem \ref{thm:generic_projections} follows now immediately. In fact let us consider the multiplication map $\mu : A^1 \times A^1 \to A^1$. This is a flat map and we have commutative diagrams

\[
\begin{array}{ccc}
\mu^* & : & z^q_{\{0\}}(X \times A^1) \\ & \downarrow & \downarrow i^*_0, i^*_1 \\ z^q_{X \times A^1 \times \{0,1\}}(X \times A^1 \times A^1) & \xrightarrow{\mu^*} & z^q(X \times A^1)
\end{array}
\]

That is $i^*_0 \mu^* = \iota$ and $i^*_1 \mu^* = p^* i_0^*$. That is $p^* i_0^*$ and $\iota$ induce the same map on homotopy groups. But $\iota$ is an equivalence by theorem \ref{thm:homotopy_groups} so $p^* i_0^*$ is an equivalence too. Since $i_0^* \mu^*$ is the identity we are done.

5. Generic projections

Now let $X \subseteq A^n_B$ be a smooth affine $B$-variety and let $\tilde{X}$ its closure in $P^n_B$. If $M_{r \times n}(B) = \text{Spec} \ A[a_{ij}]_{1 \leq i \leq r, 1 \leq j \leq n}$ is the $B$-scheme of $r \times n$ matrices we can define $U_X = \{A \in M_{r \times n}(B) \mid \text{rk} A = r, \ L_A \cap \tilde{X} = \emptyset\}$ where $L_A = \{[x : 0] \mid Ax = 0\} \subseteq P^n$ is an $(n - r)$-dimensional projective subspace. Since it is defined by the inequalities $\{Ax \neq 0\}_{[0 : x] \in \tilde{X}}, U_X$ is open. Moreover it is nonempty. In fact its complement is the image of

\[
\{(x, A) \in A^{n-1} \times M_{r \times n} \mid Ax = 0, \ [0 : x] \in X\}
\]

onto $M_{r \times n}$, which cannot be surjective since it is of dimension at most $(r - 1) + (n - r) = r$ (since all the fibers of the projection to the intersection of $\tilde{X}$ with the hyperplane at infinity have dimension $(n - 1)r$).

We say that the embedding $X \to A^n_B$ is $B$-generic if the projection $U_X \to B$ is faithfully flat.

Theorem 4 (Chow’s moving lemma ([\ref{ref:chow}], Th. 2.6.2)). Let $B$ a semi-local PID and $X$ be a finite type $B$-scheme with a $B$-generic embedding. If $C = \{C_1, \ldots, C_m\}$ is a finite collection of locally closed oprd subschemes of $X$ and $d = \{d_1, \ldots, d_m\}$ are nonnegative integers, then the inclusion

\[
z^{q,C,d}(X) \to z^q(X)
\]
is an equivalence of simplicial abelian groups.

For the rest of the section we will assume we are working with a $B$-generic embedding. In the case of a $B$-generic embedding we have a section $B' \to U_X$ where

$$B' = \text{Spec } A(a_{ij})_{1 \leq i \leq r, 1 \leq j \leq n}.$$  

This section picks out the generic point of every fiber (in fact $B'$ is the intersection of all faithfully flat opens of $M_{rxn}(B)$).

There is a map

$$f : U_X \times_B X \to U_X \times_B \mathbb{A}^r_B \quad (A, x) \mapsto (A, Ax).$$

This map is obviously finite. Moreover it is flat, since finite morphisms between Noetherian regular schemes are flat ([1, Cor. 18.17]). Since the embedding of $X$ is $B$-generic, we can pullback along the section $B' \to U_X$ to get a map

$$f : X_{B'} \to \mathbb{A}^r_{B'}$$

which is an étale map of ramification locus

$$R = \{ x \in X_{B'} \mid \ker(A|_{T_x X_{B'}}) \neq 0 \}.$$  

Our strategy to prove theorem [1] is, as before, to show that the basechange map

$$\pi^* : z^q_{C,d}(X)/z^q_{C,d-1}(X) \to z^q_{C,d}(X_{B'})/z^q_{C,d-1}(X_{B'})$$

is nullhomotopic and then invoke proposition [2.1]. We will do this by showing that $\pi^*$ is in fact homotopic to $f_* f^* \pi^*$ and so it factors through a similar quotient for $\mathbb{A}^r_{B'}$, which is zero by our previous result.

First let us study how does the ramification behave when $B$ is a field.

**Lemma 4.1.** Suppose $B = \text{Spec } k$ a field. Let $Q$ be a locally closed subset of $X \times \Delta^p$. Then

$$\dim(Q_k \cap R \times \Delta^p) \leq \dim Q - 1.$$  

In particular, by setting $p = 0$ and $Q = X$, $f$ is generically étale.

