Lecture notes on Quantum Chromodynamics

V.M. Braun [a],

[a] *Institut für Theoretische Physik, Universität Regensburg,*
D-93040 Regensburg, Germany

Version of December 6, 2023

Contents

1 The Quark Model

1.1 Historic overview

Known before 1932:

- Photon γ $m_\gamma = 0$
- Electron e^- $m_e \simeq 0.5$ MeV
- Proton p $m_P \simeq 938$ MeV

Then:

- 1932: Discovery of the neutron (Chadwick) $m_n \simeq 940$ MeV
- 1932: Isospin formalism (Heisenberg)
- 1935: Prediction of the π-meson as carrier (mediator) of strong forces (Yukawa)
- 1938: Extension of the isospin formalism to π-mesons, prediction of the π^0 (Kemmer)
- 1947: Discovery of charged π^+ and π^- (Lattes), $m_{\pi^\pm} \simeq 140$ MeV
- 1950: Discovery of the neutral π^0 $m_{\pi^0} \simeq 135$ MeV

The picture seemed to be converging, however

- 1947(?): Observation of new long-living particles in cosmic rays (V-particles); first signatures of new "strange" particles in accelerator experiments (K-mesons, Λ-hyperons)

1.1.1 Isospin formalism

One observes that p, n and also π^+, π^0, π^- have almost the same masses, why?

Recall the Hydrogen atom: the states $|n, \ell, m\rangle$ with $m = -\ell, \ldots, \ell$ have the same energies (are degenerate) because of the rotational symmetry of the Hamiltonian

? Hidden Symmetry Internal Symmetry

Analogy with spin:

$$|p\rangle, |n\rangle \Leftrightarrow |\uparrow\rangle, |\downarrow\rangle$$

Spin-rotations:

$$\begin{aligned} |\uparrow\rangle &\to \alpha|\uparrow\rangle + \beta|\downarrow\rangle \\ |\downarrow\rangle &\to \gamma|\uparrow\rangle + \delta|\downarrow\rangle \end{aligned} \quad (1.1)$$

Isospin-rotations:

$$\begin{aligned} |p\rangle &\to a|p\rangle + b|n\rangle \\ |n\rangle &\to c|p\rangle + d|n\rangle \end{aligned} \quad (1.2)$$

Symmetry group $SU(2)$:

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad \text{or} \quad U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \qquad UU^\dagger = 1\,, \qquad \det U = 1\,. \tag{1.3}$$

! Isospin: Abstract transformation in Hilbert space of the quantum states

Following this line of reasoning we introduce

$$\textbf{Nucleon} = \begin{pmatrix} p \\ n \end{pmatrix} \tag{1.4}$$

is a particle with *isospin* 1/2; It has two states with *isospin projection* +1/2 and −1/2.

For small isospin transformations

$$U = \mathbb{I} + i \sum_{a=1}^{3} \delta\phi_a \frac{\tau_a}{2}\,, \qquad \tau_a = \text{ Pauli matrices} \tag{1.5}$$

Isospin operators:

$$\begin{aligned} \hat{I}_a &= \frac{1}{2}\tau_a & &(\text{cf.}: \ \hat{S}_a = \frac{1}{2}\sigma_a) \\ \hat{I}^2 &= \hat{I}_1^2 + \hat{I}_2^2 + \hat{I}_3^2 & &(\text{cf.}: \ \hat{S}^2 = \hat{S}_1^2 + \hat{S}_2^2 + \hat{S}_3^2) \end{aligned} \tag{1.6}$$

Then

$$\begin{aligned} \hat{I}^2|p(n)\rangle &= I(I+1)|p(n)\rangle\,, \qquad I = 1/2 \\ \hat{I}_3|p\rangle &= +\frac{1}{2}|p\rangle\,, \qquad \hat{I}_3|n\rangle \ = \ -\frac{1}{2}|n\rangle \end{aligned} \tag{1.7}$$

The operator $\hat{I}^2$ is fully equivalent (mathematically) to the operator of angular momentum; possible eigenvalues are therefore (follows from group theory)

$$I = 0, 1/2, 1, 3/2, \ldots \tag{1.8}$$

Kemmer postulated that π-mesons form a system with isospin $I = 1$:

$$\pi = \begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix} \qquad \begin{aligned} I_3 &= +1 \\ I_3 &= \ \ \ 0 \\ I_3 &= -1 \end{aligned} \tag{1.9}$$

$\hookleftarrow$ This was a prediction for π^0 !

Isospin summation (cf. spin-summation):

NN-states:

$$\text{isospin}\,1/2 \otimes \text{isospin}\,1/2 = \text{isospin}\,1 + \text{isospin}\,0 \tag{1.10}$$

$$\begin{array}{llll} I=1: & I_3=+1 & pp & \\ & I_3=0 & \frac{1}{\sqrt{2}}(pn+np) & \text{triplet} \\ & I_3=-1 & nn & \\ & & & \\ I=0: & I_3=0 & \frac{1}{\sqrt{2}}(pn-np) & \text{singlet (deuteron)} \end{array} \tag{1.11}$$

πN-states:

$$\text{isospin}\,1\otimes\text{isospin}\,1/2=\text{isospin}\,1/2+\text{isospin}\,3/2 \tag{1.12}$$

$\hookleftarrow$ predictions for $\pi N\to\pi N$ scattering etc.

1.1.2 Strange particles and the "8-fold way"

Before 1953 "strange" V-particles were only seen in cosmic rays but eventually could be observed in also in accelerator experiments that allowed for their detailed study. For example

$$\pi^- + p \to K^0 + \Lambda \tag{1.13}$$

for $E_\pi \sim 1.5$ GeV one has measured, e.g.

$$\sigma(\pi^- p\to K^0\Lambda)\sim 1\,\text{mb}\equiv 10^{-27}\,\text{cm}^2\,,$$
$$\sigma_{\text{tot}}(\pi^- p\to\text{hadrons})\sim 40\,\text{mb} \tag{1.14}$$

These cross sections correspond (roughly) to geometric cross sections of hadrons

$$R^2\sim(1\,\text{fm})^2=10^{-26}\,\text{cm}^2 \tag{1.15}$$

hence K and Λ are produced via strong interaction.

"Strange" particles decay, e.g.

$$\Lambda\to p+\pi^-,\quad n+\pi^0 \tag{1.16}$$

A natural life time for decays induced by strong interaction would be

$$\tau_{\text{strong}}\sim R/c\sim 10^{-13}\,\text{cm}/3\cdot 10^{10}\,\text{cm}\,\text{s}^{-1}\sim 10^{-23}\,\text{s} \tag{1.17}$$

The experimentally measured life time is, however

$$\tau_\Lambda\simeq 2.63\cdot 10^{-10}\,\text{s} \tag{1.18}$$

hence this is a weak decay, similar to $n\to pe^-\nu$.

It seems that strong and electromagnetic decays of these particles are forbidden, why?

$\hookleftarrow$ **New quantum number — "strangeness"**

$$\begin{array}{rr} p,n,\pi^+,\pi^0,\pi^- & S=0 \\ \Lambda,\Sigma^+,\Sigma^0,\Sigma^- & S=-1 \\ \Xi^0,\Xi^- & S=-2 \\ K^0,K^+ & S=+1 \\ \bar{K}^0,K^- & S=-1 \end{array} \tag{1.19}$$

assume that strangeness is conserved in strong and electromagnetic interactions, e.g.

$$\begin{aligned} \pi^- + p &\to K^0 + \Lambda & 0+0 &\to +1-1 \quad \text{(allowed)} \\ K^- + p &\to K^0 + \Xi^0 & -1+0 &\to +1-2 \quad \text{(allowed)} \end{aligned} \tag{1.20}$$

In addition we have a conserved electric charge Q and baryon number B:

$$\begin{aligned} Baryons: \quad & B = +1 \\ Antibaryons: \quad & B = -1 \\ Mesons: \quad & B = \ \ 0 \end{aligned} \tag{1.21}$$

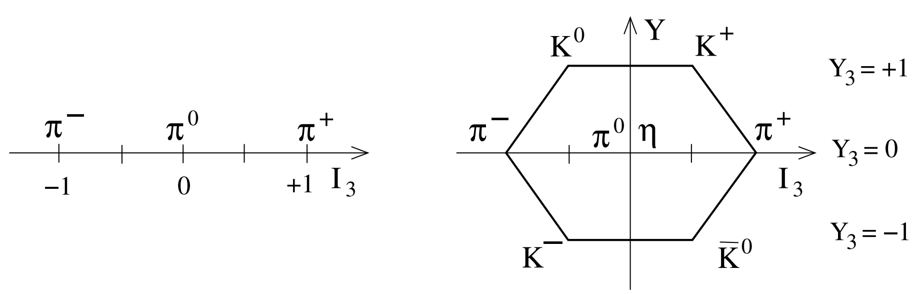
Gell-Mann and Nishijima observed that for all observed particles the following relation holds:

$$\begin{aligned} Q &= I_3 + \frac{1}{2}(S+B)\,, \\ Y &= S + B : \qquad \textbf{Hypercharge} \end{aligned} \tag{1.22}$$

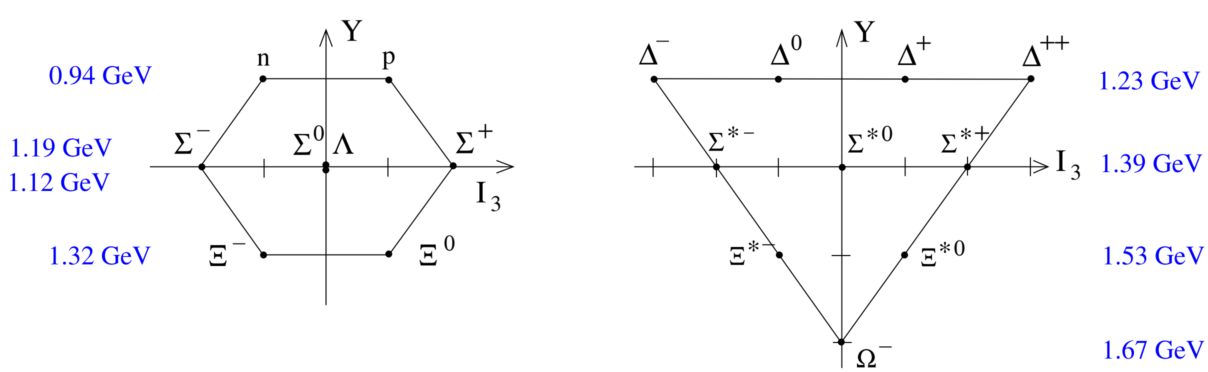
The breakthrough: Gell-Mann, Neeman 1961,1964: The "8-fold way" (← Mahajana-Buddhismus)

$$\boxed{SU(2)\ \text{(isospin)} \ \Rightarrow\ SU(3)\ \text{(isospin + hypercharge)}} \tag{1.23}$$

Mesons:



Baryons:



- ↩ Prediction for Ω^- with mass ca. 1670 MeV and $J^P = \frac{3}{2}^+$!

- Symmetry is approximate: different states are not exactly degenerate: splitting appr. 150 MeV

$$\text{Hypercharge symmetry breaking}: \quad \frac{m_\Lambda - m_p}{m_p} \sim 20\% \tag{1.24}$$

It was possible to classify all known hadrons in irreducible representations of the $SU(3)$ group (later) and predict the existence of Ω^-. However, no hadrons could be matched with the representation of the lowest dimension — the fundamental representation of $SU(3)$.

Example: isospin group $SU(2)$ $\quad I = 0, 1/2, 1, 3/2, \ldots$
Fundamenental representation

$$N = \begin{pmatrix} p \\ n \end{pmatrix}, \qquad I = 1/2 \quad I_3 = \pm 1/2 \tag{1.25}$$

Adjoint representation

$$\pi = \begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix}, \qquad I = 1 \quad I_3 = -1, 0, 1 \tag{1.26}$$

etc.
For the case of the $SU(3)$ classification the analogue of p and n was missing.

Gell-Mann, Zweig 1964 $\boxed{Quarks}$

Quarks	I	I_3	Y	S	B	Q
u	1/2	+1/2	1/3	0	1/3	2/3
d	1/2	-1/2	1/3	0	1/3	-1/3
s	0	0	-2/3	-1	1/3	-1/3

$$\boxed{Q = I_3 + \frac{1}{2}Y}$$

The $SU(3)$ transformations:

$$q = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \Rightarrow \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix} \begin{pmatrix} u \\ d \\ s \end{pmatrix}, \qquad UU^\dagger = 1, \quad \det U = 1 \tag{1.27}$$

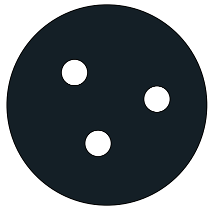
Mesons are built from a quark and antiquark. We identify:

$$\begin{aligned}
\pi^+ &= u\bar{d}, \qquad \pi^- = d\bar{u}, \\
K^+ &= u\bar{s}, \qquad K^0 = d\bar{s}, \qquad \bar{K}^0 = s\bar{d}, \qquad K^- = s\bar{u}, \\
\pi^0 &= \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}), \qquad \eta^0 = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}), \\
\eta'^0 &= \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} + s\bar{s}) \qquad \text{(singlet under } SU(3)\text{)}
\end{aligned} \tag{1.28}$$

Three different quarks u, d, s are usually called “flavors”. L2

Baryons, in turn, are built of three quarks: **The quark model**

N=



$m_u \sim m_d \sim 300$ MeV
$m_s \sim 450$ MeV

1.1.3 Quarks have color!

- **Problem 1:**
  There exist apparently no free quarks in nature — no particles with electric charge $+2/3$ or $-1/3$

$\hookleftarrow$ **Quark confinement**

- **Problem 2:**
  $\Omega^-(s = 3/2, s_3 = 3/2)$ is built of three strange quarks:

$$\Omega^- = s^\uparrow s^\uparrow s^\uparrow \tag{1.29}$$

For the ground state, one expects that the wave function describing space distribution of the three quarks in the nucleon is symmetric, $\Psi(x_1, x_2, x_3) = \Psi(x_2, x_1, x_3)$, etc.

Hence a totally symmetric wave function for a spin-3/2 particle — contradiction with Pauli principle?

Gell-Mann (1972), Fritzsch (1973): a new degree of freedom:

— Each quark exists in three versions (states), called "colors"

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \qquad d = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}, \qquad s = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}, \tag{1.30}$$

! A totally antisymmetric wave function can be built as

$$\Omega^-(3/2, 3/2) = s^\uparrow_\alpha s^\uparrow_\beta s^\uparrow_\gamma \epsilon^{\alpha\beta\gamma} \Psi(x_1, x_2, x_3) \tag{1.31}$$

! This state is invariant under rotations in the color space:

$$q_\alpha \to \sum_{\beta=1}^{3} U_{\alpha\beta} q_\beta \,; \quad \alpha = 1, 2, 3 \qquad q = (u, d, s)]$$

$$UU^\dagger = 1 \,, \quad \det U = 1 \tag{1.32}$$

— again a $SU(3)$-group.

color–$SU(3)$ $\neq$ flavor–$SU(3)$

The color–$SU(3)$ symmetry plays a fundamental role in QCD; the flavor–$SU(3)$ is (as we know now) rather accidental.

Postulate:

Only $SU(3)$-invariant ("colorless") states exist in nature.

- Baryons:

$$q_\alpha q_\beta q_\gamma \epsilon^{\alpha\beta\gamma} \to U_{\alpha\alpha'} U_{\beta\beta'} U_{\gamma\gamma'} \epsilon^{\alpha\beta\gamma} q_{\alpha'} q_{\beta'} q_{\gamma'} = \det U \, \epsilon^{\alpha'\beta'\gamma'} q_{\alpha'} q_{\beta'} q_{\gamma'} \tag{1.33}$$

- Mesons:

$$\bar{q}_\alpha q_\alpha \to \bar{q}_{\alpha'} U^\dagger_{\alpha'\alpha} U_{\alpha\beta'} q_{\beta'} = \bar{q}_{\alpha'} q_{\alpha'} \tag{1.34}$$

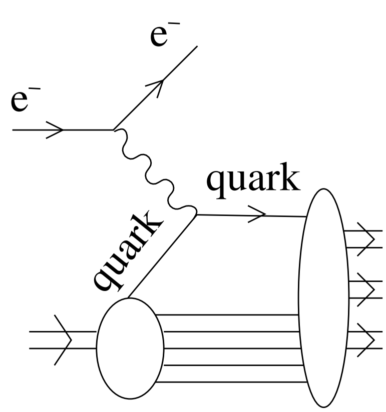
Confinement: only colorless particles exist.

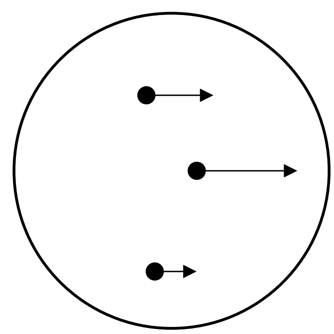
A big question: are quarks mathematical constructs only, or they indeed exist materially inside hadrons?

- 1969: Crucial evidence:
  Scattering of electrons from protons with large momentum transfer (at large angle) (Bjorken)

$$e^-(k_1) + N \to e^-(k_2) + X(\leftarrow \text{ any hadron state}) \tag{1.35}$$

! Nucleons contain quasi-free point-like particles inside them ("partons")





L3

- 1973: Gross, Wilczek, Politzer: **"Asymptotic freedom"**
  — Theory of quark-gluon interactions

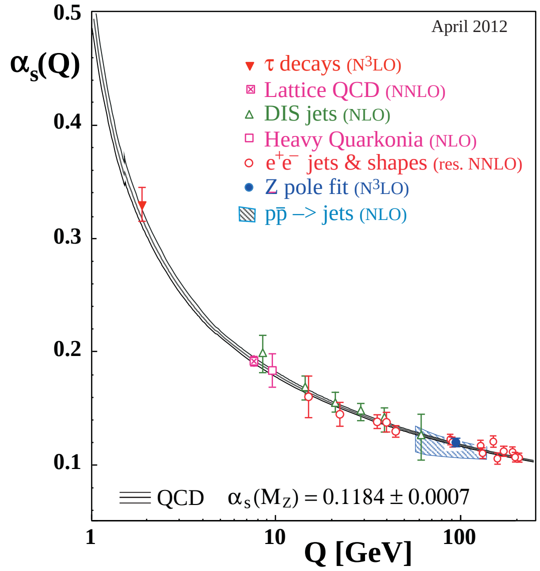
Quantum Chromodynamics

— a "nonabelian gauge theory":

QED	gauge transformations	$U(1)$	$e \to e^{i\phi(x)} e$	$\leftarrow$ photons
QCD	gauge transformations	$SU(3)$	$q \to Uq$	$\leftarrow$ gluons

QCD (color) charge (coupling constant) is small at small distances and becomes large at hadronic scales.

World summary, see [S. Bethke, arXiv:1210.0325]



1.2 Elements of group theory: The $SU(3)$ group

Consider a three-dimensional abstract Hilbert space with orthonormal basis vectors

$$|1\rangle, \quad |2\rangle, \quad |3\rangle; \qquad \langle i|k\rangle = \delta_{ik} \tag{1.36}$$

these could be e.g. $|1,2,3\rangle = |u\rangle, |d\rangle, |d\rangle$ or $|1,2,3\rangle = |u_1\rangle, |u_2\rangle, |u_3\rangle$

The $SU(3)$ group:

$$\boxed{U|i\rangle = |j\rangle U_{ji} \qquad UU^\dagger = 1\,, \quad \det U = 1} \tag{1.37}$$

Infinitesimal transformations

$$U = \mathbb{I} + i\delta\phi\, H\,, \qquad \delta\phi \in \mathbb{R} \tag{1.38}$$

Then

$$\begin{aligned} (\mathbb{I} + i\delta\phi\, H)(\mathbb{I} - i\delta\phi\, H^\dagger) \;&=\; 1 \\ \det\,(\mathbb{I} + i\delta\phi\, H) = e^{\mathrm{Tr}\ln(\mathbb{I}+i\delta\phi\, H)} = 1 + i\delta\phi \mathrm{Tr}\, H \;&=\; 1 \end{aligned} \tag{1.39}$$

yields

$$H = H^\dagger\,, \qquad \mathrm{Tr}\, H = 0 \tag{1.40}$$

i.e. H is a hermitian 3×3 matrix with Tr$= 0$.

A suitable basis (Gell-Mann matrices)

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{1.41}$$

Normalization convention

$$\mathrm{Tr}\,(\lambda_a \lambda_b) = 2\delta_{ab}\,, \qquad\qquad a,b = 1,2,\ldots,8 \tag{1.42}$$

Each infinitesimal $SU(3)$ transformation can be written as

$$U = \mathbb{I} + i\delta\phi_a \frac{\lambda_a}{2} \tag{1.43}$$

The matrices

$$t^a = \frac{1}{2}\lambda^a \tag{1.44}$$

are called *generators* of the $SU(3)$ transformations.

They satisfy the following (canonical) commutation relations:

$$\boxed{[t_a, t_b] = i f_{abc} t^c} \tag{1.45}$$

The f_{abc} symbols are called *structure constants* of the $SU(3)$.

Multiply by t^d and take the trace; since $\mathrm{Tr}\,(t_c t_d) = (1/2)\delta_{cd}$ obtain

$$i f_{abc} = 2\,\mathrm{Tr}\,([t_a, t_b] t_c) \tag{1.46}$$

From this representation one can easily see that f_{abc} are totally antisymmetric and real.

An anticommutator of two t^a matrices:

$$\{t_a, t_b\} = c_{ab}\,\mathbb{I} + d_{abc} t_c \tag{1.47}$$

Taking the trace:

$$\mathrm{Tr}\,(\{t_a, t_b\}) = 2 \cdot \frac{1}{2}\delta_{ab} = c_{ab}\mathrm{Tr}\,(\mathbb{I}) = 3c_{ab} \quad\Rightarrow\quad c_{ab} = \frac{1}{3}\delta_{ab} \tag{1.48}$$

also

$$d_{abc} = 2\mathrm{Tr}\,(\{t_a, t_b\} t_c) \tag{1.49}$$

Hence d_{abc} symbols are real and totally symmetric in indices.