**Proof.** Clearly $Q_k \cap R \times \Delta^p$ is the generic fiber of the variety

$$T_0 = \{ (x, t, A) \in Q \times U_X \mid \ker(A|_{T_x X}) \neq 0 \}.$$  

But $T_0$ is the projection onto $Q \times U_X$ of

$$T = \{ (x, t, A, v) \in Q \times U_X \times \mathbb{P}^{n-1} \mid Av = 0, \ v \in T_x X \}.$$  

Now the projection $T \to Q$ has fibers of dimension $(r - 1) + nr - r = nr - 1$ (since they are the pairs $(A, v) \in U_X \times \mathbb{P}T_x X$ such that $Av = 0$). So

$$\dim T_0 \leq \dim T = \dim Q + nr - 1.$$  

Hence

$$\dim(Q_k \cap R \times \Delta^p) \leq \dim Q - 1.$$  

Recall that our goal is to show that for every $W \in X^p_{C,d}$ the cycle $f_* f^* [W_{B'}] - [W_{B'}]$ lies in $z^q_{C,d-1}(X_{B'})$. To begin, we start studying the irreducible components of $f^{-1} f(W)$. To do so, we need first a classical lemma about secant varieties.
Lemma 4.2 ([5], Pr. 4). Let
\[ D = \{(x, y) \in X_{k'} \times X_{k'} \mid x \neq y, \ f(x) = f(y)\}. \]
Then the closure of \( D \) in \( X_{k'} \times X_{k'} \) is contained in \( D \cup \Delta \).

Proof. Since \( D \) is closed in \( X_{k'} \times X_{k'} \), it is clearly sufficient to show that if \( (x, x) \in D \) then \( x \in R \).

Now \( D \) is the generic fiber of \( D_0 = \{(x, y, A) \in X \times X \times U_X \mid x \neq y, A(x-y) = 0\} \)
so it suffices to show that if \( (x, x, A) \in D_0 \) then \( \ker A|_{T_x X} \neq 0 \). But \( D_0 \) is the projection of \( E_0 = \{(x, y, A, [v]) \in X \times X \times U_X \times \mathbb{P}_k^{n-1} \mid x \neq y, A\ast v = 0, [v] = [x - y]\} \).

So it suffices to show that if \( (x, x, A, [v]) \in E \), where \( E \) is the closure of \( E_0 \) in \( X \times X \times U_X \times \mathbb{P}_k^{n-1} \), then \( v \in T_x X \).

Let \( g \in I(X) \) be a polynomial which is zero on \( X \). If \( (x, y, A, [v]) \in E_0 \) we can write
\[ 0 = f(y) = f(x) + \sum_{i=1}^{n} \partial_i f(x)(y_i - x_i) + \sum_{|\alpha| \geq 2} a_\alpha (x-y)^\alpha \]
For every multiindex \( \alpha \) let us choose \( i_\alpha \) such that \( \alpha = (\alpha', i_\alpha) \). Then, since \( v = s(x-y) \) we can write
\[ \sum_{i=1}^{n} \partial_i f(x)v_i + \sum_{|\alpha| \geq 2} a_\alpha v_{i_\alpha}(x-u)^{\alpha'} = 0. \]
The above is a polynomial relation that holds on \( E_0 \), so it must hold on its Zariski closure \( E \). Now let \( (x, x, A, [v]) \in E \). Substituting in the above relation we get
\[ \sum_{i=1}^{n} \partial_i f(x)v_i = 0 \]
so \( v \in T_x X \). \( \square \)

Proposition 4.1 ([2] Lm. 3.5.4). Suppose that \( B = \Spec k \), with \( k \) a field. Let \( W \in X^{(p,q)} \), and \( Y \) be an irreducible component of \( f^{-1}f(W_{k'}) \) not contained in \( W_{k'} \). Then for every \( C \subseteq X \) locally closed and \( S \subseteq \{0, \ldots, p\} \)
\[ \dim(Y \cap C \times \Delta^S) \leq \max(\dim C + \#S - q, \dim(W \cap C \times \Delta^S) - 1) \] .
In particular, choosing \( C = X, Y \in X^{(p,q)} \).

Proof. Let \( T = \{(y, x, t, A) \in X \times W \times U_X \mid y \neq x, A(x-y) = 0\} \)
and let \( T_{k'} = \{(y, x, t) \in X_{k'} \times W_{k'} \mid y \neq x, f(x) = f(y)\} \)
be its fiber over the generic point of \( U_X \). Then it is clear that \( Y \cap W_{k'} \) is contained under the projection \( T_{k'} \to X_{k'} \times \Delta^p \) sending \( (y, x, t) \) to \( (y, t) \). Moreover \( Y \) is
contained in the projection of the closure of $T_\xi$ in $X_{k'} \times W_{k'}$.  

So 

$$\dim(Y \cap C_{k'} \times \Delta^S) \leq \dim(\overline{T_\xi \cap C_{k'} \times X_{k'} \times \Delta^S}).$$

Moreover the closure of $T_\xi$ in $X \times W$ is contained 

$$\overline{T_\xi} \subseteq T_\xi \cup i(W \cap R \times \Delta^p)$$

where $i : W \to X \times W$ is the map sending $(x, t)$ to $(x, x, t)$. In fact $T_\xi$ is contained in $\{(x, y) \in X_{k'} \times X_{k'} | x \neq y, f_x = f_y\} \times \Delta^p$ and by lemma 1.2 the intersection of the closure of the latter with the diagonal is precisely $R \times \Delta^p$. So 

$$\dim(Y \cap C_{k'} \times \Delta^S) \leq \max(\dim(T_\xi \cap C_{k'} \times X \times \Delta^S), \dim(W \cap (C \cap R) \times \Delta^S)) \ .$$

By lemma 4.1 we have that $\dim(W \cap (C \cap R) \times \Delta^S) \leq \dim(W \cap C \times \Delta^S) - 1$. To conclude we just need to show that 

\[ \dim(T_\xi \cap C_{k'} \times X \times \Delta^S) \leq \dim C + \#S - q . \]

Let 

$$T_0 = \{(y, x, t) \in C \times W | y \neq x, t \in \Delta^S\}$$

so that $\dim T_0 = \dim C + \dim(W \cap X \times \Delta^S) = \dim C + r + \#S - q$. There is a projection 

$$(T \cap C \times X \times \Delta^S \times U_X) \to T_0 ,$$

where all the fibers have dimension $(n-1)r$ (since the fiber consists of those $A \in U_X$ such that $A(y - x) = 0$). So 

$$\dim(T \cap C \times X \times \Delta^S \times U_X) = \dim T_0 + (n - 1)r = \dim C + nr + \#S - q .$$

But $T_\xi \cap C_{k'} \times X_{k'} \times \Delta^S$ is the generic fiber of $T \cap C \times X \times \Delta^S$, so 

$$\dim(T_\xi \cap C_{k'} \times X_{k'} \times \Delta^S) = \dim C + \#S - q$$

as promised.  

So, thanks to lemma 1.1 we have that for an arbitrary $B$, if $W \in X^{(p,q)}_{c,d}$ then all irreducible components of $f^{-1}f(W)$ different from $W$ are in $X^{(p,q)}_{c,d-1}$. So to compute $f^*f_*[W]$ we just need to study the degree of the map $W \to f(W)$.

We say that $W \in X^{(p,q)}$ is **induced** if there is a subset $F \subseteq \Delta^p$ of codimension $q$ such that $W \subseteq X \times F$. A simple dimension count implies that $W$ is an irreducible component of $X \times F$. Note that every induced $W$ is in $X^{(p,q)}_{c,d}$ for any $C$ and $d$. In fact, since $W \cap X \times \Delta^S = X \times (F \cap \Delta^S)$ has dimension $r + \#S - q$, we must have $\dim(F \cap \Delta^S) = \#S - q$ and so 

$$W \cap (C \times \Delta^S) = C \times (F \cap \Delta^S) = \dim C + \#S - q .$$

**Proposition 4.2** ([II], Lm. 5.3.3). Let $W \in X^{(p,q)}$ not induced. Then the map 

$$f : W_{B'} \to f(W_{B'})$$

is birational. In particular $f_*[W_{B'}] = [f(W_{B'})]$.

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2This is because this projection map is in fact the restriction of a projection map from $\bar{X} \times W \to X \times \Delta^p$ where $\bar{X}$ and $W$ are the projective closures of $X_{k'}$ and $W_{k'}$ in $\mathbb{P}^n_{k'}$ and $\mathbb{P}^m_{k'} \times \Delta^p$ respectively and so it is closed.
Proof. Since the thesis cares only about the map induced on the generic points of $W_B'$ and $f(W_B')$ we can basechange to the image of the generic point of $W$ in $B$ and assume that $B = \text{Spec} \ k$ is a field. Similarly, by considering only the fibers over the image of the generic point of $W$ on $\Delta^p$ we can assume that $p = 0$. In this case the condition of being non-induced is equivalent to $q > 0$.