Using trace folmulas and explicit expressions for λ_a matrices one can calculate f_{abc} and d_{abc} explicitly; these expressions are, however, rarely needed (cf.: Dirac matrices)

Useful identities:

$$\begin{aligned}
& f_{abr} f_{rcs} + f_{bcr} f_{ras} + f_{car} f_{rbs} = 0 \\
& f_{abr} d_{rcs} + f_{cbr} d_{ras} = d_{acr} f_{rbs} \\
& f_{ars} f_{brs} = 3\delta_{ab} \\
& d_{aab} = 0 \qquad \leftarrow \text{summation over "a" implied} \\
& d_{ars} d_{brs} = \frac{5}{3}\delta_{ab}
\end{aligned} \tag{1.50}$$

Very useful identities:

$$f_{abc}t^a t^b = \frac{3}{2} i t^c$$
$$t^a t^a = \frac{4}{3}\,\mathbb{I}$$
$$t^a t^b t^a = -\frac{1}{6} t^b \tag{1.51}$$

A very powerful identity:

$$(t^a)_{\alpha\beta}(t^a)_{\alpha'\beta'} = \frac{1}{2}\delta_{\alpha\beta'}\delta_{\alpha'\beta} - \frac{1}{6}\delta_{\alpha\beta}\delta_{\alpha'\beta'} \tag{1.52}$$

1.2.1 Representations of the $SU(3)$ group

- A unitary representation of the SU(3) group is a homomorphism

  $$U \to D(U)$$

  of the 3×3 matrices U onto unitary $n \times n$ matrices D,

  $$D(U)D^{\dagger}(U) = 1$$

  (in general of other dimension), which respects the group multiplication:

  $$D(U)D(V) = D(UV) \tag{1.53}$$

  $D(U)$ can also be viewed (more generally) as linear operators acting on the representation space

  $$R \xrightarrow{D(U)} R \qquad R = \mathbb{C}^n \qquad |\alpha\rangle \in R \quad \text{n-dimensional vectors} \tag{1.54}$$

- A representation is called reducible if it is block-diagonal in a certain basis

  $$D(u) = \begin{pmatrix} D_1(U) & 0 \\ 0 & D_2(U) \end{pmatrix} \quad \begin{matrix} \leftarrow \{\, n_1 \\ \leftarrow \{\, n_2 \end{matrix} \qquad n_1 + n_2 = n \tag{1.55}$$

  Otherwise it is called irreducible

  A necessary and sufficient condition:
  Represention $D(U)$ is irreducible if and only if

  $$\forall |\alpha\rangle \in R \quad \text{linear combinations of} \quad D(U)|\alpha\rangle \quad \text{span the whole space} \tag{1.56}$$

- Two representations D_1 and D_2 are called equivalent if

  $$\exists S \quad \forall U \qquad S^{-1} D_1(U) S = D_2(U) \tag{1.57}$$

L3

- Simplest representations:

  – $U \to \mathbb{I}$ [1] $\longleftarrow$ trivial representation

- $U \to U$ [3] $\longleftarrow$ fundamental representation
- $U \to U^*$ $[\bar{3}]$

- Adjoint representation [8]:

  Representation space:

$$R = \mathbb{C}^8 \qquad c = \{c_1, \dots c_8\} \qquad c_a \in \mathbb{C} \quad a = 1, \dots, 8 \tag{1.58}$$

Let

$$C = c_a t^a \qquad \longleftarrow \quad \text{a complex } 3 \times 3 \text{ matrix with Tr } = 0 \tag{1.59}$$

Define

$$C \quad \overset{D(U)}{\longrightarrow} \quad UCU^\dagger \tag{1.60}$$

or, equivalently

$$\text{Tr}\,(Ct^b) \to \text{Tr}\,(UCU^\dagger t^b) \quad \Rightarrow \quad \frac{1}{2}c_b \to c_a \text{Tr}\,(Ut^aU^\dagger t^b)$$

Infinitesimal transformations:

$$U = \mathbb{I} + i\delta\phi_a t^a \quad \longrightarrow \quad D(U) = \mathbb{I} + i\delta\phi_a T^a \tag{1.61}$$

Lie algebra:

$$[t_a, t_b] = i f_{abc} t^c \quad \longrightarrow \quad [T_a, T_b] = i f_{abc} T^c \tag{1.62}$$

(The generators in all representations obey the same commutation relations)
In our case (adjoint representation)

$$\begin{aligned} c_b \longrightarrow & \; 2\text{Tr}\left[(1 + i\delta\phi_c t^c) t^a (1 - i\delta\phi_{c'} t^{c'}) t^b\right] c_a \\ = & \; 2\text{Tr}\,[t^a t^b] c_a + 2i\delta\phi_c \big\{ \text{Tr}\,(t^c t^a t^b) - \text{Tr}\,(t^a t^c t^b) \big\} c_a \\ = & \; \big[\delta_{ab} + i\delta\phi_c i f_{cab}\big] c_a \\ \equiv & \; \big[(\mathbb{I})_{ba} + i\delta\phi_c (T^c)_{ba}\big] c_a \end{aligned} \tag{1.63}$$

It follows

$$\boxed{(T^c)_{ba} = -i f_{cba}} \tag{1.64}$$

$\hookleftarrow$ generators in the adjoint representation

<u>**Example:**</u> Classification of the $\bar{q}q$ states under $SU(3)$-flavor

Let

$$q_1 \equiv u \qquad q_2 \equiv d \qquad q_3 \equiv s$$

and assume

$$
\begin{array}{llll}
[3] & |q_i\rangle \longrightarrow |q_j\rangle U_{ji}\,, & & U \in SU(3) \\
[\bar{3}] & |\bar{q}_i\rangle \longrightarrow |\bar{q}_j\rangle U^*_{ji} & &
\end{array} \tag{1.65}
$$

Now consider quark-antiquark states. Representation space is 9-dimensional $R: \quad |q_i\rangle|\bar{q}_j\rangle$

$$
D(U)(|q_i\rangle|\bar{q}_j\rangle) = |q_{i'}\rangle|\bar{q}_{j'}\rangle U_{i'i}U^*_{j'j} \tag{1.66}
$$

This representation is <u>reducible</u>:

- $SU(3)$-invariant state:

$$
|1\rangle = \frac{1}{\sqrt{3}}|q_i\rangle|\bar{q}_i\rangle = \frac{1}{\sqrt{3}}\big(|u\rangle|\bar{u}\rangle + |d\rangle|\bar{d}\rangle + |s\rangle|\bar{s}\rangle\big) \tag{1.67}
$$

  Check

$$
\begin{aligned}
|1'\rangle \;=\; D(U)|1\rangle &= \frac{1}{\sqrt{3}}|q_{i'}\rangle|\bar{q}_{j'}\rangle U_{i'i}U^*_{j'i} \;=\; \frac{1}{\sqrt{3}}|q_{i'}\rangle|\bar{q}_{j'}\rangle U_{i'i}U^\dagger_{ij'} \\
&= \frac{1}{\sqrt{3}}|q_{i'}\rangle|\bar{q}_{j'}\rangle\delta_{i'j'} = |1\rangle
\end{aligned} \tag{1.68}
$$

- An arbitrary orthogonal state

$$
|C\rangle \;=\; C_{ij}|q_i\rangle|\bar{q}_j\rangle \qquad\qquad C_{ij} \;=\; 3\times 3 \text{ matrix with } \mathrm{Tr}\, C = 0 \tag{1.69}
$$

  Check $SU(3)$ transformation:

$$
|C'\rangle \;=\; D(U)|C\rangle = C_{ij}|q_{i'}\rangle|\bar{q}_{j'}\rangle U_{i'i}U^*_{j'j} \;=\; (UCU^\dagger)_{i'j'}|q_{i'}\rangle|\bar{q}_{j'}\rangle \tag{1.70}
$$

  Thus

$$
C' = UCU^\dagger \tag{1.71}
$$

  ! This is precisely the transformation rule of the adjoint representation

Result:

$$
\boxed{[3]\otimes[\bar{3}] \;=\; [1]+[8]} \tag{1.72}
$$

<u>**Example II:**</u> Classification of three-quark states:

$$
\boxed{[3]\otimes[3]\otimes[3] \;=\; [1]+[8]+[8]+[10]} \tag{1.73}
$$

Start with the first pair:

$$
[3]\otimes[3] \;=\; [\bar{3}]+[6]
$$

$$
\swarrow \qquad \searrow
$$

$$
\epsilon_{ijk}|q_i\rangle|q_k\rangle \qquad |q_i\rangle|q_k\rangle + |q_k\rangle|q_i\rangle
$$

Add the third quark:

$$[\bar{3}] \otimes [3] \;=\; [1] + [8]$$
$$[6] \otimes [3] \;=\; [8] + [10]$$
$$\searrow$$
$$|q_i\rangle|q_j\rangle|q_k\rangle + \text{ permutations}$$

Symmetry in quantum mechanics:

$$[\hat{H}, T^a] = 0 \tag{1.74}$$

Example: Angular momentum

$$[\hat{H}, \vec{L}] = 0\,, \qquad [\hat{L}_i, \hat{L}_j] = \frac{1}{2}\underbrace{\epsilon_{ijk}}\hat{L}_k \tag{1.75}$$
$$\downarrow$$
$$\text{structure constants of } SU(2) \sim SO(3)$$

Hence one additive quantum number:

$$L_z|\Psi\rangle = m|\Psi\rangle\,, \qquad [L_x, L_z]\neq 0 \qquad [L_y, L_z]\neq 0 \tag{1.76}$$

$\hookleftarrow$ $SO(3)$ group has rank one.
For $SU(3)$:

$$[T_3, T_8] = 0 \qquad \Leftarrow \quad \text{group has rank two} \tag{1.77}$$

therefore can require that simultaneously

$$\hat{T}_3|\Psi\rangle = t_3|\Psi\rangle\,, \qquad \hat{T}_8|\Psi\rangle = t_8|\Psi\rangle \tag{1.78}$$

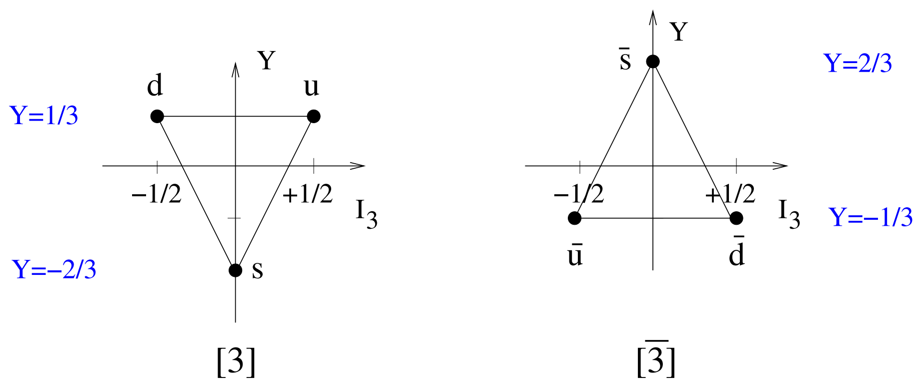
$\hookleftarrow$ two quantum numbers, $\Psi = \Psi(t_3, t_8)$.
Following Gell-Mann we identify (for $SU(3)$-flavor)

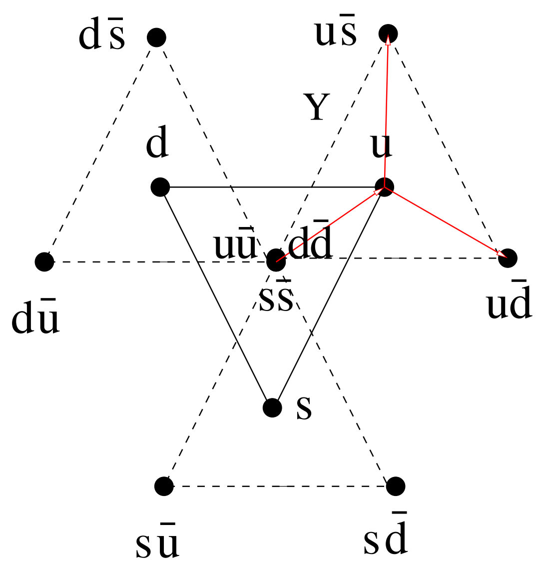
$$I_1 = t_1, \qquad I_2 = t_2, \qquad I_3 = t_3, \qquad \text{(isospin)}$$
$$Y = \frac{2}{\sqrt{3}}t_8 \qquad\qquad \text{(hypercharge)} \tag{1.79}$$

The eigenvalues of I_3 and Y that occur in a given representation can be shown as points in the isospin-hypercharge plane. Quarks u, d, s and antiquarks $\bar{u}, \bar{d}, \bar{s}$ transform according the three-dimensional fundamental representations of the $SU(3)$, called [3] and $[\bar{3}]$:

Mesons are built from a quark and antiquark:

$$[3] \otimes [\bar{3}] = [1] + [8] \tag{1.80}$$





2 Nonabelian quantum field theories

2.1 Geometry of gauge invariance

Weil (1923): gauge invariance

$$\left.\begin{aligned} \psi(x) \longrightarrow \psi'(x) &= e^{i\alpha(x)}\psi(x) \\ A_\mu(x) \longrightarrow A'_\mu(x) &= A_\mu(x) - \tfrac{1}{e}\partial_\mu\alpha(x) \end{aligned}\right\} \Longrightarrow \mathcal{L}_{\mathrm{QED}}(\psi, A) = \mathcal{L}_{\mathrm{QED}}(\psi', A') \tag{2.1}$$

[A_μ transformations introduced by Maxwell; Weil added ψ]
Modern interpretation:

Let $\psi(x) \longrightarrow \psi'(x) = e^{i\alpha(x)}\psi(x)$ and require $\mathcal{L}(\psi) = \mathcal{L}(\psi')$

What is the most general form of Lagrange density consistent with this symmetry?

- Simple:

$$m\bar\psi\psi\,, \qquad g^2(\bar\psi\psi)^2\,, \ldots \qquad \text{all allowed} \tag{2.2}$$

- Complicated: derivatives

$$n^\mu \partial_\mu \psi(x) := \lim_{\epsilon\to 0} \frac{1}{\epsilon}\Big[\psi(x+\epsilon n) - \psi(x)\Big]$$
$$\longrightarrow \lim_{\epsilon\to 0} \frac{1}{\epsilon}\Big[e^{i\alpha(x+\epsilon n)}\psi(x+\epsilon n) - e^{i\alpha(x)}\psi(x)\Big] \qquad \text{bad} \tag{2.3}$$

General solution (differential geometry)

In addition to $\psi(x)$, consider a function of two variables $U(y,x)$ with transformation property

$$\begin{aligned} U(y,x) &\longrightarrow U'(y,x) = e^{i\alpha(y)}U(y,x)e^{-i\alpha(x)} \\ U(x,x) &= 1 \end{aligned} \tag{2.4}$$

Its utility is that $\psi(y)$ and $U(y,x)\psi(x)$ have the same transformation laws.

Define **covariant derivative**:

$$n^\mu D_\mu \psi(x) := \lim_{\epsilon\to 0} \frac{1}{\epsilon}\Big[\psi(x+\epsilon n) - U(x+\epsilon n, x)\psi(x)\Big] \tag{2.5}$$

Simplest choice:

$$UU^* = 1\,, \qquad U(y,x) = e^{i\phi(y,x)} \tag{2.6}$$

Then

$$\begin{aligned} U(x+\epsilon n, x) &= 1 + i\epsilon n^\mu \boxed{\frac{\partial}{\partial y^\mu}\phi(y,x)\Big|_{y=x}} + \ldots \\ &:= 1 - i\epsilon\epsilon n^\mu \boxed{eA_\mu(x)} \qquad \text{e = arbitrary constant} \end{aligned} \tag{2.7}$$

! A new vector function $A_\mu(x)$ — **a vector field**

Math termonology: U is called a comparator of local symmetry transformations
A_μ is called a connection, it appears in a local limit of U *

Thus

$$\boxed{D_\mu \psi(x) := (\partial_\mu + ieA_\mu)\psi(x)} \tag{2.9}$$

From the transformation law

$$1 - i\epsilon n^\mu eA_\mu(x) \longrightarrow 1 - i\epsilon n^\mu eA'_\mu(x) = e^{i\alpha(x+\epsilon n)}\big[1 - i\epsilon n^\mu eA_\mu(x)\big]e^{-i\alpha(x)} \tag{2.10}$$

follows

$$A_\mu(x) \longrightarrow A'_\mu(x) = A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x) \tag{2.11}$$

so that L4

*Remark: One can choose

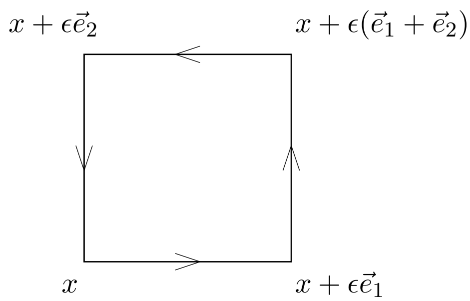
$$U(y,x) = \text{Pexp}\Big\{\int_0^1 du\,(y-x)^\mu A_\mu(ux+(1-u)y)\Big\} + \text{additional vector fields} \tag{2.8}$$

$$\begin{aligned} D_\mu\psi(x) &\longrightarrow D'_\mu\psi'(x) \\ &= \Big[\partial_\mu + ie\Big(A_\mu - \frac{1}{e}\partial_\mu\alpha\Big)\Big]e^{i\alpha(x)}\psi(x) \\ &= e^{i\alpha(x)}D_\mu\psi(x) \end{aligned} \tag{2.12}$$

Thus we are allowed to have in $\mathcal{L}$ terms like

$$\bar{\psi}\not{D}\psi\,, \qquad \bar{\psi}D_\mu D^\mu\psi\,, \quad \text{etc.} \tag{2.13}$$

What else? L5



$$W(x) = U(x, x+\epsilon\vec{e}_2)U(x+\epsilon\vec{e}_2, x+\epsilon\vec{e}_1+\epsilon\vec{e}_2)U(x+\epsilon\vec{e}_1+\epsilon\vec{e}_2, x+\epsilon\vec{e}_1)U(x+\epsilon\vec{e}_1, x) \tag{2.14}$$

A straightforward calculation for $\epsilon \to 0$

$$\begin{gathered} W(x) = 1 - i\epsilon^2 e\Big[\partial_1 A_2(x) - \partial_2 A_1(x)\Big] + \mathcal{O}(\epsilon^3) \\ W(x) = \text{ invariant} \quad \Longrightarrow \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \text{ invariant} \\ [W'(x) = W(x) \Rightarrow F'_{\mu\nu} = F_{\mu\nu}] \end{gathered} \tag{2.15}$$

Alternatively, consider

$$[D_\mu, D_\nu]\psi = [\partial_\mu, \partial_\nu]\psi + ie\Big([\partial_\mu, A_\nu] - [\partial_\nu, A_\mu]\Big)\psi - e^2[A_\mu, A_\nu]\psi \tag{2.16}$$

The first and the last terms obviously vanish. The other two:

$$[\partial_\mu, A_\nu]\psi = \partial_\mu(A_\nu\psi) - A_\nu\partial_\mu\psi = \underbrace{(\partial_\mu A_\nu(x))}_{\text{derivative only acts on A!}}\cdot\psi(x) \tag{2.17}$$

Hence

$$[D_\mu, D_\nu]\psi = ie\Big(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\Big)\psi(x) \qquad \boxed{[D_\mu, D_\nu] = ieF_{\mu\nu}} \tag{2.18}$$

Summing up, the QED Lagrangian

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\not{D})\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{2.19}$$

- need a real function with mass dimension four
- could add

$$\ldots - c\,\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta} \qquad \text{parity conservation } \Rightarrow c = 0 \tag{2.20}$$

- could add

$$\ldots + c_5\bar{\psi}D^2\psi + c_6(\bar{\psi}\psi)^2 + \ldots \qquad \bullet \text{ not renormalizable} \quad \bullet\ c_5 \sim \frac{1}{\Lambda_{UV}}\,, c_5 \sim \frac{1}{\Lambda_{UV}^2} \tag{2.21}$$

QED Lagrangian is defined (almost) uniquely by the requirements

- relativistic (Lorentz) invariance
- local gauge symmetry $U(1)$

! Photon exists "because" we require local gauge symmetry !

A very powerful idea: construct physical theories starting with geometric symmetry principles.

2.2 The Yang-Mills Lagrangian

Let

$$\psi = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} \tag{2.22}$$

Global $SU(2)$ transformations:

$$\psi \longrightarrow \psi' = e^{i\alpha^k \frac{\sigma_k}{2}}\psi; \qquad \sigma_k = \text{ Pauli matrices} \tag{2.23}$$

for example isospin — original motivation for YM

Local $SU(2)$ transformations (YM):

$$\psi(x) \longrightarrow \psi'(x) = e^{i\alpha^k(x)\frac{\sigma_k}{2}}\psi(x); \qquad \sigma_k = \text{ Pauli matrices} \tag{2.24}$$

Y&M asked: how to built a theory (Lagrangian) invariant under these trafos?

Main difference to QED: the symmetry is nonabelian:

$$e^{i\alpha}\cdot e^{i\beta} = e^{i\beta}\cdot e^{i\alpha} \qquad \text{but} \qquad e^{i\alpha^k\frac{\sigma_k}{2}}\cdot e^{i\beta^k\frac{\sigma_k}{2}} \neq e^{i\beta^k\frac{\sigma_k}{2}}\cdot e^{i\alpha^k\frac{\sigma_k}{2}} \tag{2.25}$$

Comparator of local $SU(2)$ transformations

$$U(y,x) = 2\times 2 \text{ matrix}\,, \qquad UU^\dagger = 1$$
$$U(y,x) \longrightarrow V(y)U(y,x)V^\dagger(x) \tag{2.26}$$

with

$$V(x) = e^{i\alpha^k(x)\frac{\sigma_k}{2}}, \qquad V(x)V^\dagger(x) = \mathbb{I} \tag{2.27}$$

It follows

$$U(x+\epsilon n, x) = \mathbb{I} + ig\epsilon n^\mu A^k_\mu \frac{\sigma^k}{2} + \mathcal{O}(\epsilon^2) \tag{2.28}$$

where g (arbitrary constant) will be called "coupling".
Covariant derivative

$$\boxed{D_\mu = \partial_\mu - igA^k_\mu \frac{\sigma^k}{2} \equiv \partial_\mu \cdot \mathbb{I} - igA^k_\mu \frac{\sigma^k}{2}} \tag{2.29}$$

Transformation rule for A^k_μ follows from

$$\mathbb{I} + ig\epsilon n^\mu A^k_\mu \frac{\sigma^k}{2} \longrightarrow \mathbb{I} + ig\epsilon n^\mu (A^k_\mu)' \frac{\sigma^k}{2} = V(x+\epsilon n)\Big(\mathbb{I} + ig\epsilon n^\mu A^k_\mu \frac{\sigma^k}{2}\Big)V^\dagger(x) \tag{2.30}$$

where we have to expand everything to $\mathcal{O}(\epsilon)$.

First term:

$$\begin{aligned} V(x+\epsilon n)V^\dagger(x) &= \Big(\Big[1 + \epsilon n^\mu \frac{\partial}{\partial x^\mu} + \mathcal{O}(\epsilon^2)\Big]V(x)\Big)V^\dagger(x) \\ &= \mathbb{I} + \epsilon n^\mu \left(\frac{\partial}{\partial x^\mu} V(x)\right) V^\dagger(x) + \mathcal{O}(\epsilon^2) \\ &= \mathbb{I} - \epsilon n^\mu V(x) \left(\frac{\partial}{\partial x^\mu} V^\dagger(x)\right) + \mathcal{O}(\epsilon^2) \end{aligned} \tag{2.31}$$

Therefore, comparing terms in $\sim ig\epsilon n^\mu$

$$\boxed{A^k_\mu \frac{\sigma^k}{2} \to V(x)\Big(A^k_\mu \frac{\sigma^k}{2} + \frac{i}{g}\partial_\mu\Big)V^\dagger(x)} \tag{2.32}$$

For small transformations

$$V(x) = \mathbb{I} + i\alpha^k(x)\frac{\sigma^k}{2} + \dots \tag{2.33}$$

this becomes

$$A^k_\mu \frac{\sigma^k}{2} \to A^k_\mu \frac{\sigma^k}{2} + \frac{1}{g}(\partial_\mu \alpha^k)\frac{\sigma^k}{2} + \underbrace{i\left[\alpha^k \frac{\sigma^k}{2}, A^j_\mu \frac{\sigma^j}{2}\right]}_{\text{new !}} \tag{2.34}$$

Check transformation for the covariant derivative:

$$\begin{aligned} D_\mu\psi &\longrightarrow D'_\mu\psi' = \left(\partial_\mu - ig(A')^k_\mu \frac{\sigma^k}{2}\right)\psi' \\ &= \left(\partial_\mu - igA^k_\mu \frac{\sigma^k}{2} - i(\partial_\mu\alpha^k)\frac{\sigma^k}{2} + g\left[\alpha^k \frac{\sigma^k}{2}, A^j_\mu \frac{\sigma^j}{2}\right]\right)\left(1 + i\alpha^k \frac{\sigma^k}{2}\right)\psi \\ &= \left(1 + i\alpha^k \frac{\sigma^k}{2}\right)\left(\partial_\mu - igA^k_\mu \frac{\sigma^k}{2}\right)\psi + \mathcal{O}(\alpha^2) = V(x)D_\mu\psi + \mathcal{O}(\alpha^2) \qquad \boxed{OK} \end{aligned} \tag{2.35}$$

As a consequence

$$[D_\mu, D_\nu]\psi(x) \longrightarrow V(x)[D_\mu, D_\nu]\psi(x) = \underbrace{V(x)[D_\mu, D_\nu]V^\dagger(x)}\,\underbrace{V(x)\psi(x)} \tag{2.36}$$

Define a nonabelian field strength tensor (Feldstärke) as

$$\boxed{[D_\mu, D_\nu] = -igF^k_{\mu\nu}\frac{\sigma^k}{2}\,, \qquad F^k_{\mu\nu}\frac{\sigma^k}{2} \longrightarrow V(x)F^k_{\mu\nu}\frac{\sigma^k}{2}V^\dagger(x)} \tag{2.37}$$

Inserting explicit expression for the covariant derivative this becomes

$$F^k_{\mu\nu}\frac{\sigma^k}{2} = \partial_\mu A^k_\nu\frac{\sigma^k}{2} - \partial_\nu A^k_\mu\frac{\sigma^k}{2} - ig\left[A^k_\mu\frac{\sigma^k}{2}, A^j_\nu\frac{\sigma^j}{2}\right] \tag{2.38}$$

Use

$$\left[\frac{\sigma^k}{2}, \frac{\sigma^j}{2}\right] = i\epsilon^{kjl}\frac{\sigma^l}{2} \tag{2.39}$$

multiply by σ^p and take the trace:

$$\boxed{F^k_{\mu\nu} = \partial_\mu A^k_\nu - \partial_\nu A^k_\mu + g\epsilon^{kjl}A^j_\mu A^l_\nu} \tag{2.40}$$

! $F^k_{\mu\nu}$ is not yet $SU(2)$ invariant, but this is easy to repair:

$$\mathrm{Tr}\left[(F^k_{\mu\nu}\frac{\sigma^k}{2})^2\right] = \frac{1}{2}F^k_{\mu\nu}F^{\mu\nu,k} = \text{invariant} \tag{2.41}$$

Thus, a possible Lagrangian invariant under local $SU(2)$ is (Yang-Mills)

$$\boxed{\mathcal{L}_{\rm YM} = \bar\psi(i\not{D} - m)\psi - \frac{1}{4}(F^k_{\mu\nu})^2} \tag{2.42}$$

! Very simple and very similar to QED

Euler-Lagrange equations:

- "Dirac":

$$(i\not{D} - m)\psi = 0 \qquad \text{! field hidden inside D} \tag{2.43}$$

- "Maxwell":

$$\partial^\mu F^k_{\mu\nu} + \underbrace{g\epsilon^{kjl}A^{j,\mu}F^l_{\mu\nu}} = -g\bar\psi\gamma_\nu\frac{\sigma^k}{2}\psi \equiv j^k_\nu\,, \quad [\leftarrow \text{ the } SU(2)\text{-charge curent}]$$

! non-linear equation: terms $\sim A^2, A^3$ (2.44)

Generalization to $SU(3)$, [and $SU(4)$, $\ldots SU(N)$] is trivial:

$$\psi = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \end{pmatrix} \tag{2.45}$$

Local $SU(3)$ transformations:

$$\begin{aligned} &\psi(x) \longrightarrow V(x)\psi(x) \\ &V(x) = 1 + i\alpha^a(x)t^a + \mathcal{O}(\alpha^2)\,, \qquad a = 1, 2, \ldots, 8 \end{aligned} \tag{2.46}$$

so that

$$\begin{aligned} &\frac{\sigma^a}{2} \;\longrightarrow\; t^a = \frac{\lambda^a}{2} \quad \text{Gell-Mann matrices} \\ &[\frac{\sigma^k}{2}, \frac{\sigma^j}{2}] = i\epsilon^{kjl}\frac{\sigma^l}{2} \;\longrightarrow\; [t^a, t^b] = if^{abc}t^c \end{aligned} \tag{2.47}$$

It follows

$$\begin{aligned} \psi \;&\longrightarrow\; (1 + i\alpha^a t^a)\psi \\ A^a_\mu \;&\longrightarrow\; A^a_\mu + \frac{1}{g}\partial_\mu\alpha^a + f^{abc}A^b_\mu\alpha^c \end{aligned} \tag{2.48}$$

and further

$$\begin{aligned} &[D_\mu, D_\nu] = -igF^a_{\mu\nu}t^a \\ &\quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu \\ &\quad D_\mu = \partial_\mu - igA^a_\mu t^a \end{aligned} \tag{2.49}$$

Remark: Sign of g is a convention, differs in various textbooks

and finally

$$\boxed{\mathcal{L}_{\text{QCD}} = \mathcal{L}_{\text{SU(3)}} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \sum_{\psi=u,d,s,\ldots} \bar{\psi}(i\not{D} - m_\psi)\psi} \tag{2.50}$$

L5

2.3 Quantization and Feynman rules

Assume that QCD can be quantized in the same way as QED (will have some surprises).

Generic Green functions:

$$\langle\Omega|T\{\hat{\bar{\psi}}^i_\alpha(x_1)\hat{A}^a_\mu(x_2)\hat{\psi}^j_\beta(x_3)\ldots\}|\Omega\rangle = \frac{\langle 0|T\{(\hat{\bar{\psi}}_I)^i_\alpha(x_1)(\hat{A}_I)^a_\mu(x_2)(\hat{\psi}_I)^j_\beta(x_3)\ldots e^{i\int d^4x\,\mathcal{L}_I(x)}\}|0\rangle}{\langle 0|T\{e^{i\int d^4x\,\mathcal{L}_I(x)}\}|0\rangle} \tag{2.51}$$

Here

- $\hat{\psi}$ ... on the l.h.s. are Heisenberg operators
- $|\Omega\rangle$ on the l.h.s. is the exact vacuum (ground state)
- All operators on the r.h.s. are written in the interaction representation
- $|0\rangle$ on the r.h.s. is the "perturbative" vacuum (ground state if interactions are switched off)

Then:

Propagators:

$$\langle 0|T\{\psi^i_\alpha(x)\bar{\psi}^j_\beta(y)\}|0\rangle = \delta^{ij}\int \frac{d^4p}{(2\pi)^4 i}e^{-ip(x-y)}\left(\frac{1}{m-\not{p}-i\epsilon}\right)_{\alpha\beta}$$

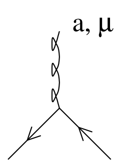
$$\langle 0|T\{A^a_\mu(x)A^b_\nu(y)\}|0\rangle = \delta^{ab}\int \frac{d^4k}{(2\pi)^4 i}e^{-ik(x-y)}\frac{1}{k^2+i\epsilon}\Big[g_{\mu\nu} - \xi\frac{k_\mu k_\nu}{k^2}\Big] \tag{2.52}$$

Interaction:

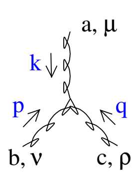
$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$$

$$\mathcal{L}_I = g\bar{\psi}A^a_\mu t^a\gamma^\mu\psi - gf^{abc}(\partial_\alpha A^a_\mu)A^{\alpha,b}A^{\mu,c} - \frac{1}{4}g^2(f^{eab}A^a_\mu A^b_\nu)(f^{ecd}A^{\mu,c}A^{\nu,d}) \tag{2.53}$$

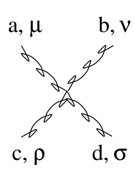
Vertices:



$$= ig\gamma_\mu t^a$$



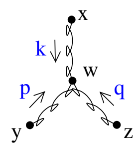
$$= gf^{abc}\big[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu\big]$$



$$\begin{aligned} = -ig^2\big[&f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\ &+ f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\ &+ f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})\big] \end{aligned} \tag{2.54}$$

For example:

$$\langle\Omega|T\{A^d_\alpha(x)A^e_\beta(y)A^f_\lambda(z)\}|\Omega =$$
$$= i\int d^4w\langle 0|T\{T\{A^d_\alpha(x)A^e_\beta(y)A^f_\lambda(z)\mathcal{L}_I(w)\}|0\rangle$$
$$= i\int d^4w\langle 0|T\{A^d_\alpha(x)A^e_\beta(y)A^f_\lambda(z)\big[-gf^{abc}(\partial_\rho A^a_\mu(w))A^{\rho,b}(w)A^{\mu,c}(w)\big]\}|0\rangle$$



$$:= \int\frac{d^4p}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}\int\frac{d^4k}{(2\pi)^4}e^{-ikx-ipy-iqz}(2\pi)^4\delta^{(4)}(p+q+k)$$
$$\times\frac{g_{\alpha\alpha'}}{k^2}\frac{g_{\beta\beta'}}{p^2}\frac{g_{\lambda\lambda'}}{q^2}\cdot(\text{Vertex})^{\alpha'\beta'\gamma'}_{def} \quad (2.55)$$

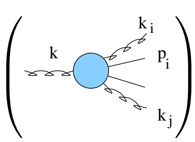
Note $\partial_\rho \longrightarrow -ik_\rho$ if the momentum points into the diagram (Einlaufender Impuls).

Set of Feynman rules defines a quantum theory, but is it selfconsistent?

2.4 Faddeev-Popov ghost fields

- In QED, selfconsistency requires that the amplitudes satisfy Ward identities

$$k^\mu\cdot\Bigg(\quad\Bigg) = 0 \quad (2.56)$$



  for arbitrary photon momenta and for electrons on mass shell $p^2 = m^2$.

  $\Longrightarrow$ electromagnetic charge is conserved in quantum theory
  $\Longrightarrow$ photon remains massless
  $\Longrightarrow$ results of calculations do not depend on gauge parameter, etc.

- In QCD, gluons are charged, so maybe we have to impose a condition $k^2 = 0$ but otherwise we expect that Ward identities must hold true.

2.4.1 Reminder: Photon polarization vectors

Consider a real photon $k^2 = 0$. Let

$$k_\mu = \{k, 0, 0, k\} \quad (2.57)$$

The four-vector $A_\mu(k)$ of a real photon can be decomposed in the basis

$$e_\mu^{(1)} = \{0,1,0,0\} \qquad e_\mu^{(2)} = \{0,0,1,0\}$$
$$e_\mu^{(+)} = \frac{1}{\sqrt{2}}\{1,0,0,1\} \qquad e_\mu^{(-)} = \frac{1}{\sqrt{2}}\{1,0,0,-1\} \tag{2.58}$$

Only first two possibilities (transverse polarizations) are physical because the other two can be disposed of by the choice of gauge:

- Lorentz gauge

$$\partial^\mu A_\mu(x) = 0 \implies k^\mu A_\mu(k) = 0$$
$$\implies A_\mu(k) = e_\mu^{(-)} A_-(k) \;\; \text{not allowed} \;\; [k^\mu e_\mu^{(-)} = 2] \tag{2.59}$$

- For the special case $k^2 = 0$ Lorentz condition does not specify the gauge uniquely

$$0 = k^\mu A_\mu(k) = k^\mu A'_\mu(k) = k^\mu(A_\mu + k_\mu \lambda(k))$$
$$\implies A_\mu(k) = e_\mu^{(+)} A_+(k) \sim k_\mu \;\; \text{can be gauged away} \tag{2.60}$$

Thus, emission of “plus” or “minus” photons cannot influence any observable quantities
Note: it does not mean that we always take Lorentz gauge.

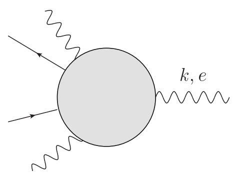
2.4.2 Reminder: Current conservation and Ward Identity in QED

- $U(1)$ gauge symmetry $\Rightarrow$ conserved current (Noether theorem)

$$j^\mu(x) = -e\bar{\psi}(x)\gamma^\mu\psi(x)\,, \qquad \partial_\mu j^\mu(x) = 0$$

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - m)\psi - j_\mu(x)A^\mu(x) \tag{2.61}$$

- Photon emission in QED:



$$iM(k) = iM^\mu(k)\epsilon^*_\mu(k)$$

$$M^\mu(k) = \int d^4x\, e^{ikx}\langle f|\hat{j}^\mu(x)|i\rangle \tag{2.62}$$

- Ward Identity = Current conservation in quantum theory:

$$0 = k_\mu M^\mu(k) = \int d^4x\,\langle f|\hat{j}^\mu(x)|i\rangle\left(-i\frac{\partial}{\partial x^\mu}\right)e^{ikx} = i\int d^4x\, e^{ikx}\langle f|\partial_\mu\hat{j}^\mu(x)|i\rangle \tag{2.63}$$

- Unitarity (conservation of probability) in quantum mechanics:

$$\frac{d}{dt}\int d^3x\,|\Psi(x,t)|^2 = 0 \qquad \Longleftarrow \text{ Hamiltonian is a hermitian operator} \tag{2.64}$$

- Unitarity in QED: unphysical photons cannot be produced in collisions of "physical" particles

Total cross section for photon emission:

$$\sigma \sim \sum_{\substack{phys.\\polar.}} |M|^2 = \sum_{\substack{phys.\\polar.}} \epsilon^{(\lambda),*}_\mu \epsilon^{(\lambda)}_\nu M^\mu M^{\nu,*} = |M^1|^2 + |M^2|^2 \tag{2.65}$$

However

$$k_\mu M^\mu = 0 \implies k_0 M^0 - k_3 M^3 = 0 \implies M^0 = M^3 \tag{2.66}$$

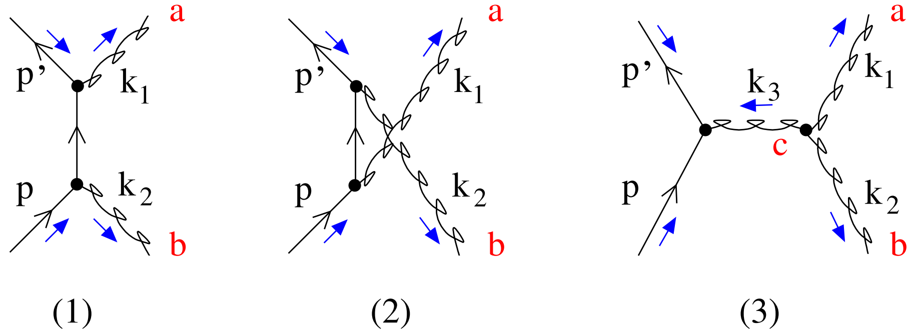
Therefore can write also

$$\sum_{\substack{phys.\\polar.}} \epsilon^{(\lambda),*}_\mu \epsilon^{(\lambda)}_\nu M^\mu M^{\nu,*} = |M^1|^2 + |M^2|^2 + |M^3|^2 - |M^0|^2 = -g_{\mu\nu}M^\mu M^{\nu,*} = \sum_{\substack{all\\polar.}} \epsilon^{(\lambda),*}_\mu \epsilon^{(\lambda)}_\nu M^\mu M^{\nu,*} \tag{2.67}$$

i.e. the sum over transverse polarizations is equal to the sum over all polarizations.
Accepted terminology: The QED S-matrix is unitary

2.4.3 Quark-antiquark annihilation into a pair of gluons

Let us check what happens in QCD on a simple example:



$$q(p) + \bar{q}(p') \to g^a(k_1) + g^b(k_2) \tag{2.68}$$

- The first two diagrams together:

$$\begin{aligned} iM^{\mu\nu}_{1,2}\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2) &= \epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2) \\ &\times (ig)^2\bar{v}(p')\left\{\gamma^\mu t^a \frac{i}{\not{p} - \not{k}_2 - m}\gamma^\nu t^b + \gamma^\nu t^b \frac{i}{\not{k}_2 - \not{p}' - m}\gamma^\mu t^a\right\} u(p) \end{aligned} \tag{2.69}$$

Replace

$$\epsilon^*_\nu(k_2) \longrightarrow k_{2\nu} \tag{2.70}$$

Obtain

$$iM^{\mu\nu}_{1,2}\epsilon^*_\mu(k_1)k_{2\nu} = \epsilon^*_\mu(k_1)(ig)^2\bar{v}(p')\left\{\gamma^\mu t^a \frac{i}{\not{p} - \not{k}_2 - m}\not{k}_2 t^b + \not{k}_2 t^b \frac{i}{\not{k}_2 - \not{p}' - m}\gamma^\mu t^a\right\} u(p) \tag{2.71}$$

Thanks to Dirac equation can replace

$$\begin{aligned} (\not{p} - m)u(p) = 0, &\qquad \text{in the first term} \qquad \not{k}_2 u(p) = (\not{k}_2 - \not{p} + m)u(p) \\ \bar{v}(p')(\not{p}' + m) = 0, &\qquad \text{in the second term} \qquad \bar{v}(p')\not{k}_2 = \bar{v}(p')(\not{k}_2 - \not{p}' - m) \end{aligned} \tag{2.72}$$

The propagators cancel and we get

$$\begin{aligned} iM^{\mu\nu}_{1,2}\epsilon^*_\mu(k_1)k_{2\nu} &= \epsilon^*_\mu(k_1)(ig)^2\bar{v}(p')\Big\{ - i\gamma^\mu[t^a, t^b]\Big\}u(p) \\ &= -g^2\epsilon^*_\mu(k_1)\bar{v}(p')\gamma^\mu u(p) f^{abc}t^c \end{aligned} \tag{2.73}$$

L6

- The third diagram:

$$\begin{aligned} iM^{\mu\nu}_3\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2) &= \epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2) \\ &\times (ig)\bar{v}(p')\gamma_\rho t^c u(p)\frac{-i}{k_3^2} g f^{abc}\Big[g^{\mu\nu}(k_2 - k_1)^\rho + g^{\nu\rho}(k_3 - k_2)^\mu + g^{\rho\mu}(k_1 - k_3)\Big] \end{aligned} \tag{2.74}$$

$$\text{now replace} \qquad \epsilon^*_\nu(k_2) \longrightarrow k_{2\nu}$$

$$\text{and use} \qquad k_1 + k_2 + k_3 = 0 \longrightarrow k_2 = -k_1 - k_3 \tag{2.75}$$

Then

$$\begin{aligned}
\epsilon^*_\nu(k_2)[***] &\longrightarrow k_{2\nu}[***] \\
&= k_2^\mu(k_2 - k_1)^\rho + k_2^\rho(k_3 - k_2)^\mu + g^{\rho\mu}(k_1 - k_3)\cdot k_2 \\
&= (-k_1 - k_3)^\mu(-2k_1 - k_3)^\rho + (-k_1 - k_3)^\rho(2k_3 + k_1)^\mu + g^{\rho\mu}(k_1 - k_3)\cdot(-k_1 - k_3) \\
&= g^{\rho\mu}k_3^2 - k_3^\rho k_3^\mu - g^{\rho\mu}k_1^2 + k_1^\rho k_1^\mu
\end{aligned} \tag{2.76}$$

Hence

$$iM_3^{\mu\nu}\epsilon^*_\mu(k_1)k_{2\nu} = \epsilon^*_\mu(k_1)(ig)\bar{v}(p')\gamma_\rho t^c u(p)\frac{-i}{k_3^2}gf^{abc}\Big[g^{\rho\mu}k_3^2 - k_3^\rho k_3^\mu - g^{\rho\mu}k_1^2 + k_1^\rho k_1^\mu\Big]$$

Assume $k_1^2 = 0$ (on-shell) <u>and</u> $\epsilon^*_\mu(k_1)k_1^\mu = 0$ (physical polarization). Then:

- the last two terms vanish
- the second term vanishes as well:

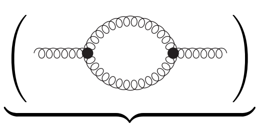
$$(-k_3^\rho)\bar{v}(p')\gamma_\rho u(p) = \bar{v}(p')[(\not{p}' + m) + (\not{p} - m)]u(p) = 0 \tag{2.77}$$

- the first term gives:

$$iM_3^{\mu\nu}\epsilon^*_\mu(k_1)k_{2\nu} = \epsilon^*_\mu(k_1)g^2\bar{v}(p')\gamma^\mu u(p)f^{abc}t^c \tag{2.78}$$

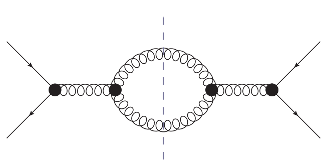
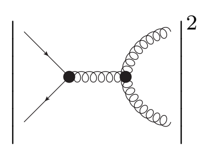
  and exactly cancels the contribution of the first two diagrams!

Happy end? — **No!** — a disaster in loop diagrams (true quantum effects)

$q_\nu \cdot$  $\neq 0$

↘ all polarizations in intermediate state (2.79)

Gluons with unphysical polarizations can be produced $\longrightarrow$ unitarity is broken:

2 Im  $\neq \int d\,(\text{phase space})$  (2.80)

2.4.4 Faddeev-Popov ghosts

Solution: Faddeev, Popov (1967):
Modify QCD Lagrangian

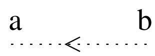
$$\mathcal{L}_{QCD} \longrightarrow \mathcal{L}_{QCD} + \bar{c}^a(x)\Big(-\partial^\mu D^{ab}_{\mu\nu}\Big)c^b(x)$$

$$D^{ab}_\mu = \partial_\mu + f^{abc}A^c_\mu \qquad \text{covariant derivative in adjoint representation} \tag{2.81}$$

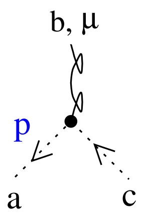
ghost field $c^a(x)$:

- spin-0 field (scalar)
- $a = 1, 2, \ldots, 8$
  (adjoint representation, like gluon)
- Fermi-statistics (!?)
  $\{c(\vec{x}, t), \bar{c}(\vec{y}, t)\} = 0$

(new) Feynman rules:



$$= \frac{i\delta^{ab}}{p^2 + i\epsilon}$$



$$= -g f^{abc} p_\mu \tag{2.82}$$

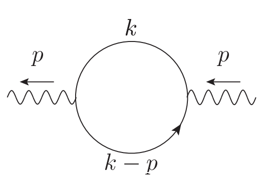
- Derivation uses path integral formalism ⇒ QFT lectures in SS
- The role of ghosts is to subtract "wrong" gluon polarizations, hence "wrong" spin-statistics relation
- Ward identities are modified by terms including ghosts ⇒ Slavnov-Taylor identities
- We will see how this works in practical calculations (exercises)

3 Renormalization and Regularization in QED

Two big issues in quantum field theories:

- Make all expressions mathematically well defined — **Regularization**
- Make sense of the theory that contains infinities — **Renormalization**

3.1 Vacuum polarization



$$\Pi_{\mu\nu} = -e_0^2 \int \frac{d^4k}{(2\pi)^4 i} \mathrm{Tr}\left\{\gamma_\mu \frac{1}{m - \not{k}} \gamma_\nu \frac{1}{m - \not{k} + \not{p}}\right\}$$

$$= -e_0^2 \int \frac{d^4k}{(2\pi)^4 i} \frac{\mathrm{Tr}\left[\gamma_\mu (m + \not{k}) \gamma_\nu (m + \not{k} - \not{p})\right]}{(m^2 - k^2 - i\epsilon)(m^2 - (k-p)^2 - i\epsilon)} \tag{3.1}$$

The trace:

$$\begin{aligned} \mathrm{Tr} &= m^2 \mathrm{Tr}\left[\gamma_\mu \gamma_\nu\right] + \mathrm{Tr}\left[\gamma_\mu \not{k} \gamma_\nu (\not{k} - \not{p})\right] \\ &= 4m^2 g_{\mu\nu} + 4[k_\mu (k-p)_\nu + (k_\nu (k-p)_\mu - g_{\mu\nu} k \cdot (k-p)] \end{aligned} \tag{3.2}$$

Main trick (Feynman):

$$\frac{1}{A \cdot B} = \int_0^1 d\alpha \frac{1}{[\alpha A + (1-\alpha)B]^2} \tag{3.3}$$

$$\frac{\Gamma(a)\Gamma(b)}{A^a \cdot B^b} = \int_0^1 d\alpha\, \alpha^{a-1}(1-\alpha)^{b-1} \frac{\Gamma(a+b)}{[\alpha A + (1-\alpha)B]^{a+b}} \tag{3.4}$$

Use for $A = m^2 - k^2 - i\epsilon$ and $B = m^2 - (k-p)^2 - i\epsilon$

The denominator:

$$\begin{aligned}
\alpha\,(m^2 - k^2) + (1-\alpha)\,[m^2 - (k-p)^2] &= m^2 - \alpha k^2 - (1-\alpha)k^2 + 2(1-\alpha)(kp) - (1-\alpha)p^2 \\
&= m^2 - [k - (1-\alpha)p]^2 + (1-\alpha)^2 p^2 - (1-\alpha)p^2 \\
&= m^2 - [k - (1-\alpha)p]^2 - \alpha(1-\alpha)p^2
\end{aligned} \tag{3.5}$$

Useful notation for future:

$$\boxed{\bar{\alpha} := 1 - \alpha} \tag{3.6}$$

Thus we obtain

$$\Pi_{\mu\nu} = -4e_0^2 \int_0^1 d\alpha \int \frac{d^4k}{(2\pi)^4 i} \frac{g_{\mu\nu}[m^2 - k(k-p)] + k_\mu (k-p)_\nu + (k_\nu (k-p)_\mu}{[m^2 - [k - \bar{\alpha}p]^2 - \alpha\bar{\alpha}p^2 - i\epsilon]^2} \tag{3.7}$$

Change of integration variable: $\quad k \to k' = k - \bar{\alpha}p; \qquad d^4k \equiv d^4k'$

The main advantage: denominator only depends on k'^2; in the numerator $k \to k' + \bar{\alpha}p$:

$$g_{\mu\nu}[m^2 - (k' + \bar{\alpha}p)(k' - \alpha p)] + (k' + \bar{\alpha}p)_\mu (k' - \alpha p)_\nu + (k' + \bar{\alpha}p)_\nu (k' - \alpha p)_\mu \tag{3.8}$$

! Can delete all linear terms in k' because

$$\int d^4k' \frac{k'_\rho}{[k'^2 + X]^2} = 0 \qquad \text{no preferred direction in space} \tag{3.9}$$

Thus, changing notation back to $k' \to k$

$$\Pi_{\mu\nu} = -4e_0^2 \int_0^1 d\alpha \int \frac{d^4k}{(2\pi)^4 i} \frac{g_{\mu\nu}[m^2 - k^2 + \alpha\bar{\alpha}p^2)] + 2k_\mu k_\nu - 2\alpha\bar{\alpha}p_\nu p_\nu}{[m^2 - k^2 - \alpha\bar{\alpha}p^2 - i\epsilon]^2} \tag{3.10}$$