So, we have $X \subseteq A^k$ smooth variety of dimension $r$, $W \subseteq X$ closed irreducible subset of codimension $q > 0$ and we want to show that 

$$f : W_{k'} \to f(W_{k'})$$

is birational. Let us consider

$$S = \{ (w_1, w_2, A) \in W \times W \times U_X \mid w_1 \neq w_2, A(w_1 - w_2) = 0 \} .$$

The projection onto $(W \times W \setminus \Delta W)$ has all fibers of dimension $(n-1)r$, so

$$\dim S = 2(\dim W) + (n-1)r = (n+1)r - 2q .$$

Hence, its fiber $S_\xi$ over the generic point $\xi$ of $U_X$ has dimension

$$\dim S_\xi = r - 2q < \dim W .$$

In particular the map $S_\xi \to W_{k'} \to f(W_{k'})$ cannot be surjective, since $f$ is finite and so preserves the dimension of subvarieties. Let $V_1 \subseteq f(W_{k'})$ be a dense open subset disjoint from the image.

Similarly, the map $W_{k'} \cap R \to f(W_{k'})$ cannot be surjective and we can find $V_2$ dense open of $f(W_{k'})$ disjoint from the image. Let $V = V_1 \cap V_2$. Then $(f|_W)^{-1}V \to V$ is an étale map (since being étale is local on the source and $(f|_W)^{-1}V$ is a connected component of $f^{-1}V$) and the diagonal

$$(f|_W)^{-1}V \to (f|_W)^{-1}V \times_V (f|_W)^{-1}V$$

is an isomorphism (since it is a surjective open embedding). Recall that the diagonal is an isomorphism if and only if the map is a monomorphism. But the only étale monomorphisms in the category of schemes are open embedding. So $f$ restricted to $W \cap f^{-1}V$ must be an open embedding and in particular birational. \hfill \qed

Ok finally we can conclude that for every $W \in X_{c,d}^{(p,q)}$, the cycle $f^* f_* [W] - [W]$ lies in $z_{c,d-1}^q(X)$. In fact, if $W$ is not induced $f^* f_* [W_{B'}] - [W_{B'}]$ is just the sum of $[Y]$ for $Y$ an irreducible component of $f^{-1}f(W_{B'})$ not contained in $W_{B'}$ and so lies in $z_{c,d-1}^q(X_{B'})$ by the argument after proposition 4.1. Otherwise, if $W$ is contained in $X \times F$ for $F$ of codimension $q$, $f_* [W_{B'}] = (\deg f) [A^k_{B'} \times F]$ and so $f^* f_* [W_{B'}]$ is a sum of induced cycles that are in $(X_{B'})_{c,d-1}^{(p,q)}$ just from it being induced. Summing all up, we have that

$$f^* f_* \pi^* - \pi^* : z_{c,d}^q(X)/z_{c,d-1}^q(X) \to z_{c,d}^q(X_{B'})/z_{c,d-1}^q(X_{B'})$$

is the zero map. So to prove that $\pi^*$ is nullhomotopic is enough to prove that $f^* f_*$ is nullhomotopic. To do we need a last lemma.

Lemma 4.3 (2. Lm. 3.5.2). Let $f : X \to Y$ be a morphism of smooth $B$-schemes. Then there are $c', d'$ such that for every $W \in Y^{(p,q)}$ we have $f^{-1}W \in X_{c',d'}^{(p,q)}$ if and only if $W \in Y_{c',d'}^{(p,q)}$. 

Proof. As usual we can reduce through lemma 1.1 to the case where \( B \) is the spectrum of a field. For every \( j \) and every \( C_i \) let us write
\[
C_{i,j} = \{ y \in Y \mid \dim(X_y \cap C_i) = j \}
\]
Then for every \( W \in X^{(p,q)} \) we have
\[
\dim(f^{-1}W \cap C_i \times \Delta^S) = \max_j (\dim(f^{-1}(C_{i,j} \times \Delta^S \cap W) \cap C_i \times \Delta^S)) = \max_j \dim(W \cap C_{i,j} \times \Delta^S) + j
\]
So we can take \( C' = \{ C_{i,j} \}_{i,j} \) and \( d_{ij} = \dim C_i - \dim C_{i,j} - j \). \( \square \)

Note that, assuming the moving lemma, this allows us to construct a functor from \( \text{Sm}_B^{op} \) to pro-systems of simplicial abelian groups such that every arrow is an equivalence, sending \( X \) to \( \{ z^{\mathbb{Q}}_{C,d}(X) \}_{C,d} \). By taking homotopy limits this gives a functor \( z^{\mathbb{Q}}(-) \) from \( \text{Sm}_B^{op} \) to the positive derived category of \( \mathbb{Z} \).

Finally, by the argument after proposition 4.1 the map \( f^* f_* \pi^* \) factors through
\[
z^{\mathbb{Q}}_{C,d,f}(\mathbb{A}^*_B)/z^{\mathbb{Q}}_{C,d-1,f}(\mathbb{A}^*_B)
\]
and lemma 4.3 and theorem 2 imply that the latter is contractible. So the cofiber
\[
z^{\mathbb{Q}}_{C,d}(X)/z^{\mathbb{Q}}_{C,d-1}(X)
\]
is contractible and we have proven our thesis.

References