The integral with two powers $k_\mu k_\nu$ can only be $\sim g_{\mu\nu}$:

$$\int d^4k \frac{k_\mu k_\nu}{[k^2 + X]^2} = I(p^2)\, g_{\mu\nu}$$

$$\otimes\ g^{\mu\nu}: \qquad \int d^4k \frac{k^2}{[k^2 + X]^2} = 4I(p^2) \tag{3.11}$$

Therefore

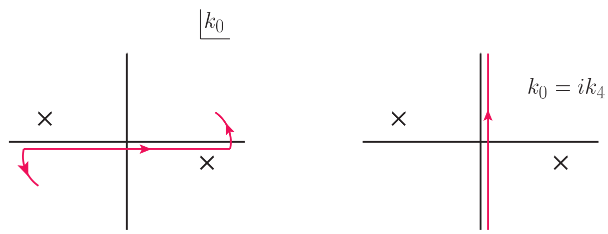
$$\int d^4k \, \frac{k_\mu k_\nu}{[k^2+X]^2} = \frac{1}{4} g_{\mu\nu} \int d^4k \, \frac{k^2}{[k^2+X]^2}$$

$$\text{or effectively} \quad \left[k_\mu k_\nu \quad \Rightarrow \quad \frac{1}{4} g_{\mu\nu} k^2\right] \quad \text{under the integral} \tag{3.12}$$

Thus, finally

$$\Pi_{\mu\nu} = -4e_0^2 \int_0^1 d\alpha \int \frac{d^4k}{(2\pi)^4 i} \left[p_\mu p_\nu \frac{-2\alpha\bar{\alpha}}{[m^2 - k^2 - \alpha\bar{\alpha}p^2 - i\epsilon]^2} + \mathcal{O}(g_{\mu\nu})\right] \tag{3.13}$$

Time has come to calculate the integral. Analytic continuation (Wick rotation):



(3.14)

In this case

$$d^4k = dk_0 d^3\vec{k} = i dk_1 dk_2 dk_3 dk_4 = i d^4 k_E$$

$$k^2 = k_0^2 - \vec{k}^2 = -(k_1^2 + k_2^2 + k_3^2 + k_4^2) = -k_E^2 \tag{3.15}$$

! Space and euclidian time coordinates build a usual Euclidian space (in 4 dim.)
! All factors "i" cancel

$$\Pi_{\mu\nu} = -4e_0^2 \int_0^1 d\alpha \int \frac{d^4k_E}{(2\pi)^4} \left[p_\mu p_\nu \frac{-2\alpha\bar{\alpha}}{[m^2 + k_E^2 - \alpha\bar{\alpha}p^2 - i\epsilon]^2} + \mathcal{O}(g_{\mu\nu})\right] \tag{3.16}$$

Euler:

$$\int d^N k \, f(k^2) = \int d\Omega_N \int_0^\infty k^{N-1} dk \, f(k^2) = \underbrace{\frac{2\pi^{N/2}}{\Gamma(N/2)}}_{\int d\Omega_N} \frac{1}{2} \int_0^\infty dk^2 \, k^{N-2} f(k^2) \tag{3.17}$$

In our case (N=4)

$$\Pi_{\mu\nu} = 8e_0^2 \int_0^1 d\alpha \frac{1}{(2\pi)^4} \pi^2 \int_0^\infty dk^2 \, k^2 \left[\frac{\alpha\bar{\alpha}p_\mu p_\nu}{[m^2 + k^2 - \alpha\bar{\alpha}p^2 - i\epsilon]^2} + \mathcal{O}(g_{\mu\nu})\right] \tag{3.18}$$

?! The integral is divergent at $k^2 \to \infty$ (UV divergence)

- The simplest *regularization* is to introduce a cutoff

$$\int_0^\infty dk^2 \quad \Longrightarrow \quad \int_0^{M^2} dk^2 \tag{3.19}$$

We assume that $M^2 \gg p^2, m^2$, in this case

$$\int_0^{M^2} dk^2 \frac{k^2}{[k^2+m^2-\alpha\bar{\alpha}p^2]^2} = -\frac{M^2}{M^2+m^2-\alpha\bar{\alpha}p^2} + \ln\frac{M^2+m^2-\alpha\bar{\alpha}p^2}{m^2-\alpha\bar{\alpha}p^2}$$
$$\simeq -1 + \ln\frac{M^2}{m^2-\alpha\bar{\alpha}p^2} \tag{3.20}$$

and

$$\Pi_{\mu\nu} = -8p_\mu p_\nu \frac{e_0^2}{16\pi^2}\int_0^1 d\alpha\,\alpha\bar{\alpha}\left[1+\ln\frac{m^2-\alpha\bar{\alpha}p^2-i\epsilon}{M^2}\right] + \mathcal{O}(g_{\mu\nu}) \tag{3.21}$$

Gauge invariance $\Rightarrow$ Ward identity implies

$$p^\mu \Pi_{\mu\nu} = p^\nu \Pi_{\mu\nu} = 0 \qquad \Longrightarrow \qquad \Pi_{\mu\nu} = (g_{\mu\nu} - p_\mu p_\nu)\Pi(p^2) \tag{3.22}$$

If this property holds, the calculation of $\mathcal{O}(g_{\mu\nu})$ contribution is not necessary. One obtains

$$\begin{aligned}\Pi(p^2) &= \frac{2\alpha_0}{\pi}\int_0^1 d\alpha\,\alpha\bar{\alpha}\left[1+\ln\frac{m^2-\alpha\bar{\alpha}p^2-i\epsilon}{M^2}\right] \\ &= \frac{\alpha_0}{9\pi}\left[-2+3\ln\frac{m^2}{M^2}-\frac{12m^2}{p^2}+3\left(1+\frac{2m^2}{p^2}J\ln\frac{J+1}{J-1}\right)\right], \qquad \boxed{J=\sqrt{1-\frac{4m^2}{p^2}}} \\ \Pi(0) &= \frac{\alpha_0}{3\pi}\left[1+\ln\frac{m^2}{M^2}\right]\end{aligned} \tag{3.23}$$

We will discuss how to make sense of the dependence on M^2 in great detail. Before that, there is still another issue to address:

Introducing a cutoff we have solved our mathematical problem to make integrals well defined, but at a high cost: It is easy to see that this procedure actually *breaks* gauge invariance.

Indeed, the complete expression for $\Pi_{\mu\nu}$ is

$$\Pi_{\mu\nu} = -4e_0^2\int_0^1 d\alpha\int\frac{d^4k_E}{(2\pi)^4}\left[p_\mu p_\nu\frac{-2\alpha\bar{\alpha}}{[m^2+k_E^2-\alpha\bar{\alpha}p^2]^2} + g_{\mu\nu}\frac{m^2+\frac{1}{2}k_E^2+\alpha\bar{\alpha}p^2}{[m^2+k_E^2-\alpha\bar{\alpha}p^2]^2}\right] \tag{3.24}$$

Contribution of $k_E^2 \gg p^2, m^2$ is therefore of the form

$$\Pi_{\mu\nu} = -4e_0^2\int_0^1 d\alpha\int\frac{d^4k_E}{(2\pi)^4}\left[p_\mu p_\nu\frac{-2\alpha\bar{\alpha}}{k_E^4}+\frac{1}{2}g_{\mu\nu}\frac{1}{k_E^2}\right] \tag{3.25}$$

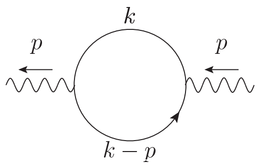
The second contribution is *quadratically* divergent, hence $\sim M^2$, not $\sim \ln M^2$

Note also that

$$\int^{M^2}\frac{d^4k}{(2\pi)^4} \neq \int^{M^2}\frac{d^4(k+p)}{(2\pi)^4} \tag{3.26}$$

that we used above and also in the proof of the Ward identity (for scalar QED).

Let us recall how it works on our case:



$$\Pi_{\mu\nu} = -e_0^2 \int \frac{d^4k}{(2\pi)^4 i} \mathrm{Tr}\left\{\gamma_\mu \frac{1}{m - \not{k}} \gamma_\nu \frac{1}{m - \not{k} + \not{p}}\right\}$$

$$p^\mu \Pi_{\mu\nu} = -e_0^2 \int \frac{d^4k}{(2\pi)^4 i} \mathrm{Tr}\left\{\not{p} \frac{1}{m - \not{k}} \gamma_\nu \frac{1}{m - \not{k} + \not{p}}\right\} \tag{3.27}$$

Write

$$\not{p} = (m - \not{k} + \not{p}) - (m - \not{k}) \tag{3.28}$$

Then, using cyclic property of the trace

$$p^\mu \Pi_{\mu\nu} = -e_0^2 \int \frac{d^4k}{(2\pi)^4 i} \mathrm{Tr}\left\{\frac{1}{m - \not{k}} \gamma_\nu - \gamma_\nu \frac{1}{m - \not{k} + \not{p}}\right\} \tag{3.29}$$

This vanishes *if* one can change integration variable from p to $p - k$.

L7

There exist other, better, regularizations that avoid this problem.

- Pauli-Villars regularization:

$$\frac{1}{m^2 - k^2} - \frac{1}{M^2 - k^2} \tag{3.30}$$

— subtract the same diagram with a "heavy" electron with mass M

- Dimensional regularization ('t Hooft, Veltman, 1971-73), Nobel prize 1999

Lessons:

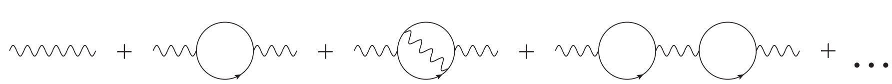
- Calculations of loop diagrams only make sense with a certain UV regularization
- It is possible to choose regularization to maintain Lorentz and gauge invariance
- However, all results depend on an (unphysical) number — the UV cutoff; what to do?

3.2 Photon self-energy and wave function

Free photon propagator in Feynman gauge

$$D^{(0)}_{\mu\nu}(x) = \int \frac{d^4k}{(2\pi)^4 i} \frac{g_{\mu\nu}}{k^2 + i\epsilon} e^{-ikx}, \qquad D^{(0)}_{\mu\nu}(k) = \frac{g_{\mu\nu}}{k^2 + i\epsilon} \tag{3.31}$$

Exact photon propagator

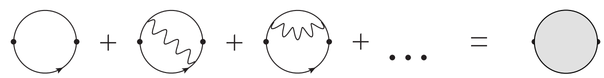
$D_{\mu\nu}(k) =$  (3.32)

The last pictured contribution is the repetition of the second one; can happen separated by large time interval

Such contributions are called "one-particle reducible", they are simple and can be summed up

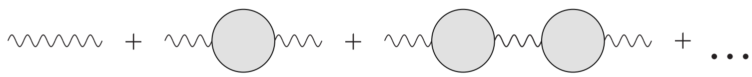
One defines *photon self energy* as the sum of all 1PI diagrams (amputated):

$$\Pi_{\mu\nu}(k) = \tag{3.33}$$



Then

$$D_{\mu\nu}(k) = \tag{3.34}$$



or

$$D_{\mu\nu}(k) \;=\; D^{(0)}_{\mu\nu}(k) + D^{(0)}_{\mu\mu_1}(k)\Pi^{\mu_1\mu_2}(k)D^{(0)}_{\mu_2\nu}(k) + \dots \tag{3.35}$$

⇒ Dyson equation:

$$D_{\mu\nu}(k) \;=\; D^{(0)}_{\mu\nu}(k) + D^{(0)}_{\mu\mu_1}(k)\Pi^{\mu_1\mu_2}(k)\underbrace{D_{\mu_2\nu}}_{\swarrow\ \text{exact!}}(k) \tag{3.36}$$

Using $D^{(0)}_{\mu\nu}(k) = g_{\mu\nu}/k^2$ this yields an equation

$$k^2 D_{\mu\nu}(k) \;=\; g_{\mu\nu} + \Pi_\mu{}^{\mu_2}(k)D_{\mu_2\nu}(k) \qquad \Longrightarrow \qquad \Big[k^2 g_{\mu\mu_1} - \Pi_{\mu\mu_1}\Big]D_{\mu_1\nu} \;=\; g_{\mu\nu} \tag{3.37}$$

Let (Lorentz invariance)

$$\begin{aligned} \Pi_{\mu\nu}(k) &= g_{\mu\nu}\, a_1(k^2) + k_\mu k_\nu\, a_2(k^2) \\ D_{\mu\nu}(k) &= g_{\mu\nu}\, d_1(k^2) + k_\mu k_\nu\, d_2(k^2) \end{aligned} \tag{3.38}$$

Using first the expansion for $\Pi_{\mu\nu}$:

$$\big[k^2 - a_1(k^2)\big]D_{\mu\nu} - k_\mu k^{\mu_1} D_{\mu_1\nu}\, a_2(k^2) = g_{\mu\nu} \tag{3.39}$$

and second for $D_{\mu\nu}$, obtain

$$g_{\mu\nu} \;=\; \big[k^2 - a_1\big]d_1\, g_{\mu\nu} + \big[k^2 - a_1\big]k_\mu k_\nu\, d_2 - k_\mu k_\nu\, d_1 a_1 - k^2 d_2 a_2\, k_\mu k_\nu \tag{3.40}$$

Collecting the terms $\propto g_{\mu\nu}$:

$$1 \;=\; \big[k^2 - a_1\big]d_1 \qquad \Longrightarrow \qquad d_1(k^2) \;=\; \frac{1}{k^2 - a_1(k^2)} \tag{3.41}$$

Therefore

$$D_{\mu\nu}(k) = \frac{g_{\mu\nu}}{k^2 - a_1(k^2) + i\epsilon} + \mathcal{O}(k_\mu k_\nu) \tag{3.42}$$

If $a_1(k^2=0)\neq 0$ we are in a big trouble because photon acquires a mass (or just disappears). Indeed

$$D_{\mu\nu}(x) = \int \frac{d^4k}{(2\pi)^4 i}\frac{g_{\mu\nu}}{k^2 - a_1(k^2) + i\epsilon} e^{-ikx} = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\vec{x}} \int \frac{dk_0}{(2\pi)i}\frac{g_{\mu\nu}e^{-ik_0x_0}}{k_0^2 - \vec{k}^2 - a_1(k^2) + i\epsilon} \quad (3.43)$$

Einstein's relation $k_0 = |\vec{k}|$ (or $k_0 = \sqrt{m^2+\vec{k}^2}$) emerges when the k_0 integral is taken by residues. Thus

— If the pole position is shifted (from zero), particle acquires a mass

— If there is no pole but, say, a cut (e.g. $1/\sqrt{k_0 - m}$) there is no particle at all (dissipation)

! Gauge invariance saves the day (as always)

$$k^\mu \Pi_{\mu\nu} = 0 \Rightarrow k^\mu\big[g_{\mu\nu}\, a_1(k^2) + k_\mu k_\nu\, a_2(k^2)\big] = 0 \quad \Rightarrow \quad a_1(k^2) = -k^2 a_2(k^2) \quad (3.44)$$

We used this relation in the previous Section:

$$\Pi_{\mu\nu}(k) \;=\; \big(g_{\mu\nu}k^2 - k_\mu k_\nu\big)\,\Pi(k^2)\,, \qquad a_1(k^2) = k^2\Pi(k^2)\,, \qquad a_2(k^2) = -\Pi(k^2) \quad (3.45)$$

or

$$\boxed{D_{\mu\nu}(k) \;=\; \frac{g_{\mu\nu}}{k^2[1-\Pi(k^2)]}} \quad (3.46)$$

Thus, *unless* $\Pi(k^2) \sim 1/k^2$, *(Higgs mechanism)* the photon remains massless in quantum theory.

For $k^2 \to 0$ (almost real photon) we can write

$$D_{\mu\nu}(k) \;\stackrel{k^2\to 0}{=}\; \frac{Z_3 g_{\mu\nu}}{k^2 + i\epsilon}\,, \qquad Z_3 = \frac{1}{[1-\Pi(0)]} \quad (3.47)$$

that is, the pole of the propagator remains at $k^2 = 0$, but the *residue* at the pole changes.

Recall that the photon propagator arise from the product of free photon wave functions and $g_{\mu\nu}$ originates from the sum over polarizations:

$$-g_{\mu\nu} \;=\; \sum_\lambda e_\mu^{(\lambda)*}(k) e_\nu^{(\lambda)}(k) \;+\; \text{unphysical polarizations} \quad (3.48)$$

Therefore effectively

$$g_{\mu\nu} \;\to\; Z_3 g_{\mu\nu} \qquad \Longrightarrow \qquad e_\nu^{(\lambda)}(k) \;\to\; \sqrt{Z_3}\, e_\nu^{(\lambda)}(k) \quad (3.49)$$

Interpretation:

The physical photon spends part of its life as a e^+e^- pair (or more complicated state).

Its wave function is a sum of many components

$$\Psi^{\text{phys}}_{\text{photon}} \;=\; \Psi_\gamma \;+\; \Psi_{e^+e^-} \;+\; \Psi_{e^+e^-\gamma} \;+\; \dots \quad (3.50)$$

If we require one physical photon in space

$$\int d^3x\, |\Psi^{\text{phys}}_{\text{photon}}(x)|^2 = 1 \qquad \Longrightarrow \qquad \int d^3x\, |\Psi_\gamma(x)|^2 = Z_3 < 1 \quad (3.51)$$

Redefinition of the normalization of polarization vectors ($e_\mu e^\mu = -1$) is not convenient.
Better keep Z_3 factors explicitly in which case they also enter the relation between the Green function and scattering matrix:

$$
\begin{aligned}
T^{\lambda_1,\ldots,\lambda_n}(k_1,\ldots,k_n) =& e^{(\lambda_1)}_{\mu_1}\ldots e^{(\lambda_n)}_{\mu_n}\widetilde{G}^{\mu_1,\ldots,\mu_n}_{amp}(k_1,\ldots,k_n)\Big|_{\text{on-shell}} \\
&\Rightarrow \left(\sqrt{Z_3}\,e^{(\lambda_1)}_{\mu_1}\right)\ldots\left(\sqrt{Z_3}\,e^{(\lambda_n)}_{\mu_n}\right)\widetilde{G}^{\mu_1,\ldots,\mu_n}_{amp}(k_1,\ldots,k_n)\Big|_{\text{on-shell}}
\end{aligned} \tag{3.52}
$$

— an extra $\sqrt{Z_3}$ factor for each external photon line

Last but not least, we can rewrite the expression for the propagator in the following way:

$$
D_{\mu\nu}(k) \;=\; \frac{g_{\mu\nu}}{k^2[1-\Pi(0)-(\Pi(k^2)-\Pi(0))]} \;:=\; \frac{Z_3 g_{\mu\nu}}{k^2[1-\Pi^{(r)}(k^2)]} \;:=\; Z_3\, D^{(r)}_{\mu\nu}(k) \tag{3.53}
$$

where

$$
\begin{aligned}
Z_3 &= \frac{1}{1-\Pi(0)} && \text{photon WF renormalization constant} \\
\Pi^{(r)}(k^2) &= \frac{\Pi(k^2)-\Pi(0)}{1-\Pi(0)} && \text{photon renormalized self-energy}
\end{aligned} \tag{3.54}
$$

By construction $\Pi^{(r)}(k^2) = \mathcal{O}(k^2)$ so that for $k^2 \to 0$ the propagator is that of a free photon (up to the Z_3 factor)

Let us use the expressions that we have just derived.
To $\mathcal{O}(\alpha)$ accuracy:

$$
\begin{aligned}
Z_3 &= \frac{1}{1-\Pi(0)} \simeq 1+\Pi(0)+\mathcal{O}(\alpha^2) \;=\; \begin{cases} 1-\frac{\alpha}{3\pi}\ln\frac{\mu^2_{\overline{\text{MS}}}}{m^2} & \overline{\text{MS}} \\ 1-\frac{\alpha}{3\pi}\left[\ln\frac{M^2}{m^2}-1\right] & \text{cutoff} \end{cases} \\
\Pi^{(r)} &= \frac{\Pi(k^2)-\Pi(0)}{1-\Pi(0)} \simeq \Pi(k^2)-\Pi(0)+\mathcal{O}(\alpha^2) \\
&= -\frac{2\alpha_0}{\pi}\int_0^1 d\alpha\,\alpha\bar{\alpha}\left\{\left[\ln\frac{M^2}{m^2-\alpha\bar{\alpha}k^2}-1\right]-\left[\ln\frac{M^2}{m^2}-1\right]\right\} \\
&= -\frac{2\alpha_0}{\pi}\int_0^1 d\alpha\,\alpha\bar{\alpha}\left[\ln\frac{m^2}{m^2-\alpha\bar{\alpha}k^2}\right] \qquad \leftarrow \quad \text{in both schemes}
\end{aligned} \tag{3.55}
$$

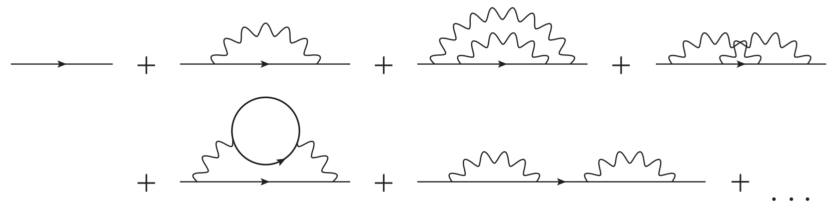
Thus:
- The renormalization constant Z_3 depends on the regularization and scheme
- The renormalized propagator does not depend on regularization

We have been able to localize the problem of divergences in photon propagator — include all of them in *one* constant. Let us see whether we can do the same for other cases.

3.3 Electron mass and wave function renormalization

Exact electron propagator

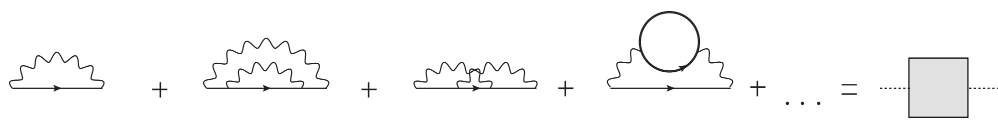
$S(p) =$



(3.56)

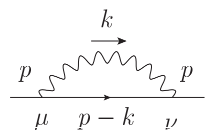
Define *electron self-energy* as the sum of all 1PI (amputated) diagrams:

$-\Sigma(p) =$



(3.57)

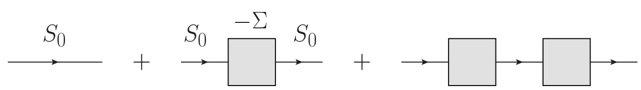
E.g.



$$= \int \frac{d^4k}{(2\pi)^4 i} e\gamma_\mu \frac{m_0 + \not{p} - \not{k}}{m_0^2 - (p-k)^2 - i\epsilon} e\gamma_\nu \frac{g^{\mu\nu}}{k^2 + i\epsilon} \tag{3.58}$$

Then

$S(p) =$



$$\begin{aligned}
&= \frac{1}{m_0 - \not{p}} + \frac{1}{m_0 - \not{p}}\Big(-\Sigma\Big)\frac{1}{m_0 - \not{p}} + \ldots \\
&= \frac{1}{m_0 - \not{p}}\left[1 + \left(-\Sigma\,\frac{1}{m_0 - \not{p}}\right) + \left(-\Sigma\,\frac{1}{m_0 - \not{p}}\right)^2 + \ldots\right] \\
&= \frac{1}{m_0 - \not{p}}\left[\frac{1}{1 + \Sigma\,\frac{1}{m_0 - \not{p}}}\right] \;=\; \frac{1}{m_0 - \not{p} + \Sigma(p)}
\end{aligned} \tag{3.59}$$

⇐ To the last step:
One can show that

$$\Sigma(p) \;=\; \Sigma_1(p^2)\cdot \mathbb{I} + \not{p}\Sigma_2(p^2) \;=\; \Sigma(\not{p}) \qquad \leftarrow \not{p}^2 = p^2 \tag{3.60}$$

Therefore $\Sigma(p)\not{p} = \not{p}\Sigma(p)$.

Mass of a particle corresponds to the pole position in the propagator. L8
In order to find position of the pole it is convenient to project the matrix $S(p)$ on the free electron state.
Let m be the "true" mass, and $u_m(p)$ the corresponding Dirac spinor. i.e.

$$(\not{p} - m)\, u_m(p) \;=\; 0 \tag{3.61}$$

Then

$$\frac{1}{m_0 - \not{p} + \Sigma(\not{p})}\, u_m(p) \;=\; \frac{1}{m_0 - m + \Sigma(m)}\, u_m(p) \tag{3.62}$$

Vanishing of the denominator implies the equation for m:

$$m_0 - m + \Sigma(m) = 0 \qquad \Longrightarrow \qquad m = m(m_0, e_0) \tag{3.63}$$

Since m_0 is not directly observable, it makes sense to eliminate it in favor of $m = 0.511$ MeV:

$$S(p) \;=\; \frac{1}{m_0 - \not{p} + \Sigma(\not{p})} = \frac{1}{m - \not{p} + \Sigma(\not{p}) - \Sigma(m)} \tag{3.64}$$

As we discussed for the photon, the residue at the pole is also important

$$S(x-y) \sim \sum_\lambda \int \frac{d^3p}{(2\pi)^3} \Psi_e(x)\bar{\Psi}_e(y) \qquad\qquad S_0(p) = \frac{\overset{\swarrow\ \bar{u}(p,\lambda)u(p,\lambda) = 2m}{1}}{m_0 - \not{p}} \tag{3.65}$$

In the present case we can write

$$S(p) \;=\; \frac{1}{(m - \not{p})(1 - \Sigma'(m)) + \underbrace{[\Sigma(\not{p}) - \Sigma(m) - (\not{p} - m)\Sigma'(m)]}_{\mathcal{O}((\not{p}-m)^2)}} \tag{3.66}$$

so that close to mass shell

$$S(p) \overset{\not{p}\to m}{=} \frac{1}{(1 - \Sigma'(m))} \cdot \frac{1}{(m - \not{p})} \tag{3.67}$$

Define

$$\begin{aligned} Z_2 \;&=\; \frac{1}{1 - \Sigma'(m)} \\ \Sigma^{(r)} \;&=\; \frac{1}{1 - \Sigma'(m)} \big[\Sigma(\not{p}) - \Sigma(m) - (\not{p} - m)\Sigma'(m)\big] \end{aligned} \tag{3.68}$$

Then

$$\boxed{S(p) \;=\; \frac{Z_2}{m - \not{p} + \Sigma^{(r)}(p)} \;=\; Z_2\, S^{(r)}(p)} \tag{3.69}$$

— *renormalized* electron self-energy and *renormalized* propagator
— Z_2 is called electron WF renormalization constant

Note that

$$\Sigma^{(r)}(\not{p}) \overset{\not{p}\to m}{=} \mathcal{O}\Big((\not{p} - m)^2\Big) \tag{3.70}$$

Therefore for slow electrons $\vec{p} \ll 0.5$ MeV/c

$$\langle\Omega|T\{\psi(x)\bar{\psi}(y)\}|\Omega\rangle\Big|_{x_0>y_0} = \int \frac{dp_0 d^3\vec{p}}{(2\pi)^4} e^{-ip(x-y)} \frac{iZ_2}{\not{p}-m}$$
$$= \sum_s \int \frac{d^3p}{(2\pi)^3 2E_p} \underbrace{\left[\sqrt{Z_2}u(p,s)e^{-iE_p x_0 + i\vec{p}\cdot\vec{x}}\right]}_{\Psi_{\vec{p}}(\vec{x},t)} \underbrace{\left[\sqrt{Z_2}\bar{u}(p,s)e^{iE_p y_0 - i\vec{p}\cdot\vec{y}}\right]}_{\Psi^*_{\vec{p}}(\vec{y},t)} \quad (3.71)$$

Interpretation: (similar to photon)
The physical electron is accompanied by photons (or e^+e^- pairs).
Its wave function is a sum of many components

$$\Psi^{\text{phys}}_{\text{electron}} = \Psi_e + \Psi_{e\gamma} + \Psi_{e\gamma\gamma} + \ldots \quad (3.72)$$

If we require one physical electron in space

$$\int d^3x\, |\Psi^{\text{phys}}_{\text{electron}}(x)|^2 = 1 \quad \Longrightarrow \quad \int d^3x\, |\Psi_e(x)|^2 = Z_2 < 1 \quad (3.73)$$

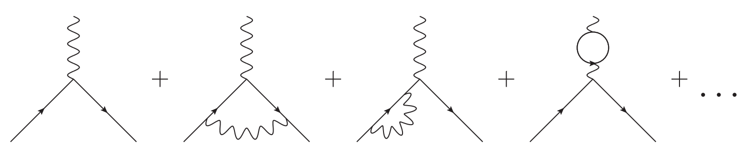
Redefinition of the normalization of Dirac spinors ($\bar{u}u = 2m$) is not convenient.
Better keep Z_2 factors explicitly in which case they also enter the relation between the Green function and scattering matrix.

— an extra $\sqrt{Z_2}$ factor for each external fermion (electron or positron) line

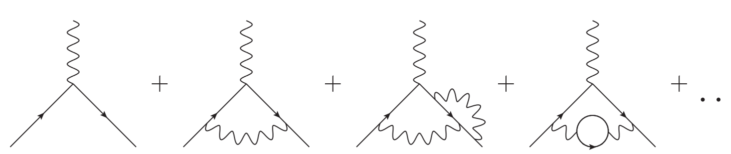
One can show that at least in one-loop calculation (→ exercises)
• $\Sigma(p)$ is UV divergent and must be calculated using a certain regularization
• All UV divergences are localized in Z_2 and the relation $m = m(m_0, e_0)$; the renormalized self energy and propagator are finite

3.4 Renormalized interaction vertex

The three-particle Green function corresponding to photon emission contains both 1PI and 1PR contributions:


(3.74)

We define the vertex function as the sum of all 1PI (amputated) Feynman diagrams:

$\Gamma_\mu(p_1, p_2) =$

(3.75)

It is convenient to separate the leading term

$$\Gamma_\mu(p_1, p_2) = \gamma_\mu + \Lambda_\mu(p_1, p_2) \quad (3.76)$$

For $p_1^2 \to m^2$, $p_2^2 \to m^2$ Lorentz + gauge inv. $\Rightarrow$ $\Lambda_\mu \sim \gamma_\mu$ or $\sim \sigma_{\mu\nu} q^\nu$.
If in addition $q = p_2 - p_1 \to 0$ then $\sigma_{\mu\nu} q^\nu \to 0$ and only $\sim \gamma_\mu$ is possible:

$$\Lambda_\mu(\not{p}_1 \to m, \not{p}_2 \to m) \; = \; \gamma_\mu \Lambda(m, m)$$

Adding and subtracting

$$\begin{aligned}\Gamma_\mu(p_1, p_2) &= \gamma_\mu\big[1 + \Lambda(m,m)\big] + \Big[\Lambda_\mu(p_1,p_2) - \gamma_\mu\Lambda(m,m)\Big] \\ &= \big[1 + \Lambda(m,m)\big]\left[\gamma_\mu + \frac{\Lambda_\mu(p_1,p_2) - \gamma_\mu\Lambda(m,m)}{1+\Lambda(m,m)}\right] \qquad (3.77)\end{aligned}$$

Define

$$\begin{aligned}Z_1 &= \frac{1}{1+\Lambda(m,m)} \\ \Lambda_\mu^{(r)} &= \frac{\Lambda_\mu(p_1,p_2) - \gamma_\mu\Lambda(m,m)}{1+\Lambda(m,m)} \qquad (3.78)\end{aligned}$$

Then

$$\boxed{\Gamma_\mu(p_1,p_2) \; = \; Z_1^{-1}\Big[\gamma_\mu + \Lambda_\mu^{(r)}\Big] \; := \; Z_1^{-1}\Gamma_\mu^{(r)}(p_1,p_2)} \qquad (3.79)$$

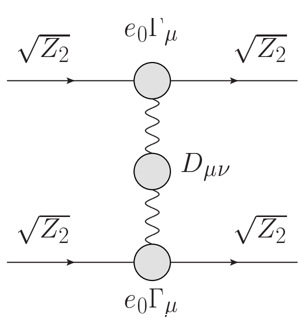
By construction, $\Lambda_\mu^{(r)}$ vanishes when electrons are close to the mass shell and the photon momentum goes to zero. In this limit the exact vertex function looks as the leading-order one apart from the Z_1^{-1} factor.

One-loop calculation:
- $\Gamma(p_1, p_2)$ is UV divergent and must be calculated using a certain regularization
- All UV divergences are localized in Z_1; the renormalized vertex is finite

3.5 Effective charge and renormalizability

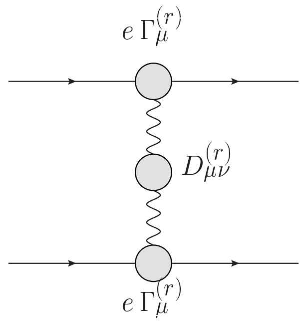
Consider electron-electron scattering at very small angles $t \to 0$

Main contribution comes from the diagrams where the electrons are connected by *one* photon line:



$$\begin{aligned}\Gamma_\mu &= Z_1^{-1}\,\Gamma_\mu^{(r)} \\ D_{\mu\nu} &= Z_3\,D_{\mu\nu}^{(r)} = \sqrt{Z_3}\sqrt{Z_3}D_{\mu\nu}^{(r)} \qquad (3.80)\end{aligned}$$

Combine all Z-factors with e_0: $\Rightarrow$

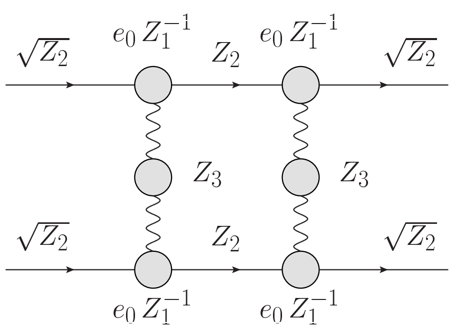


$$\boxed{e \;=\; e_0 Z_2 Z_1^{-1}\sqrt{Z_3}}$$

If $t = (p_1 - p_2)^2 \to 0$:
— $\Gamma_\mu^{r)} \to \Gamma_\mu^{0)}$, $D_{\mu\nu}^{(r)} \to D_{\mu\nu}^{(0)}$ $\Leftarrow$ LO diagram with e_0 replaced by e;
— These diagrams $\sim 1/t$, hence dominate at $t \to 0$: $d\sigma/d\Omega \sim 1/\sin^4\theta/2$;

$\Rightarrow$ e is the true electric charge that enters Coulomb law!

More complicated diagrams:



always $e_0 Z_2 Z_1^{-1}\sqrt{Z_3}$ in each vertex

Thus:
- all physical amplitudes (Green functions on mass shell) can be written in terms of

$$m(m_0, e_0), \qquad e(m_0, e_0), \qquad D_{\mu\nu}^{(r)}(m, e), \qquad S^{(r)}(m, e), \Gamma_\mu^{(r)}(m, e)$$

- The renormalized propagators and interaction vertex expressed in terms of renormalized mass and electric charge are *finite*, i.e. do not depend on regularization of UV divergences to all orders in perturbation theory [to LO $D_{\mu\nu}^{(r)}(m, e) = D_{\mu\nu}^{(r)}(m_0, e_0)$].

- The UV divergences only affect the relation between renormalized (physical) and "bare" mass and coupling. If m, e are substituted by their experimental values, UV divergences disappear from all expressions

- A quantum field theory with such properties is called *renormalizable*

In a different language:
A QFT is always defined with an UV cutoff: $|k_\mu| < M$ (divergences, ignorance of true theory at short distances). How such a theory make sense? Let

$$\begin{aligned} \text{Theory } 1 &= \{m_0, e_0, M\} \\ \text{Theory } 2 &= \{m_0', e_0', M'\} \end{aligned} \tag{3.81}$$

and require that the parameters are adjusted such that

$$m(e_0, m_0, M) = m(e_0', m_0', M')$$
$$e(e_0, m_0, M) = e(e_0', m_0', M') \tag{3.82}$$

This means that the bare mass and charge depend on the cutoff:

$$e_0 = e_0(M)\,, \qquad\qquad m_0 = m_0(M) \tag{3.83}$$

Accepted terminology "The theory is defined at the scale M"

A theory in which such tuning is possible is called "a renormalizable theory"

- special for QED:

$$Z_1 = Z_2 \implies e = e_0\sqrt{Z_3} \qquad [\Leftarrow \text{ Ward–Takahashi Identity}] \tag{3.84}$$

3.6 ∗∗∗ *Generalized Ward Identity* ∗∗∗

Electron is not the only electrically charged particle:

$$e^{\pm}: \quad m_e \simeq 0.511\,\text{MeV}, \qquad \mu^{\pm}: \quad m_\mu \simeq 106\,\text{MeV}, \qquad p,\bar{p}: \quad m_p \simeq 940\,\text{MeV}, \quad \ldots \tag{3.85}$$

They have exactly the same electric charge, why?

We could put all "bare" charges equal e_0 by hand, but what happens after renormalization?

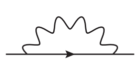
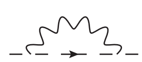
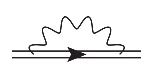
The propagators depend on the mass, so they are all different:

$S^{(e)}(p)$ electron/positron e^+e^-

$S^{(\mu)}(p)$ muon/antimuon $\mu^+\mu^-$ (3.86)

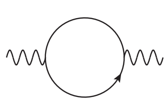
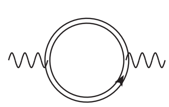
$S^{(p)}(p)$  proton/antiproton $p, \bar{p}$

Hence self-energies and Z_2-factors are also different:

$Z_2^{(e)}(p)$  $Z_2^{(\mu)}(p)$  $Z_2^{(p)}(p)$  (3.87)

and similarly we have three different $Z_1^{(e)}$, $Z_1^{(\mu)}$ $Z_1^{(p)}$.

In contrast, there is only *one* Z_3 which contains a sum over all charged particles:

$Z_3 =$  +  +  + . . . (3.88)

Therefore

$$e^{(e)} = e_0\,(Z_1^{(e)})^{-1}\,Z_2^{(e)}\,\sqrt{Z_3},$$
$$e^{(\mu)} = e_0\,(Z_1^{(\mu)})^{-1}\,Z_2^{(\mu)}\,\sqrt{Z_3}, \qquad \text{etc.} \tag{3.89}$$

(?!) Why $e^{(e)} = e^{(\mu)}$ (experiment)?

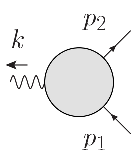
This result follows from the *generalized Ward identity* (below) which implies that for every charged particle in QED

$$\boxed{Z_1 = Z_2} \qquad \text{i.e.} \quad Z_1^{(e)} = Z_2^{(e)}\,, \quad Z_1^{(\mu)} = Z_2^{(\mu)} \tag{3.90}$$

As a consequence

$$e^{(e)} \;=\; e^{(\mu)} \;=\; e^{(\bar{p})} \;=\; \ldots \;=\; e_0\sqrt{Z_3} \qquad \Longleftarrow \qquad \text{universal coupling} \tag{3.91}$$

Generalized Ward identity

$\Gamma_\mu =$ 

- For real particle $\not{p}_1 = m, \quad \not{p}_2 = m$

$$k^\mu \Gamma_\mu(p_1, p_2) \;=\; 0\,, \qquad k = p_1 - p_2 \tag{3.92}$$

- For virtual particles $\not{p}_1 \neq m, \quad \not{p}_2 \neq m$

$$k^\mu \Gamma_\mu(p_1, p_2) \;=\; S^{-1}(p_2) - S^{-1}(p_1)$$

Let us first show that $Z_1 = Z_2$ follows from this result and then prove it.

In the limit $k_\mu \to 0, \not{p}_{1,2} \to m$

$$\begin{aligned} \Gamma_\mu &= Z_1^{-1}[\gamma_\mu + \Lambda^{(r)}(p_1, p_2)] \quad \to \quad Z_1^{-1}\gamma_\mu \\ S(p_1) &= \frac{Z_2}{m - \not{p}_1 + \Sigma^{(r)}(p_1)} \quad \to \quad \frac{Z_2}{m - \not{p}_1} \end{aligned} \tag{3.93}$$

Therefore in this limit

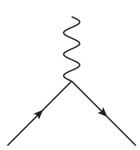
$$\begin{aligned} k^\mu \Gamma_\mu \to Z_1^{-1} \not{k} \;&=\; Z_1^{-1}(\not{p}_1 - \not{p}_2) \\ &=\; Z_1^{-1}\left[(m - \not{p}_2) - (m - \not{p}_1)\right] \\ &=\; Z_1^{-1}\left[S^{-1}(p_2) - S^{-1}(p_1)\right] Z_2 \end{aligned} \tag{3.94}$$

It follows that

$$Z_1^{-1} Z_2 = 1 \qquad \longrightarrow \qquad Z_1 = Z_2 \tag{3.95}$$

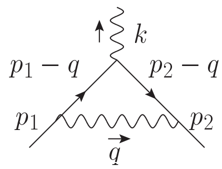
Now let us prove the generalized Ward identity itself.

1) Leading order $\mathcal{O}(1)$



$$\Gamma_\mu^{(0)} \equiv \gamma_\mu\,, \qquad k^\mu \Gamma_\mu^{(0)} = \not{p}_1 - \not{p}_2 = (m - \not{p}_2) - (m - \not{p}_1) = S_0^{-1}(p_2) - S_0^{-1}(p_1) \tag{3.96}$$

2) Next-to-Leading order $\mathcal{O}(e^2)$



$$\Lambda_\mu^{(1)} = e_0^2 \int \frac{d^4q}{(2\pi)^4 i} \gamma^\alpha \frac{1}{m_0 - \not{p}_2 + \not{q}} \gamma_\mu \frac{1}{m_0 - \not{p}_1 + \not{q}} \gamma^\beta \frac{g_{\alpha\beta}}{q^2} \tag{3.97}$$

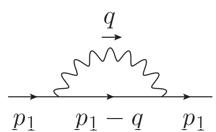
Write

$$k^\mu \gamma_\mu = (p_1 - p_2)^\mu \gamma_\mu = (m_0 - \not{p}_2 + \not{q}) - (m_0 - \not{p}_1 + \not{q}) \tag{3.98}$$

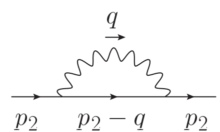
Then

$$k^\mu \Lambda_\mu^{(1)} = e_0^2 \int \frac{d^4q}{(2\pi)^4 i} \gamma_\alpha \frac{1}{m_0 - \not{p}_1 + \not{q}} \gamma^\alpha \frac{1}{q^2} - e_0^2 \int \frac{d^4q}{(2\pi)^4 i} \gamma_\alpha \frac{1}{m_0 - \not{p}_2 + \not{q}} \gamma^\alpha \frac{1}{q^2}$$

=



−



$$= \big[-\Sigma^{(1)}(p_1) \big] - \big[-\Sigma^{(1)}(p_2) \big] \tag{3.99}$$

On the other hand

$$S^{-1}(p) = m_0 - \not{p} + \Sigma(p) \quad \overset{\mathcal{O}(e^2)}{\Longrightarrow} \quad S^{-1}(p_2) - S^{-1}(p_1) = \not{p}_1 - \not{p}_2 + \Sigma^{(1)}(p_2) - \Sigma^{(1)}(p_1) \tag{3.100}$$

This is exactly what we want to prove.

Another form of the same identity:
Consider the limit $k = p_1 - p_2 \to 0$

$$\begin{aligned} k^\mu \Gamma_\mu(p_1, p_2) &= S^{-1}(\not{p}_1 - \not{k}) - S^{-1}(\not{p}_1) = -\frac{dS^{-1}(p_1)}{d\not{p}_1} \not{k} \\ &\Rightarrow \Gamma_\mu(p_1, p_1) = -\frac{dS^{-1}(p_1)}{d\not{p}_1} \gamma_\mu = -\frac{dS^{-1}(p_1)}{dp_1^\mu} \\ &\qquad \nwarrow \quad \gamma_\mu = \gamma_\nu \frac{dp_1^\nu}{dp_1^\mu} = \frac{d\not{p}_1}{dp_1^\mu} \end{aligned} \tag{3.101}$$

Thus

$$\boxed{\Gamma_\mu(p, p) = -\frac{dS^{-1}}{dp^\mu}} \tag{3.102}$$

This is an exact relation to all orders in perturbation theory.

3.7 Renormalization group (QED)

An explicit calculation gives (QED lectures)
Photon self energy, one loop:

$$\Pi(k) = 2\frac{\alpha_0}{\pi} \int_0^1 du\, u(1-u) \left[\ln \frac{m_0^2 - u(1-u)k^2 - i\epsilon}{\Lambda_{UV}^2} + 1 \right] \tag{3.103}$$

For large momenta

$$m_0^2 \ll |k^2| \ll \Lambda_{UV}^2 \tag{3.104}$$

this expression simplifies to logarithmic accuracy to

$$\Pi(k) = -\frac{\alpha_0}{3\pi} \ln \frac{\Lambda^2}{-k^2 - i\epsilon} + \mathcal{O}(1)\,, \qquad \alpha_0 = \frac{e_0^2}{4\pi} \tag{3.105}$$

Let us clarify the meaning of renormalization procedure on this simple example. L9
Start with

$$D_{\mu\nu} = \frac{g_{\mu\nu}}{k^2 + i\epsilon} \frac{1}{1 - \Pi(k^2)} = \frac{g_{\mu\nu}}{k^2 + i\epsilon} \frac{1}{1 + \frac{\alpha_0}{3\pi} \ln \frac{\Lambda^2}{-k^2}} \tag{3.106}$$

and rewrite

$$\begin{aligned}
\frac{1}{1 + \frac{\alpha_0}{3\pi} \ln \frac{\Lambda^2}{-k^2}} &= \frac{1}{1 + \frac{\alpha_0}{3\pi} \ln \frac{\Lambda^2}{m^2} - \frac{\alpha_0}{3\pi} \ln \frac{-k^2}{m^2}} \\
&= \frac{1}{1 + \frac{\alpha_0}{3\pi} \ln \frac{\Lambda^2}{m^2}} \times \frac{1}{1 - \frac{\frac{\alpha_0}{3\pi} \ln \frac{-k^2}{m^2}}{1 + \frac{\alpha_0}{3\pi} \ln \frac{\Lambda^2}{m^2}}} \\
&= \frac{Z_3}{1 - Z_3 \frac{\alpha_0}{3\pi} \ln \frac{-k^2}{m^2}} = \frac{Z_3}{1 - \frac{\alpha}{3\pi} \ln \frac{-k^2}{m^2}}
\end{aligned} \tag{3.107}$$

where

$$Z_3 = Z_3(\Lambda, m) = \frac{1}{1 + \frac{\alpha_0}{3\pi} \ln \frac{\Lambda^2}{m^2}} \tag{3.108}$$

and I used that

$$e = \sqrt{Z_3} e_0 \Longrightarrow \alpha = Z_3 \alpha_0 \tag{3.109}$$

Thus we obtain

$$\begin{aligned}
D_{\mu\nu} &= Z_3\, D_{\mu\nu}^{(r)} \\
D_{\mu\nu}^{(r)} &= \frac{g_{\mu\nu}}{k^2 + i\epsilon} \frac{1}{1 - \frac{\alpha}{3\pi} \ln \frac{-k^2}{m^2}}
\end{aligned} \tag{3.110}$$

- $D_{\mu\nu}^{(r)} = D_{\mu\nu}^{(r)}(e, m)$ as we want
- This is a good approximation so far as

$$\frac{\alpha}{3\pi} \ln \frac{-k^2}{m^2} \ll 1\,, \qquad \alpha = \frac{1}{137} \tag{3.111}$$

Question: L9
What to do if $\alpha << 1$ but $\alpha \ln \frac{-k^2}{m^2} \sim 1$?
— [In QED of academic interest, in QCD ($\alpha_{QCD} \sim 0.3$) important]

— One can show that two-loop diagrams produce $\alpha^2 \ln^2 \frac{-k^2}{m^2}$, three-loop $\alpha^3 \ln^3 \frac{-k^2}{m^2}$ etc

We can try to reorganize the perturbation theory:

- Fixed-order perturbation theory (usual):

  1-st order: α
  2-nd order: α^2, etc.

- Resummed perturbation theory:

$$\begin{aligned} \text{LO (Leading Order)}: \quad & \alpha \ln \frac{k^2}{m^2}\,,\ \alpha^2 \ln^2 \frac{k^2}{m^2}\,,\ \ldots\ ,(\alpha \ln \frac{k^2}{m^2})^n \\ \text{NLO (next-to-LO)}: \quad & \alpha^2 \ln \frac{k^2}{m^2}\,,\ \alpha^3 \ln^2 \frac{k^2}{m^2}\,,\ \ldots\ ,\alpha(\alpha \ln \frac{k^2}{m^2})^n \end{aligned} \tag{3.112}$$

but would need to calculate leading (subleading etc.) parts of Feynman diagrams to all orders.

A very powerful approach — **the renormalization group**

3.7.1 The running coupling, β-function

The idea:
Renormalization procedure is, basically, splitting the large logarithms in two parts:

$$\ln \frac{-k^2}{\Lambda^2} = \underbrace{\ln \frac{-k^2}{m^2}}_{\swarrow \atop \Pi^{(r)},\Sigma^{(r)},\Lambda^{(r)}} + \underbrace{\ln \frac{m^2}{\Lambda^2}}_{\searrow \atop Z-\text{factors}} \tag{3.113}$$

What we achieve by doing this:

1. Dependence on Λ disappears from physical observables; only enters $e = e(e_0, m_0, \Lambda)$
2. Renormalized e, m are the charge and mass for free electrons, as measured in low-energy expts.

Renormalized $\Pi^{(r)}, \Sigma^{(r)}, \Lambda^{(r)}$ are zero for free particles
$\Longrightarrow$ Renormalized propagators = free propagators for $k^2 = 0$, $p^2 = m^2$.

Note that property (1.) is crucial, (2.) is convenient

Let us change the prescription of how we renormalize:

$$\ln \frac{-k^2}{\Lambda^2} = \underbrace{\ln \frac{-k^2}{M^2}}_{\swarrow \atop \Pi^{(r)},\Sigma^{(r)},\Lambda^{(r)}} + \underbrace{\ln \frac{M^2}{\Lambda^2}}_{\searrow \atop Z-\text{factors}} \tag{3.114}$$

where M is an arbitrary mass parameter. For simplicity we will assume

$$m \ll M \ll \Lambda \tag{3.115}$$

In other words

$$\begin{aligned} & \Pi^{(r)}(k^2) := \Pi(k^2) - \Pi(k^2 = 0) \\ \Longrightarrow \quad & \Pi^{(r)}(k^2) := \Pi(k^2) - \Pi(-k^2 = M^2) \end{aligned} \tag{3.116}$$

and similar

$$\Lambda_\mu^{(r)}(p_1,p_2) := \Lambda_\mu(p_1,p_2) - \Lambda_\mu(p_1^2 = p_2^2 = (p_1-p_2)^2 = -M^2)$$
$$\Sigma^{(r)}(p) := \Sigma^{(r)}(p) - \Sigma(\not p = M) - (\not p - M)\Sigma'((\not p = M) \qquad (3.117)$$

What happens in this situation?

- We must always specify the value of M explicitly, i.e.

$$\Pi^{(r)} = \Pi^{(r)}(k^2, M^2, e(M), m(M)) \text{ etc.}$$

- The first property remains valid
- The second property is lost; replaced by

Renormalized propagators (vertices) = free propagators (vertices) for $-k^2 = M^2$, $-p^2 = M^2$

$\hookleftarrow$ Accepted terminology: "the theory is renormalized on the scale M"

We can overtake all results with a simple substitution $m \to M$:

$$Z_3(\Lambda, M, \alpha_0) = \frac{1}{1 + \frac{\alpha_0}{3\pi}\ln\frac{\Lambda^2}{M^2}} \equiv Z_3(M)$$

$$\Pi^{(r)}(k) = -\frac{\alpha}{3\pi}\ln\frac{M^2}{-k^2} \qquad D_{\mu\nu}^{(r)}(k) = \frac{g_{\mu\nu}}{k^2}\frac{1}{1 - \frac{\alpha}{3\pi}\ln\frac{-k^2}{M^2}}$$

$$\alpha(M) = \frac{\alpha_0}{1 + \frac{\alpha_0}{3\pi}\ln\frac{\Lambda^2}{M^2}} \equiv Z_3(M)\,\alpha_0\,, \qquad \alpha_{\text{Coulomb}} \simeq \alpha(M = m) \qquad (3.118)$$

- ! If we choose $M \sim |k|$ there are no large logs in the renormalized propagator

  — Problem solved? Not quite: we do not know the value of $\alpha(M)$ for large M

  — What happens when we change $M_1 \to M_2$?

Consider

$$\alpha(M_1) = \frac{\alpha_0}{1 + \dfrac{\alpha_0}{3\pi}\ln\dfrac{\Lambda^2}{M_1^2}} \qquad \alpha(M_2) = \frac{\alpha_0}{1 + \dfrac{\alpha_0}{3\pi}\ln\dfrac{\Lambda^2}{M_2^2}} \qquad (3.119)$$

- both expressions are valid if $\ln(\Lambda^2/M_{1,2}^2) \ll 1/\alpha$

Idea: eliminate α_0:

$$\alpha(M_2) = \frac{\alpha_0}{1 + \dfrac{\alpha_0}{3\pi}\ln\dfrac{\Lambda^2}{M_1^2} + \dfrac{\alpha_0}{3\pi}\ln\dfrac{M_1^2}{M_2^2}} = \frac{\alpha_0}{1 + \dfrac{\alpha_0}{3\pi}\ln\dfrac{\Lambda^2}{M_1^2}} \times \frac{1}{1 + \dfrac{\dfrac{\alpha_0}{3\pi}\ln\dfrac{M_1^2}{M_2^2}}{1 + \dfrac{\alpha_0}{3\pi}\ln\dfrac{\Lambda^2}{M_1^2}}}$$

$$= \frac{\alpha(M_1)}{1 + \dfrac{\alpha(M_1)}{3\pi}\ln\dfrac{M_1^2}{M_2^2}} \qquad (3.120)$$

• this relation is valid if $\ln(M_1^2/M_2^2) \ll 1/\alpha$; Λ can be arbitrary as it falls out

What to do if $M_1^2 \lll M_2^2$?
— Split the interval in smaller ones, $M_1^2 \leq \mu_1^2 \leq \mu_2^2 \leq \ldots \leq M_2^2$
— Apply the above relation for each smaller interval!

An intelligent way to do this: a differential equation

$$\Longrightarrow \quad \begin{aligned} \frac{1}{\alpha(M)} &= \frac{1}{\alpha_0}\left(1 + \frac{\alpha_0}{3\pi}\ln\frac{\Lambda^2}{M^2}\right) \\ M\frac{d}{dM}\frac{1}{\alpha(M)} &= -\frac{1}{\alpha^2(M)}M\frac{d}{dM}\alpha(M) = -\frac{2}{3\pi} \end{aligned} \tag{3.121}$$

or

$$M\frac{d}{dM}\alpha(M) = +\frac{2}{3\pi}\alpha^2(M) \tag{3.122}$$

A very important concept:

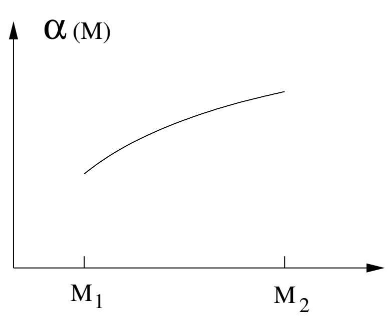
Beta-function **(Gell-Mann–Low function)**

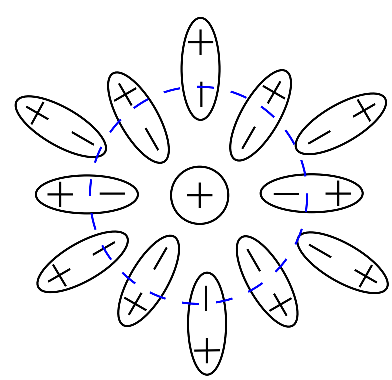
$$\boxed{M\frac{d}{dM}\alpha(M) = \beta(\alpha(M)) = \beta_0\alpha^2(M) + \beta_1\alpha^3(M) + \beta_2\alpha^4(M) + \ldots} \tag{3.123}$$

We have calculated

$$\boxed{\beta_0^{QED} = \frac{2}{3\pi} > 0} \tag{3.124}$$

Terminology: the running coupling
Interpretation: QED charge increases at small distances/large scales due to vacuum polarization





3.7.2 Electron propagator at large momenta; Callan-Symanzik and renormalization group equations

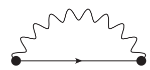
Consider electron propagator for $p \gg m$

$$S(p) = \frac{iZ_2}{\not{p} - m + \Sigma^{(r)}(\not{p})}\,, \qquad \Sigma(\not{p}) = \not{p}\Sigma_1(p^2) + m\Sigma_2(p^2)\,, \qquad p^2 = \not{p}^2 \tag{3.125}$$

In this calculation we can neglect the electron mass $m \to 0$, thus

$$S(p) \simeq \frac{iZ_2}{\not{p}(1 + \Sigma_1^{(r)}(p^2))} \; = \; ? \tag{3.126}$$

First order:

$$\Sigma_1^{\text{bare}}(p) =$$

$$= \frac{\alpha_0}{4\pi} \ln \frac{\Lambda^2}{-p^2} \qquad \text{[exercises]} \tag{3.127}$$

Renormalization:

$$\Sigma_1^{(r)}(p^2, M) = \Sigma_1(p^2) - \Sigma_1(-p^2 = M^2) = \frac{\alpha_0}{4\pi} \ln \frac{M^2}{-p^2} \simeq \frac{\alpha}{4\pi} \ln \frac{M^2}{-p^2}$$

$$Z_2(M) = \frac{1}{1 + \frac{\alpha_0}{4\pi} \ln \frac{\Lambda^2}{M^2}} \tag{3.128}$$

• Both expressions are valid to $\mathcal{O}(\alpha^2)$

In general: L10

$$S^{\text{bare}}(p, \Lambda, \alpha_0) = Z_2(\Lambda, M, \alpha_0) S^{\text{ren}}(p, M, \alpha(M)) \tag{3.129}$$

Note that the l.h.s. does not depend on M. Thus

$$0 = M \frac{d}{dM} S^{\text{bare}} = \left(M \frac{d}{dM} Z_2 \right) S^{\text{ren}} + Z_2 \left[M \frac{\partial}{\partial M} + \left(M \frac{d}{dM} \alpha(M) \right) \frac{\partial}{\partial \alpha} \right] S^{\text{ren}} \tag{3.130}$$

We define

$$M \frac{d}{dM} \alpha(M) \; := \; \beta(\alpha) \qquad \text{beta-function}$$

$$\boxed{\frac{1}{Z_2} M \frac{d}{dM} Z_2 \; := \; 2\gamma(\alpha) \qquad \text{anomalous dimension}} \tag{3.131}$$

Callan-Symanzik equation:

$$\boxed{\left[M \frac{\partial}{\partial M} + \beta(\alpha) \frac{\partial}{\partial \alpha} + 2\gamma(\alpha) \right] S^{\text{ren}}(p, M, \alpha(M)) = 0} \tag{3.132}$$

• this is an example; the CS equation can written for other objects (later)
• why $2\gamma(\alpha)$: the number of external fermion legs (convenient)

In perturbation theory

$$\beta(\alpha) = \beta_0 \alpha^2 + \beta_1 \alpha^3 + \dots \qquad \beta_0 = \frac{2}{3\pi}$$

$$\gamma(\alpha) = \gamma_0 \alpha + \gamma_1 \alpha^2 + \dots \qquad \gamma_0 = \frac{1}{4\pi} \tag{3.133}$$

The self energy can depend on M through $\alpha(M)$, or dimensionless ratio p/M: L10

$$S^{\text{ren}}(p, M, \alpha) = \frac{i}{\not{p}(1 + \Sigma_1^{(r)}(p^2, M^2, \alpha))} = \frac{i}{\not{p}(1 + \Sigma_1^{(r)}(p^2/M^2, \alpha))} \tag{3.134}$$

This implies that if we rescale $p_\mu \to t p_\mu$ and $M \to tM$ <u>for fixed α</u>:

$$S^{\text{ren}}(tp, tM, \alpha) = t^{-1} S^{\text{ren}}(p, M, \alpha) \tag{3.135}$$

Euler's Homogeneous Function Theorem:

$$\left[p_\mu \frac{\partial}{\partial p_\mu} + M \frac{\partial}{\partial M} \right] S^{\text{ren}} = (-1)\, S^{\text{ren}} \tag{3.136}$$

We can use this to convert partial derivatives in M into derivatives in p_μ! Obtain Renormalization Group (RG) equation:

$$\boxed{\left[p_\mu \frac{\partial}{\partial p_\mu} - \beta(\alpha) \frac{\partial}{\partial \alpha} + 1 - 2\gamma(\alpha) \right] S^{\text{ren}}(p, M, \alpha(M)) = 0} \tag{3.137}$$

- this is an example; the RG equation can written for other objects (later)
- this is a linear differential equation describing the momentum dependence

General solution:

$$S^{\text{ren}}(p, M, \alpha(M)) = \frac{i}{\not{p}}\, \tilde{S}(\alpha(p)) \exp\left\{ 2 \int_{\alpha(M)}^{\alpha(p)} d\alpha' \, \frac{\gamma(\alpha')}{\beta(\alpha')} \right\}$$

$$\tilde{S}(\alpha(p)) := S^{\text{ren}}(p = M, \alpha(M)) \qquad \text{boundary condition, only function of } \alpha(p) \tag{3.138}$$

where $\alpha(p) = \alpha(M^2 = -p^2)$ is the running coupling

$$p_\mu \frac{\partial}{\partial p_\mu} \alpha(p) = \beta(\alpha) \tag{3.139}$$

[insert in the equation and check that it is satisfied]

One-loop approximation:

$$\tilde{S}(\alpha(p)) = 1\,, \qquad \gamma(\alpha) = \gamma_0 \alpha\,, \qquad \beta(\alpha) = \beta_0 \alpha^2 \tag{3.140}$$

Then

$$\exp\{**\} = \exp\left\{ 2 \int_{\alpha(M)}^{\alpha(p)} d\alpha' \, \frac{\gamma_0}{\beta_0 \alpha'} \right\} = \exp\left\{ 2 \frac{\gamma_0}{\beta_0} \ln \frac{\alpha(p)}{\alpha(M)} \right\}$$

$$= \left(\frac{\alpha(p)}{\alpha(M)} \right)^{2\gamma_0/\beta_0} = \left(\frac{\alpha(p)}{\alpha(M)} \right)^{3/4} \tag{3.141}$$

so that

$$\boxed{S^{\text{ren}}(p, M, \alpha(M)) = \frac{i}{\not p}\left(\frac{\alpha(p)}{\alpha(M)}\right)^{3/4}} \tag{3.142}$$

If $|p|$ is not very much different from M, can use

$$\alpha(p) = \frac{\alpha(M)}{1 + \frac{\alpha(M)}{3\pi}\ln\frac{M^2}{-p^2}} \tag{3.143}$$

In this case

$$\begin{aligned}\left(\frac{\alpha(p)}{\alpha(M)}\right)^{2\gamma_0/\beta_0} &= \left(1 + \frac{\alpha(M)}{3\pi}\ln\frac{M^2}{-p^2}\right)^{-2\gamma_0/\beta_0} = \left(1 + \frac{\beta_0}{2}\alpha(M)\ln\frac{M^2}{-p^2}\right)^{-2\gamma_0/\beta_0} \\ &= 1 - \gamma_0\alpha(M)\ln\frac{M^2}{-p^2} + \mathcal{O}(\alpha^2\ln^2(\ldots)) = 1 - \frac{\alpha(M)}{4\pi}\ln\frac{M^2}{-p^2}\end{aligned} \tag{3.144}$$

— in agreement with one-loop calculation

Our new result is also applicable for $p^2 \ggg M^2$:
The difference:

- The one-loop result valid if

$$\alpha \ll 1\,, \qquad \alpha\ln\frac{M^2}{p^2} \ll 1 \tag{3.145}$$

- The "RG-improved" result valid if

$$\alpha \ll 1\,, \qquad \alpha\ln\frac{M^2}{p^2} = \mathcal{O}(1) \tag{3.146}$$

- At the end can choose $M \to m$ if desired.

For a better (NLO) approximation have to calculate three new constants:

$$\begin{aligned}\beta(\alpha) &= \beta_0\alpha^2 + \beta_1\alpha^3 \\ \gamma(\alpha) &= \gamma_0\alpha + \gamma_1\alpha^2 \\ \tilde{S}(\alpha(p)) &= 1 + s_1\alpha(p)\end{aligned} \tag{3.147}$$

— sum up all terms $\sim \alpha(\alpha\ln M^2/p^2)^k$, $k = 0, 1, \ldots$

4 Dimensional regularization and minimal subtraction

4.1 Polarization operator in dimensional regularization

Basic idea: analytic continuation in the number of space-time dimensions

$$d = 4 \qquad \Longrightarrow \qquad d = 4 - 2\epsilon\,, \qquad \epsilon \to 0_+ \tag{4.1}$$

Then

$$\int d^d k_E = \frac{\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty dk^2 \, (k^2)^{d/2-1} \quad \nearrow 1-\epsilon<1 \tag{4.2}$$

In addition the charge becomes slightly modified

$$e_0 \qquad \Longrightarrow \qquad \mu^{2-d/2} e_0 \qquad [\mu \text{ has dimension GeV}] \tag{4.3}$$

in order that and  have the same dimension.

Let us calculate $\Pi_{\mu\nu}$ in dimensional regularization. We can start with

$$\begin{aligned} \Pi_{\mu\nu} &= -4e_0^2 \mu^{4-d} \int_0^1 d\alpha \int \frac{d^d k}{(2\pi)^d i} \frac{1}{[m^2 - k^2 - \alpha\bar{\alpha}p^2 - i\epsilon]^2} \\ &\quad \times \Big\{ g_{\mu\nu}[m^2 - k^2 + \alpha\bar{\alpha}p^2] - 2\alpha\bar{\alpha}p_\mu p_\nu + 2k_\mu k_\nu \Big\} \end{aligned} \tag{4.4}$$

Master-formulas for loop integrals in d dimensions (see Appendix **??**)

$$\int d^d k \, \frac{\Gamma(a)}{[-k^2 - A - i\epsilon]^a} = i\pi^{\frac{d}{2}} \frac{\Gamma\left(a - \frac{d}{2}\right)}{[-A]^{a-\frac{d}{2}}} \tag{4.5}$$

$$\int d^d k \, \frac{\Gamma(a)}{[-k^2 - A - i\epsilon]^a} \, k_\mu k_\nu = i\pi^{\frac{d}{2}} \left(-\frac{g_{\mu\nu}}{2}\right) \frac{\Gamma\left(a - 1 - \frac{d}{2}\right)}{[-A]^{a-1-\frac{d}{2}}} \tag{4.6}$$

Obtain

$$\begin{aligned} \Pi_{\mu\nu} &= -4e_0^2 \mu^{4-d} \int_0^1 d\alpha \frac{1}{(4\pi)^{d/2}} \Big\{ \Big[g_{\mu\nu}[m^2 + \alpha\bar{\alpha}p^2] - 2\alpha\bar{\alpha}p_\mu p_\nu \Big] \frac{\Gamma(2 - d/2)}{[m^2 - \alpha\bar{\alpha}p^2]^{2-d/2}} \\ &\qquad\qquad + \Big[d \, \frac{g_{\mu\nu}}{2} - g_{\mu\nu} \Big] \frac{\Gamma(1 - d/2)}{[m^2 - \alpha\bar{\alpha}p^2]^{1-d/2}} \Big\} \\ &\qquad\qquad -k^2 g_{\mu\nu} \nearrow \qquad \nwarrow 2k_\mu k_\nu \end{aligned} \tag{4.7}$$

In the second term

$$\Big[d \, \frac{g_{\mu\nu}}{2} - g_{\mu\nu} \Big] = (-g_{\mu\nu})(1 - d/2) \, , \qquad\qquad (1 - d/2)\Gamma(1 - d/2) = \Gamma(2 - d/2) \tag{4.8}$$

Therefore

$$\begin{aligned} \Pi_{\mu\nu} &= -\frac{4e_0^2 \mu^{4-d}}{(4\pi)^{d/2}} \int_0^1 d\alpha \, \frac{\Gamma(2 - d/2)}{[m^2 - \alpha\bar{\alpha}p^2]^{2-d/2}} \Big\{ g_{\mu\nu}[\cancel{m^2} + \alpha\bar{\alpha}p^2] - 2\alpha\bar{\alpha}p_\mu p_\nu - (\cancel{m^2} - \alpha\bar{\alpha}p^2) g_{\mu\nu} \Big\} \\ &= -\frac{8e_0^2 \mu^{4-d}}{(4\pi)^{d/2}} \int_0^1 d\alpha \, \alpha\bar{\alpha} \frac{\Gamma(2 - d/2)}{[m^2 - \alpha\bar{\alpha}p^2]^{2-d/2}} \Big(g_{\mu\nu}p^2 - p_\mu p_\nu \Big) \end{aligned} \tag{4.9}$$

This indeed has the structure required by gauge invariance! Therefore

$$\Pi(p^2) = -\frac{8e_0^2}{(4\pi)^2} \int_0^1 d\alpha \, \alpha\bar{\alpha} \left[\frac{1}{(4\pi)^{d/2-2}} \frac{\mu^{4-d}\Gamma(2 - d/2)}{[m^2 - \alpha\bar{\alpha}p^2]^{2-d/2}} \right] \tag{4.10}$$

Now let us take the limit $d = 4 - 2\epsilon, \quad \epsilon \to 0$:

$$\frac{1}{(4\pi)^{d/2-2}} = \frac{1}{(4\pi)^{-\epsilon}} = e^{\epsilon \ln 4\pi} = 1 + \epsilon \ln 4\pi + \mathcal{O}(\epsilon^2)$$
$$\mu^{4-d} = \mu^{2\epsilon} = e^{2\epsilon \ln \mu} = 1 + \epsilon \ln \mu^2 + \mathcal{O}(\epsilon^2)$$
$$\Gamma(2 - d/2) = \Gamma(\epsilon) = \frac{1}{\epsilon} \cdot \epsilon\Gamma(\epsilon) = \frac{1}{\epsilon}\Gamma(1+\epsilon) = \frac{1}{\epsilon}(1 - \epsilon\gamma_E) \qquad \boxed{\gamma_E \simeq 0.5772} \tag{4.11}$$

Thus we obtain

$$\Big[\, * \,\Big] = \frac{1}{\epsilon}(1 - \epsilon\gamma_E)(1 + \epsilon \ln 4\pi)\left(1 + \epsilon \ln \mu^2\right)\left(1 + \epsilon \ln \frac{1}{m^2 - \alpha\bar{\alpha}p^2}\right)$$
$$= \frac{1}{\epsilon} - \gamma_E + \ln 4\pi + \ln \frac{\mu^2}{m^2 - \alpha\bar{\alpha}p^2} + \mathcal{O}(\epsilon) \tag{4.12}$$

Regularization: subtraction of divergent term at $\epsilon \to 0$ L11

$$\begin{array}{lll} \text{MS}: & \text{Minimal Subtraction} & \cancel{\frac{1}{\epsilon}} \\ \overline{\text{MS}}: & \text{Modified Minimal Subtraction} & \cancel{\frac{1}{\epsilon} - \gamma_E + \ln 4\pi} \end{array} \tag{4.13}$$

One often writes μ^2_{MS} or $\mu^2_{\overline{\text{MS}}}$ to distinguish between these two standard choices.

Note that choice of the subtraction scheme can be compensated by the choice of μ:

$$\mu^2_{\overline{\text{MS}}} = \mu^2_{\text{MS}}\, 4\pi\, e^{-\gamma_E} \tag{4.14}$$

This is usually referred to as scheme-dependence.

We obtain

$$\Pi^{\overline{\text{MS}}}(p^2) = -\frac{2\alpha_0}{\pi} \int_0^1 d\alpha\, \alpha\bar{\alpha}\, \ln \frac{\mu^2_{\overline{\text{MS}}}}{m^2 - \alpha\bar{\alpha}p^2}$$
$$\Pi^{\text{cutoff}}(p^2) = -\frac{2\alpha_0}{\pi} \int_0^1 d\alpha\, \alpha\bar{\alpha} \left[\ln \frac{M^2}{m^2 - \alpha\bar{\alpha}p^2} - 1\right] \tag{4.15}$$

The two expressions are formally equivalent if we identify

$$M^2 \;=\; e\, \mu^2_{\overline{\text{MS}}} \;=\; 2.71828\, \mu^2_{\overline{\text{MS}}} \tag{4.16}$$

Last but not least:
In dimensional regularization, e.g.,

$$Z_2 = 1 - \gamma_0\alpha_0 \ln \frac{\Lambda^2}{M^2} \;\longrightarrow\; 1 - \frac{1}{4-D}\gamma_0\alpha_0 \tag{4.17}$$

Therefore, formally

$$\ln \frac{\Lambda^2}{M^2} \;\longrightarrow\; \frac{1}{4-D} \;+\; \text{const} \tag{4.18}$$

[Details $\longrightarrow$ exercises]

4.2 Asymptotic freedom: QCD Beta–function

$$\alpha_s(\mu) \equiv \frac{g^2(\mu)}{4\pi} = Z_2^2(\mu) Z_1^{-2}(\mu) Z_3(\mu) \alpha_0 \,, \qquad \alpha_0 \equiv \alpha_s^{(0)}$$
$$\mu \frac{d}{d\mu} \alpha_s(\mu) = \beta(\alpha_s) = \beta_0 \alpha_s^2 + \beta_1 \alpha_s^3 + \ldots \tag{4.19}$$

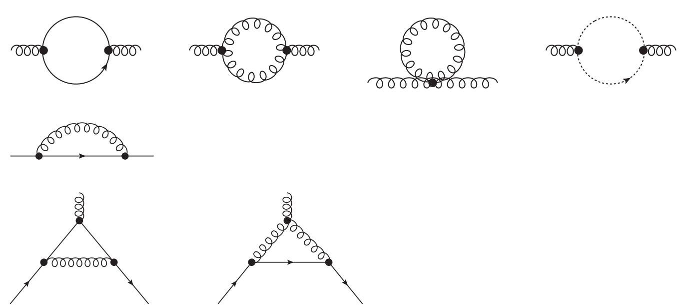
Difference to QED:

- Renormalization at $k^2 \to 0$, $\not p \to m$ makes no sense because of confinement
- $Z_1 \neq Z_2$, therefore have to calculate everything:

Z_3 :

Z_2 :

Z_1 :



(4.20)

Let

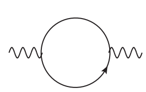
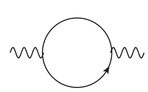
$$Z_3(\mu) = 1 - \delta_3 \alpha_0 \ln \frac{\Lambda^2}{\mu^2}$$
$$Z_2(\mu) = 1 - \delta_2 \alpha_0 \ln \frac{\Lambda^2}{\mu^2}$$
$$Z_1(\mu) = 1 - \delta_1 \alpha_0 \ln \frac{\Lambda^2}{\mu^2} \tag{4.21}$$

Then

$$\beta_0^{\mathrm{QCD}} = 2(\delta_3 + 2\delta_2 - 2\delta_1) \tag{4.22}$$

- Calculation using explicit UV cutoff is at best inconvenient
  $\Longrightarrow$ Dimensional regularization

The simplest case

 $\Big|_{\mathrm{QCD}} = n_f \mathrm{Tr}(t^a t^b) \times$  $\Big|_{\mathrm{QED},\, e \to -g}$ (4.23)

where n_f is the number of existing quarks $u, d, s, \ldots$, $\mathrm{Tr}(t^a t^b) = 1/2 \delta^{ab}$

Compare:
1) Calculation with a cutoff:

$$\begin{aligned}\Pi^{\text{cutoff}}(k^2) &= 2\frac{\alpha_0}{\pi}\int_0^1 d\alpha\,\alpha(1-\alpha)\left[\ln\frac{m^2-\alpha(1-\alpha)k^2}{\Lambda^2}-1\right]\\ &= \frac{\alpha_0}{3\pi}\ln\frac{M^2}{\Lambda^2}\ +\ \text{finite terms}\\ &= \alpha_0\delta_3^{QED}\ln\frac{M^2}{\Lambda^2}\ +\ \text{finite terms}\end{aligned}\tag{4.24}$$

$\Longrightarrow$ δ_3 is a coefficient of $\alpha_0 \ln\frac{1}{\Lambda^2}$

2) Calculation in dim. reg.:

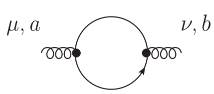
$$\begin{aligned}\Pi^{\text{dim.reg.}}(k^2) &= 2\frac{\alpha_0}{\pi}\int_0^1 d\alpha\,\alpha(1-\alpha)\left[-\frac{2}{4-D}+\gamma_E-\ln 4\pi+\ln\frac{m^2-\alpha(1-\alpha)k^2}{\mu^2}-1\right]\\ &= \frac{\alpha_0}{3\pi}\left(-\frac{2}{4-D}\right)\ +\ \text{finite terms}\end{aligned}\tag{4.25}$$

$\Longrightarrow$ δ_3 is a coefficient of $\alpha_0\left(-\frac{2}{4-D}\right)$

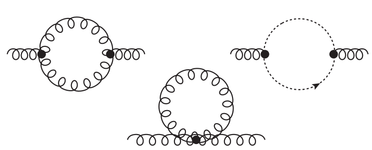
In both cases we need a divergent part only, with the correspondence

$$\boxed{\frac{2}{4-D}\ \Longleftrightarrow\ \ln\Lambda_{UV}^2}\tag{4.26}$$

Explicit calculation (exercises) gives:



$$=(k^2g^{\mu\nu}-k^\mu k^\nu)\delta^{ab}\left(-\frac{\alpha_s}{4\pi}\cdot\frac{2}{3}n_f\right)\Gamma(2-D/2)$$



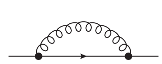
$$=(k^2g^{\mu\nu}-k^\mu k^\nu)\delta^{ab}\left(+\frac{\alpha_s}{4\pi}\cdot\frac{5}{3}N_c\right)\Gamma(2-D/2)\tag{4.27}$$

$$\Gamma(2-D/2)=\frac{1}{2-D/2}\cdot(2-D/2)\Gamma(2-D/2)=\frac{1}{2-D/2}\Gamma(3-D/2)=\frac{2}{4-D}\ +\ \text{finite terms}\tag{4.28}$$

so that

$$\delta_3=\frac{1}{4\pi}\left(-\frac{5}{3}N_c+\frac{2}{3}n_f\right)\tag{4.29}$$

Next:



$$:\qquad \Sigma(p,m=0)=\frac{\alpha_0}{4\pi}\not{p}\,C_F\,\Gamma(2-D/2)\tag{4.30}$$

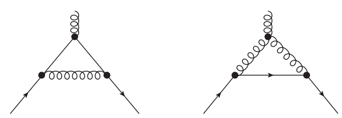
where for $SU(N_c)$

$$\sum_a t^a t^a := C_F \, \mathbb{I} \,, \qquad\qquad C_F = \frac{N_c^2 - 1}{2N_c} \to \frac{4}{3} \quad (N_c = 3) \tag{4.31}$$

$$S(p) = \frac{1}{\not p \left(1 + \frac{\alpha_0}{4\pi} C_F \, \Gamma(2 - D/2)\right)} \implies Z_2 = 1 - \frac{\alpha_0}{4\pi} C_F \, \Gamma(2-D/2) + \dots$$

$$\implies \delta_2 = \frac{1}{4\pi} C_F \tag{4.32}$$

Finally (exercises)



$$\implies \delta_1 = \frac{1}{4\pi}(C_F + N_c) \tag{4.33}$$

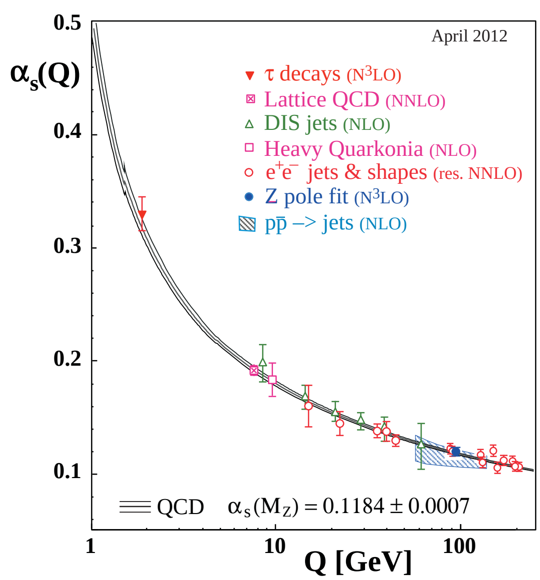
and we obtain (Gross, Wilczek, Politzer, 1973)

$$\beta_0^{\text{QCD}} = 2(\delta_3 + 2\delta_2 - 2\delta_1) = \frac{2}{4\pi}\Big[\underbrace{-\frac{5}{3}N_c + \frac{2}{3}n_f}_{\delta_3} + \underbrace{2C_F}_{\delta_2} \underbrace{- 2C_F - 2N_c}_{\delta_1}\Big] \tag{4.34}$$

- Results for individual δ_k in Feynman gauge, the sum is gauge-invariant

$$\boxed{\beta_0^{\text{QCD}} = \frac{1}{2\pi}\left[\frac{2}{3}n_f - \frac{11}{3}N_c\right] < 0 \qquad \beta_0^{\text{QED}} = \frac{2}{3\pi} > 0} \tag{4.35}$$

Asymptotic freedom:



- Often a different definition is used: L11

$$\mu\frac{d}{d\mu}g(\mu) = \beta(g)\,, \qquad \frac{1}{g}\beta(g) = -b_0\frac{\alpha_s}{4\pi} - b_1\left(\frac{\alpha_s}{4\pi}\right)^2 + \ldots$$
$$b_0 = \frac{11}{3}N_c - \frac{2}{3}n_f \tag{4.36}$$

- five terms in this expansion are known (five loops) [*Baikov, Chetyrkin, Kühn, 2017*]
- A simple parametrization

$$\mu\frac{d}{d\mu}\alpha_s(\mu) = \beta_0\alpha_s^2 = -\frac{b_0}{2\pi}\alpha_s^2 \implies \mu\frac{d}{d\mu}\frac{1}{\alpha_s(\mu)} = -\frac{b_0}{2\pi} \implies \alpha_s(\mu) = \frac{2\pi}{b_0\ln(\mu/\Lambda_{\rm QCD})} \tag{4.37}$$

with $\Lambda_{\rm QCD} \simeq 200 - 250$ MeV (not to be mixed with UV cutoff)

One can rewrite this relation as

$$\Lambda_{\rm QCD} = \mu\, e^{-2\pi/(b_0\alpha_s(\mu))} = \Lambda_{\rm UV} e^{-2\pi/(b_0\alpha_0)} \tag{4.38}$$

! The scale parameter arises because a field theory "remembers" about the UV cutoff
This phenomenon is called dimensional transmutation.
! Note that $e^{-1/(\beta_0\alpha_s(\mu))}$ has zero perturbative expansion

4.3 Renormalization on a Lagrangian level

All UV divergencies in QED can be isolated in three renormalization constants Z_1 (interaction vertex), Z_2 (electron propagator), and Z_3 (photon propagator). To obtain a finite result involving divergent diagrams, the suggested procedure was to calculate the diagrams using bare parameters m_0, e_0 with a certain regulator M to make the expressions well-defined (regularization), and reexpress the results in terms of "physical" parameters m, e (renormalization). The resulting expression should be finite in the limit $\Lambda_{UV} \to \infty$. A more convenient procedure (especially in higher orders) is to implement the renormalization on the Lagrangian level.

As an example, consider scalar field theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m_0^2\phi^2 - \frac{\lambda_0}{24}\phi^4 \tag{4.39}$$

where we will also disregard the mass term for simplicity. The (exact) propagator of the ϕ field will be divergent (similar to electron propagator in QED) and close to the mass shell can be written as

$$\int d^4x\, e^{ipx}\langle\Omega|\phi(x)\phi(0)|\Omega\rangle = \frac{iZ_\phi^2}{p^2 - m^2} \quad + \text{terms regular at } p^2 = m^2 \tag{4.40}$$

We can eliminate the Z_ϕ^2 factor from this equation by rescaling the field

$$\phi = Z_\phi(\epsilon,\mu)\phi_r(\mu) \tag{4.41}$$

where μ is the renormalization scale (before I used M). Then obviously

$$\int d^4x\, e^{ipx}\langle\Omega|\phi_r(x)\phi_r(0)|\Omega\rangle = \frac{i}{p^2 - m^2} \quad + \text{terms regular at } p^2 = m^2 \tag{4.42}$$

is free from any divergences, by construction. In addition, we will have divergences related to the vertex function and can isolate them in a similar manner by introducing the renormalized coupling

$$\lambda_0 = \mu^{2\epsilon} Z_\lambda(\epsilon,\mu)\lambda_r(\mu) \tag{4.43}$$

The Lagrangian of our theory can then be written in terms of the renormalized field and coupling as L12

$$\mathcal{L} = \frac{1}{2} Z_\phi^2 (\partial_\mu \phi_r)^2 - \mu^{2\epsilon} Z_\lambda \frac{\lambda_r}{24} Z_\phi^4 \phi_r^4 \tag{4.44}$$

Here we already imply using dimensional regularization and the appearance of the $\mu^{2\epsilon}$ factor needs explanation.

We require that the action of the theory is dimensionless. In four dimensions, this requires $[\phi] = 1$ (field canonical dimension) and the coupling λ is dimensionless. Changing $\int d^4x\,\mathcal{L} \mapsto \int d^dx\,\mathcal{L}$ requires to modify both. From the kinetic term it follows that we need to assign $[\phi] = d/2-1 = 1-\epsilon$, and then the bare coupling constant cannot remain dimensionless, we need $[\lambda_0] = 4-d = 2\epsilon$. Following 't Hooft we introduce the factor $\mu^{2\epsilon}$ in the bare coupling in order that the renormalized coupling λ_r remains dimensionless. One can write

$$Z_\phi^2 = 1 + \delta Z_\phi^2\,, \qquad\qquad \mu^{2\epsilon} Z_\lambda Z_\phi^4 = 1 + \delta Z \tag{4.45}$$

and

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_r)^2 - \frac{\lambda_r}{24}\phi_r^4 + \frac{1}{2}\delta Z_\phi^2 (\partial_\mu \phi_r)^2 - \delta Z \frac{\lambda}{24}\phi_r^4 \tag{4.46}$$

so that the addenda (called counterterms) produces extra contributions that cancel the divergences that appear in the Feynman diagrams corresponding to the first two terms (identical to original Lagrangian). In other words, in our old formulation the renormalzation was done by dropping the divergent $1/\epsilon$ terms in Feynman diagrams, in the new formulation the divergent terms are cancelled by adding the contribution of the counterterms.

By definition of the minimal subtraction, the counterterms can only involve poles in $1/\epsilon$, so the they have the following generic structure

$$\begin{aligned} Z(\epsilon,\mu) &= 1 + \frac{1}{\epsilon}\Big[z_{11}a + z_{12}a^2 + z_{13}a^3 + \ldots\Big] + \frac{1}{\epsilon^2}\Big[z_{22}a^2 + z_{22}a^3 + \ldots\Big] + \frac{1}{\epsilon^3}\Big[z_{33}a^3 + \ldots\Big] + \ldots \\ &= 1 + \sum_{k\le n} z_{nk}\frac{a^n}{\epsilon^k}\,, \qquad\qquad a(\mu) = \frac{\lambda_r(\mu)}{(4\pi)^2} \end{aligned} \tag{4.47}$$

One obtains by a direct calculation

$$\begin{aligned} Z_\phi^2 &= 1 + \frac{1}{\epsilon}\Big[-\frac{a^2}{24} + \frac{a^3}{48}\Big] + \frac{1}{\epsilon^2}\Big[-\frac{a^3}{24}\Big] + \mathcal{O}(a^4) \\ Z_\lambda &= 1 + \frac{1}{\epsilon}\Big[\frac{3a}{2} - \frac{17a^2}{12}\Big] + \frac{1}{\epsilon^2}\Big[-\frac{9a^2}{4}\Big] + \mathcal{O}(a^3) \end{aligned} \tag{4.48}$$

The β-function and the anomalous dimension of the scalar field are defined as follows:

$$\beta(a) = \mu\frac{d}{d\mu}\,a = \frac{da}{d\ln\mu}\,, \qquad\qquad \gamma_\phi = \frac{1}{2Z_\phi^2}\mu\frac{d}{d\mu}Z_\phi^2 = \frac{1}{2}\frac{d\ln Z_\phi^2}{d\ln\mu} \tag{4.49}$$

Let us calculate the β-function first. The starting observation is that the bare coupling does not depend on the renormalization scale. Thus

$$\begin{aligned} 0 &= \frac{d}{d\ln\mu}\Big[\mu^{2\epsilon}Z_\lambda(\epsilon,\mu)a(\mu)\Big] = 2\epsilon\mu^{2\epsilon}Z_\lambda(\epsilon,\mu)a(\mu) + \mu^{2\epsilon}\left(\frac{d}{d\ln\mu}Z_\lambda(\epsilon,\mu)\right)a(\mu) + \mu^{2\epsilon}Z_\lambda(\epsilon,\mu)\frac{da(\mu)}{d\ln\mu} \\ &= 2\epsilon\mu^{2\epsilon}Z_\lambda(\epsilon,\mu)a(\mu) + \mu^{2\epsilon}\frac{dZ_\lambda}{da}\frac{da(\mu)}{d\ln\mu}a(\mu) + \mu^{2\epsilon}Z_\lambda(\epsilon,\mu)\frac{da(\mu)}{d\ln\mu} \end{aligned} \tag{4.50}$$

where I used that $Z(\epsilon,\mu$ only depends on μ through powers of $a(\mu)$. Using the definition of the β function and dividing out the factor $\mu^{2\epsilon}Z_\lambda(\epsilon,\mu)$ one gets

$$0 = 2\epsilon\, a(\mu) + a\frac{d\ln Z_\lambda}{da}\beta(a) + \beta(a) \tag{4.51}$$

or

$$\beta(a) = -\frac{2\epsilon\, a(\mu)}{1 + a\frac{d\ln Z_\lambda}{da}} = -2\epsilon\, a(\mu)\Big[1 - a\frac{d\ln Z_\lambda}{da} + \left(a\frac{d\ln Z_\lambda}{da}\right)^2 + \dots\Big] \tag{4.52}$$

Note that the β-function on the l.h.s. of this equation is a finite quantity in the $\epsilon \to 0$ limit (as a derivative of a finite renormalized coupling), but $\frac{d\ln Z_\lambda}{da}$ is a sum of poles:

$$a\frac{d\ln Z_\lambda}{da} = \frac{a}{\epsilon}\left(z_{11} + 2z_{21}a\right) + \frac{a^2}{\epsilon^2}[-z_{11}^2 + 2z_{22}] + \mathcal{O}(a^3) \tag{4.53}$$

Thus the only way how a finite l.h.s. can arise is that all $1/\epsilon^2$ and higher power contributions from the expansion of the Z-factor must cancel (which implies that there are some nontrivial relations between the coefficients of higher poles), and only the $1/\epsilon$ term will contribute (and the singularity cancels thanks to the prefactor ϵ). Thus we get

$$\beta(a) = -2\epsilon\, a\Big[1 - \frac{1}{\epsilon}\left(z_{11}a + 2z_{21}a^2\right) + \mathcal{O}(a^3)\Big] = -2\epsilon\, a + 2z_{11}a^2 + 4z_{21}a^3 + \mathcal{O}(a^4) \tag{4.54}$$

From the above expression for Z_λ we read off $z_{11}^{(\lambda)} = 3/2$ and $z_{21}^{(\lambda)} = -17/12$ so that

$$\beta(a) = -2\epsilon\, a + 3a^2 - \frac{17}{3}a^3 + \mathcal{O}(a^4) \tag{4.55}$$

Next, calculate γ_ϕ:

$$\begin{aligned} \gamma_\phi &= \frac{1}{2}\frac{d\ln Z_\phi^2}{d\ln\mu} = \frac{1}{2}\frac{d\ln Z_\phi^2}{da}\beta(a) \\ &= \frac{1}{2}\left\{\frac{1}{\epsilon}\left(2z_{21}^{(\phi)} + 3z_{31}^{(\phi)}a\right) + \text{ higher poles } + \mathcal{O}(a^2)\right\}\left\{-2\epsilon\, a + 3a^2 - \frac{17}{3}a^3 + \mathcal{O}(a^4)\right\} \end{aligned} \tag{4.56}$$

Here again, the only way to obtain a finite contribution is from the product of the single pole terms in the Z_ϕ^2-factor and the $-2\epsilon a$ term in the β-function; the higher poles must cancel. Thus

$$\gamma_\phi = \frac{1}{2}(-2\epsilon a)\frac{1}{\epsilon}\left(2z_{21}^{(\phi)} + 3z_{31}^{(\phi)}a\right) = -2z_{11}^{(\phi)} - 3z_{21}^{(\phi)}a^2 = \frac{1}{12}a - \frac{1}{16}a^2 \tag{4.57}$$

To summarize

- The β-function in a $4-2\epsilon$ dimensional theory has a term $\beta(a) = -2\epsilon a + \mathcal{O}(a^2)$. In renormalized quantities we set $\epsilon \to 0$ so that this term can be omitted (at the end of calculation!)
- Only simple pole terms $\sim 1/\epsilon$ in the Z-factors are important, because all higher-order terms must cancel in proper combinations. One can show that all higher poles can be restored without calculation from this condition, they do not contain new physical information.

The construction of the renormalized action/Lagrangian in gauge theories is very similar. We start with the gauge-fixed QCD action in $d = 4 - 2\epsilon$ in terms of bare fields

$$S = \int d^d x \left[\bar{q}(\not{\partial} - i\mu^\epsilon g_0 \not{A})q + \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu,a} - \bar{c}^a (\partial_\mu D^\mu c)^a + \frac{1}{2\xi}(\partial_\mu A^{a,\mu})^2 \right] \tag{4.58}$$

Here

$$\begin{aligned} F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\mu A^a_\nu + \mu^\epsilon g_0 f^{abc} A^b_\mu A^c_\nu \\ D_\mu c &= \partial_\mu c - i\mu^\epsilon g_0 [A_\mu, c] \qquad D^{ab}_\mu = \partial_\mu + \mu^\epsilon g_0 f^{abc} A^c_\mu \end{aligned} \tag{4.59}$$

and make the replacements

$$q \to Z_q q_r\,, \qquad A \to Z_A A_r\,, \qquad c \to Z_c c_r\,, \qquad g \to Z_g g_r\,, \qquad \xi \to Z_\xi \xi_r \tag{4.60}$$

Converting to our previous notation (standard)

$$Z_2 = Z_q^2\,, \qquad Z_3 = Z_A^2\,, \qquad Z_1^{-1} = Z_g Z_A^{-1} Z_q^{-2} \tag{4.61}$$

The corresponding anomalous dimensions are defined as

$$\gamma_g = \mu\partial_\mu \ln Z_g \qquad \gamma_q = \mu\partial_\mu \ln Z_q \qquad \gamma_A = \mu\partial_\mu \ln Z_A \tag{4.62}$$

etc. We will use

$$\alpha_s = \frac{g^2}{4\pi}\,, \qquad a = \frac{\alpha_s}{4\pi} = \frac{g^2}{(4\pi)^2} \tag{4.63}$$

The β-functions:

$$\begin{aligned} &\beta(a) = \mu\partial_\mu a = 2a(-\epsilon - \gamma_g) = 2a(-\epsilon - \bar{\beta}(a))\,, \qquad \bar{\beta}(a) = \beta_0 a + \beta_1 a^2 + \dots\,, \qquad \beta_0 = \frac{11}{3}N_c - \frac{2}{3}n_f\,, \\ &\beta_\xi(\xi, g) = \mu\partial_\mu \xi = \mu\partial_\mu \xi_0 Z_\xi^{-1} = -\xi\mu\partial_\mu \ln Z_\xi = -\xi\mu\partial_\mu \ln Z_A^2 = -2\xi\mu\partial_\mu \ln Z_A = -2\xi\gamma_A\,. \end{aligned} \tag{4.64}$$

where I used that $Z_\xi = Z_A^2$ since the gauge fixing term is not renormalized.

Note that in Landau gauge $\xi = 0$ the beta-function β_ξ vanishes identically for arbitrary coupling.

4.4 * * * *The strategy of regions: A simple example* * * *

Dimensional regularization offers very powerful tools to calculate Feynman diagrams. The strategy of regions is a technique which allows one to carry out asymptotic expansions of loop integrals around various limits. The expansion is obtained by splitting the integration in different regions and appropriately expanding the integrand in each case. We will later formulate an effective theory, called SCET, where the different regions will be represented by different effective theory fields. My presentation follows a book by T.Becher, A.Broggio, A.Ferroglia: *Introduction to Soft-Collinear Effective Theory*. For a higher-level discussion see B. Jantzen, arXiv:1111.2589.

Our present goal is only to illustrate the main idea. To this end we consider a simple integral, which we will expand using different methods, first using a cutoff to separate two different regions and then with dimensional regularization:

$$I = \int_0^\infty dk \frac{k}{(k^2+m^2)(k^2+M^2)} = \frac{1}{M^2-m^2} \ln \frac{M}{m} \,. \tag{4.65}$$

We assume $m^2 \ll M^2$ and will discuss an expansion in the small parameter m/M. Obviously

$$I = \frac{1}{M^2}\left(1 + \frac{m^2}{M^2} + \frac{m^4}{M^4} + \cdots\right) \ln \frac{M}{m} \,. \tag{4.66}$$

Our goal is to reproduce this result by expanding the integrand in Eq. (4.65) before carrying out the integration. This is helpful in cases where the full result is not available.

A naive expansion of the integrand

$$\frac{k}{(k^2+m^2)(k^2+M^2)} = \frac{k}{k^2(k^2+M^2)}\left(1 - \frac{m^2}{k^2} + \frac{m^4}{k^4} + \cdots\right) \tag{4.67}$$

does not work as it gives rise to IR divergent integrals. This was to be expected: If one could simply Taylor expand the integrand in m/M and integrate term by term, the result would necessarily be an analytic function of m because the integrals would simply give the Taylor coefficients of the expansion in m. But the result for I is not analytic in m/M, it contains a logarithm.

The problem with the naive expansion is obviously that it is valid only for $k \gg m^2$ while the integration domain in Eq. (4.65) includes a region in which $k^2 \sim m^2$. As a warm up, we can avoid this problem introducing a scale $m \ll \Lambda \ll M$ to separate the two regions:

$$I = \underbrace{\int_0^\Lambda dk \frac{k}{(k^2+m^2)(k^2+M^2)}}_{I_{(I)}} + \underbrace{\int_\Lambda^\infty dk \frac{k}{(k^2+m^2)(k^2+M^2)}}_{I_{(II)}} \,. \tag{4.68}$$

In the first region we use that $k, m \ll M$ so that

$$I_{(I)} = \int_0^\Lambda dk \frac{k}{(k^2+m^2)(k^2+M^2)} = \int_0^\Lambda dk \frac{k}{(k^2+m^2)M^2}\left(1 - \frac{k^2}{M^2} + \frac{k^4}{M^4} + \cdots\right) \,. \tag{4.69}$$

In the second region we use $m \ll k, M$ so that

$$I_{(II)} = \int_\Lambda^\infty dk \frac{k}{(k^2+m^2)(k^2+M^2)} = \int_\Lambda^\infty dk \frac{k}{k^2(k^2+M^2)}\left(1 - \frac{m^2}{k^2} + \frac{m^4}{k^4} + \cdots\right) \,. \tag{4.70}$$

Taking into account the first two terms in $I_{(I)}$ and the leading term only in $I_{(II)}$

$$\begin{aligned} I_{(I)} &\simeq \frac{M^2+m^2}{2M^4} \ln\left(1 + \frac{\Lambda^2}{m^2}\right) - \frac{\Lambda^2}{2M^4} = -\frac{1}{M^2} \ln\left(\frac{m}{\Lambda}\right) - \frac{\Lambda^2}{2M^4} + \mathcal{O}\left(\frac{\Lambda^4}{M^6}, \frac{m^2}{M^4} \log\left(\frac{\Lambda}{m}\right)\right) \,, \\ I_{(II)} &\simeq \frac{1}{2M^2} \ln\left(1 + \frac{M^2}{\Lambda^2}\right) = -\frac{1}{M^2} \ln\left(\frac{\Lambda}{M}\right) + \frac{\Lambda^2}{2M^4} + \mathcal{O}\left(\frac{\Lambda^4}{M^6} \log\left(\frac{M}{\Lambda}\right)\right) \,. \end{aligned} \tag{4.71}$$

and summing up

$$I = I_{(I)} + I_{(II)} = -\frac{1}{M^2} \ln\left(\frac{m}{M}\right) + \mathcal{O}\left(\frac{m^2}{M^4} \log\left(\frac{M}{m}\right)\right) , \tag{4.72}$$

which is the expected result. All terms depending on Λ cancel as they must.

Thus the trick works, but it is well known that the use of hard cutoffs is impractical in calculations of Feynman diagrams (apart from the simplest cases). We want to find out whether a separation of different integration regions can be achieved using dimensional regularization. To this end, consider

$$I = \int_0^\infty dk\, k^{-\varepsilon} \frac{k}{(k^2+m^2)(k^2+M^2)} , \tag{4.73}$$

where we will eventually send $\varepsilon \to 0$ at the end of the calculation.

Using a low-energy expansion of the integrand, $k, m \ll M$, as in (4.69)

$$I_{(I)} = \int_0^\infty dk\, k^{-\varepsilon} \frac{k}{(k^2+m^2)M^2} \left(1 - \frac{k^2}{M^2} + \frac{k^4}{M^4} + \cdots\right) . \tag{4.74}$$

Here, for each term separately, we can choose the dimensional regulator $\varepsilon > 0$ such that the integral will converge both for $k \to 0$ (IR finite) and for $k \to \infty$ (UV finite).

Similarly, by performing a large-energy expansion $k \gg \Lambda$ (cf. (4.70)) one obtains

$$I_{(II)} = \int_0^\infty dk\, k^{-\varepsilon} \frac{k}{k^2(k^2+M^2)} \left(1 - \frac{m^2}{k^2} + \frac{m^4}{k^4} + \cdots\right) . \tag{4.75}$$

and we can choose $\varepsilon < 0$, to avoid IR divergences in the region where $k \to 0$. Taking into account the first terms only one finds

$$I_{(I)} = \frac{m^{-\varepsilon}}{2M^2} \Gamma\left(1 - \frac{\varepsilon}{2}\right) \Gamma\left(\frac{\varepsilon}{2}\right) = \frac{1}{M^2}\left(\frac{1}{\varepsilon} - \ln m + \mathcal{O}(\varepsilon)\right) . \tag{4.76}$$

$$I_{(II)} = -\frac{M^{-\varepsilon}}{2M^2} \Gamma\left(1 - \frac{\varepsilon}{2}\right) \Gamma\left(\frac{\varepsilon}{2}\right) = \frac{1}{M^2}\left(-\frac{1}{\varepsilon} + \ln M + \mathcal{O}(\varepsilon)\right) . \tag{4.77}$$

The poles in ε cancel in the sum, and the final result is again as expected!

This looks like magic, because at first sight there are at least two suspect issues:

• First: can we choose $\varepsilon > 0$ in the low-energy region and $\varepsilon < 0$ in the high-energy region and then combine the two?

— This is legitimate, because dimensionally regularized expressions are defined for *arbitrary* ε: we only choose $\varepsilon > 0$ to be able to evaluate $I_{(I)}$ as a standard integral, but by analytic continuation the resulting function on the right-hand side is uniquely defined for any complex-valued ε and can be combined with $I_{(II)}$.

• Second: The integration domain in both Eq. (4.74) and Eq. (4.75) is not restricted to a low/high energy region. Since we integrate the high-energy part over the low-energy region (and vice versa), one could worry that this leads to a double counting?

— To see that this does not happen, observe that the low-energy integral $I_{(I)} \sim m^{-\varepsilon}$, while the high-energy integral $I_{(II)} \sim M^{-\varepsilon}$. This statement remains true also for the subleading terms. Keeping the complete dependence on m and M the result for our integral (for finite ε) is

$$I = \frac{1}{2}\Gamma\left(1-\frac{\varepsilon}{2}\right)\Gamma\left(\frac{\varepsilon}{2}\right)\frac{m^{-\varepsilon}-M^{-\varepsilon}}{M^2-m^2}\,. \tag{4.78}$$

and the low-energy/high-energy parts just pick up the pieces $\sim m^{-\varepsilon}$ and $\sim M^{-\varepsilon}$, respectively. Even though we integrate twice over the full integration domain, there is no double counting, since the two pieces scale differently: the low-energy integrals can never produce a term $M^{-\varepsilon}$ since they depend analytically on the large scale, and vice-versa.

— Let us see what happens if we insist in restricting the integration domain of the low- and high-energy region integrals when using dimensional regularization. The integral in the low-energy region would become in this case

$$\begin{aligned} I^{\Lambda}_{(I)} &= \int_0^{\Lambda} dk\, k^{-\varepsilon}\frac{k}{(k^2+m^2)M^2}\left(1-\frac{k^2}{M^2}+\frac{k^4}{M^4}+\dots\right) \\ &= \left[\int_0^{\infty} dk - \int_{\Lambda}^{\infty} dk\right] k^{-\varepsilon}\frac{k}{(k^2+m^2)M^2}\left(1-\frac{k^2}{M^2}+\frac{k^4}{M^4}+\dots\right) \\ &= I_{(I)} - R_{(I)}\,. \end{aligned} \tag{4.79}$$

To calculate the remainder $R_{(I)}$ we can use that $k \geq \Lambda \gg m^2$ to expand in the small m limit:

$$\begin{aligned} R_{(I)} &= \int_{\Lambda}^{\infty} dk\, k^{-\varepsilon}\frac{k}{(k^2+m^2)M^2}\left(1-\frac{k^2}{M^2}+\dots\right) \\ &= \int_{\Lambda}^{\infty} dk\, k^{-\varepsilon}\frac{k}{k^2M^2}\left(1-\frac{m^2}{k^2}-\frac{k^2}{M^2}+\dots\right)\,. \end{aligned} \tag{4.80}$$

Note that in this way we performed two expansions already: First, expanding the integrand in the limit $M \to \infty$ to separate part $I_{(I)}$, and, second, expanding the result for $m \to 0$.

Similarly, consider the high-energy integral $I_{(II)}$ in Eq. (4.75) with a lower cutoff Λ on the integration. As above, it can be written as an integral without a cutoff and a remainder

$$\begin{aligned} R_{(II)} &= \int_0^{\Lambda} dk\, k^{-\varepsilon}\frac{k}{k^2(k^2+M^2)}\left(1-\frac{m^2}{k^2}+\dots\right) \\ &= \int_0^{\Lambda} dk\, k^{-\varepsilon}\frac{k}{k^2M^2}\left(1-\frac{m^2}{k^2}-\frac{k^2}{M^2}+\dots\right)\,. \end{aligned} \tag{4.81}$$

In this remainder, we have again expanded the integrand in both the limit of small m and also in the limit of large M, but in the opposite order as in $R_{(I)}$. However, the two expansions commute so that the integrands of $R_{(I)}$ and $R_{(II)}$ are identical. Adding up the two pieces, we find that

$$R = R_{(I)} + R_{(II)} = \int_0^{\infty} dk\, k^{-\varepsilon}\frac{k}{k^2M^2}\left(1-\frac{m^2}{k^2}-\frac{k^2}{M^2}+\dots\right)\,. \tag{4.82}$$

The dependence on Λ disappears, as expected, and the remaining integrals are of the type

$$\int_0^{\infty}\frac{dk}{k}\,\frac{1}{k^{n+\varepsilon}} =? \qquad n = 0, \pm 2, \dots \tag{4.83}$$

Integrals of this type are called *scaleless integrals* as they do not involve dimensionful parameters.

Take $n = 0$ as example (the leading term). As written, the integral is ill-defined since a nonzero $\varepsilon > 0$ does not help to eliminate the divergence for $k \to 0$ and $k \to \infty$ simultaneously. To define this integral properly one needs to introduce an additional regulator — e.g. restore the scale separation

$$\int_0^\infty \frac{dk}{k}\, k^{-\epsilon} = \int_0^\Lambda \frac{dk}{k}\, k^{-\epsilon} + \int_\Lambda^\infty \frac{dk}{k}\, k^{-\epsilon} \tag{4.84}$$

such that the first and the second pieces can be regularized choosing $\varepsilon = \varepsilon_{IR} < 0$ or $\varepsilon = \varepsilon_{UV} > 0$, respectively:

$$\int_0^\Lambda \frac{dk}{k}\, k^{-\epsilon_{IR}}\Big|_{\varepsilon_{IR}<0} = -\frac{1}{\varepsilon_{IR}}\Lambda^{-\varepsilon_{IR}}, \text{ and } \qquad \int_\Lambda^\infty \frac{dk}{k}\, k^{-\epsilon_{UV}}\Big|_{\varepsilon_{UV}>0} = +\frac{1}{\varepsilon_{UV}}\Lambda^{-\varepsilon_{UV}} \tag{4.85}$$

The main point is that by analytic continuation the resulting functions on the r.h.s. are uniquely defined for any complex-valued ε so that we can take $\varepsilon_{IR} = \varepsilon_{UV}$ at the end, such that the whole integral vanishes. This observation is crucial for the applications of dimensional regularization:

all scaleless integrals can be put to zero

To summarize, the remainder (4.82) vanishes because it is given by a series of scaleless integrals, so that there is no double counting. L13

5 Unitarity and Feynman diagrams

5.1 Optical Theorem and cut diagrams

A transition amplitude (probability amplitude) to observe a particular state in a scattering experiment can formally be thought of as a matrix element of the unitary operator in the Hilbert space

$$\langle q_1 \dots q_n | S | k_1 k_2 \rangle \equiv \lim_{T\to\infty} \langle q_1 \dots q_n | e^{-i\hat{H}_I T} | k_1 k_2 \rangle \tag{5.1}$$

Unitarity means

$$S^\dagger S = \mathbb{I} \tag{5.2}$$

i.e.

$$\sum_{q_f} \langle k_1' k_2' | S^\dagger | q_f \rangle \langle q_f | S | k_1 k_2 \rangle = \langle k_1' k_2' | S^\dagger S | k_1 k_2 \rangle = (2\pi)^6 \delta^{(3)}(k_1 - k_1') \delta^{(3)}(k_2 - k_2') \tag{5.3}$$

We write

$$S := \mathbb{I} + iT \qquad \text{T-matrix}$$

$$\langle A | iT | B \rangle := (2\pi)^4 \delta^{(4)}(p_A - p_B)\, iM(B \to A) \qquad \text{Amplitude} \tag{5.4}$$

Unitarity of the S-matrix implies

$$S^\dagger S = \mathbb{I} \implies (\mathbb{I} - iT^\dagger)(\mathbb{I} + iT) = \mathbb{I}$$

$$\implies \boxed{-i(T - T^\dagger) = T^\dagger T} \tag{5.5}$$

- a) Take matrix element $\langle p_1 p_2 | \dots | k_1 k_2 \rangle$
- b) Insert a complete set of states $T^\dagger T \to \sum_{q_f} T^\dagger |q_f\rangle\langle q_f| T$, i.e.

$$\langle p_1 p_2 | T^\dagger T | k_1 k_2 \rangle = \sum_n \left(\prod_{f=1}^n \int \frac{d^3 q_f}{(2\pi)^3} \frac{1}{2E_f} \right) \langle p_1 p_2 | T^\dagger | q_1, \dots, q_n \rangle \langle q_1, \dots, q_n | T | k_1 k_2 \rangle \tag{5.6}$$

One obtains:

$$-i\Big[M(k_1 k_2 \to p_1 p_2) - M^*(p_1 p_2 \to k_1 k_2)\Big](2\pi)^4 \delta^{(4)}(k_1 + k_2 - p_1 - p_2) =$$

$$= \sum_n \left(\prod_{f=1}^n \int \frac{d^3 q_f}{(2\pi)^3} \frac{1}{2E_f} \right) M^*(p_1 p_2 \to q_f) M(k_1 k_2 \to q_f)$$

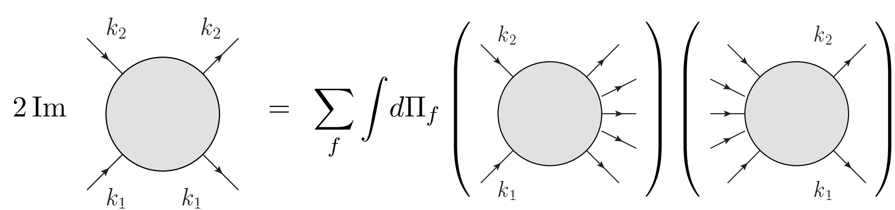
$$\times (2\pi)^4 \delta^{(4)}(k_1 + k_2 - \sum_f q_f)(2\pi)^4 \delta^{(4)}(k_1 + k_2 - p_1 - p_2) \tag{5.7}$$

where the energy-conservation δ-functions can be cancelled on both sides.

An important special case is $(k_1, k_2) = (p_1, p_2)$ (forward scattering)

$$\boxed{2\,\mathrm{Im}\, M(k_1 k_2 \to k_1 k_2) = \sum_n \left(\prod_{f=1}^n \int \frac{d^3 q_f}{(2\pi)^3} \frac{1}{2E_f} \right) |M(k_1 k_2 \to q_f)|^2 (2\pi)^4 \delta^{(4)}(k_1 + k_2 - \sum_f q_f) \ = \ 4\sqrt{s}|p_{cm}|\sigma_{tot}} \tag{5.8}$$

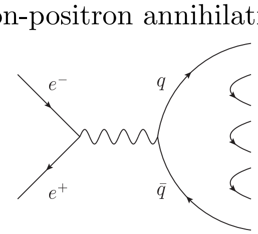
— <u>The optical theorem</u>



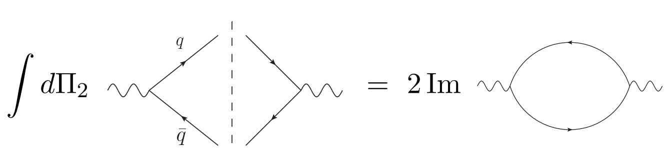
(5.9)

- this is an exact relation
- valid in each order of perturbation theory separately

<u>Example:</u> hadron production in electron-positron annihilation



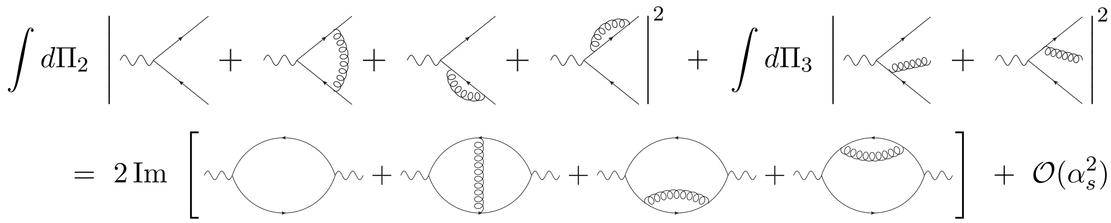
On the quark-gluon level, to leading order:



(5.10)

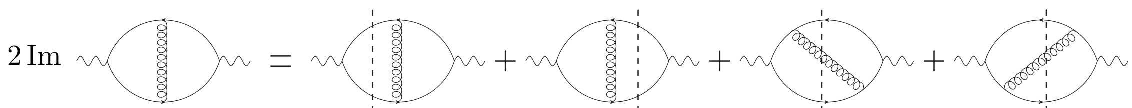
This explains why we always had $\ln(-k^2 - i\epsilon)$: Imaginary part can only occur for $k^2 > 0$ in which case real decay processes are possible

To the first order in α_s we can have quark-antiquark or quark-antiquark-gluon final states:



(5.11)

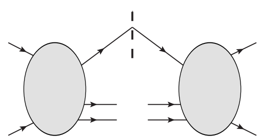
Very importantly, these rules are valid on a diagram-per-diagram level:



(5.12)

Cut diagrams

Consider phase space integration over one particular fermion line:



$$\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\sum_s M^*(p,\ldots)u(p,s)\bar{u}(p,s)M(p,\ldots)$$

$$= \int \frac{d^4p}{(2\pi)^3}\delta(p^2-m^2)\theta(p_0)\, M^*(p,\ldots)(\not{p}+m)M(p,\ldots) \tag{5.13}$$

Note that

$$\frac{1}{p^2-m^2+i\epsilon} = \mathrm{PV}\frac{1}{p^2-m^2} - i\pi\delta(p^2-m^2) \tag{5.14}$$

so that this can be written as

$$\ldots = \int \frac{d^4p}{(2\pi)^4}M^*(p,\ldots)\Big[-2\mathrm{Im}\,\frac{(\not{p}+m)}{p^2-m^2+i\epsilon}\Big]\theta(p_0)M(p,\ldots) \tag{5.15}$$

Similarly for a gluon (photon) line:

$$\int \frac{d^3k}{(2\pi)^3}\frac{1}{2|k|}\sum_\lambda M^{\nu,*}(k,\ldots)\epsilon^{(\lambda)}_\nu(k)\epsilon^{*(\lambda)}_\mu(k)M^\mu(k,\ldots)$$

$$= \int \frac{d^4k}{(2\pi)^3}\delta(k^2)\theta(k_0)M^{\nu,*}(k,\ldots)(-g_{\mu\nu})M^\mu(k,\ldots)$$

$$= \int \frac{d^4k}{(2\pi)^4}M^{\nu*}(k,\ldots)\Big[-2\mathrm{Im}\,\frac{-g_{\mu\nu}}{k^2+i\epsilon}\Big]\theta(p_0)M^\mu(k,\ldots) \tag{5.16}$$

— Thus, expressions for $\int d\Pi_n\, M^*\, M$ can be written similar to usual Feynman diagrams, adding new rules for "cut" propagators — the "cut diagrams"

— Thanks to the optical theorem, the sum over all cuts gives the total imaginary part for the particular Feynman diagram contribution to the forward scattering amplitude.

— This statement is actually more general and applies to arbitrary Feynman diagram. The imaginary part of a Feynman diagram can be calculated using the following algorithm:

- Cut through the diagram in all possible ways such that cut propagators can simultaneously be put on shell
- Use $k^2 + i\epsilon$ prescription for all propagators to the left of the cut and $k^2 - i\epsilon$ prescription to the right of the cut
- Use usual vertices to the left and complex conjugated to the right of the cut
- For all cut propagators replace

$$\frac{1}{p^2 - m^2 + i\epsilon} \Longrightarrow -2\pi i \delta(p^2 - m^2) \tag{5.17}$$

  and add positive energy conditions $\theta(p_0)$ for all lines

- Sum the contributions of all cuts

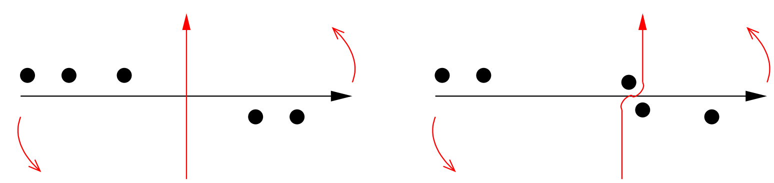
These rules are often referred to as Cutkosky cutting rules (which they are not)

5.2 * * * *Singularities of Feynman diagrams* * * *

It is easy to convince oneself that various "i" factors in propagators and vertices combine in such a way that the Feynman diagram is real unless the integrations become singular because the denominators vanish.

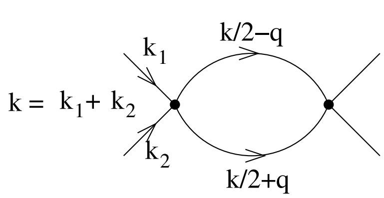
• Vanishing of one denominator is not enough because the integration contour can be moved away

• Imaginary parts arise when two poles come together and trap the integration countour so it cannot be moved



• This phenomenon is called "pinching"

Example: Scalar field theory with quartic interaction $\lambda\phi^4$



$$iM = \frac{\lambda^2}{2}\int \frac{d^4q}{(2\pi)^4}\frac{1}{[(k/2-q)^2 - m^2 + i\epsilon][(k/2+q)^2 - m^2 + i\epsilon]} \tag{5.18}$$

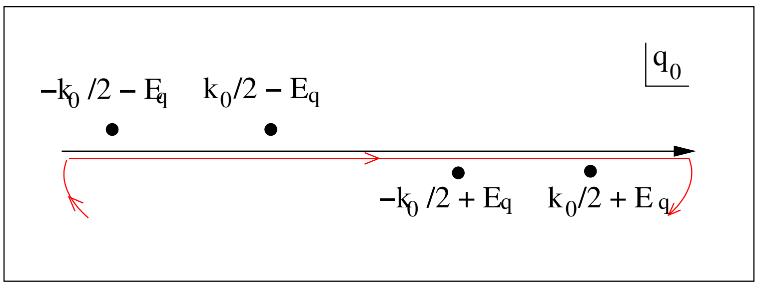
— 1/2 is a symmetry factor for the diagram
— $\int d^4q/[(2\pi)^4 i]$ is real and positive after rotation to Euclidian space; thus M is naturally a real function

Let $k^2 > 0$, choose

$$k_\mu = (k_1 + k_2)_\mu = (k_0, 0, 0, 0) \tag{5.19}$$

and consider integration over q_0. The integrand has four poles located at

$$q_0 = \frac{1}{2}k_0 \pm (E_q - i\epsilon)\,, \qquad q_0 = -\frac{1}{2}k_0 \pm (E_q - i\epsilon)\,, \qquad E_q = \sqrt{\vec{q}^2 + m^2} \tag{5.20}$$



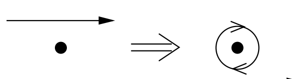
We choose to take the integral by closing the contour in the lower half-plane
— Thus have to sum the contributions of two poles at $q_0 = -\frac{1}{2}k_0 + E_q - i\epsilon$ and $q_0 = +\frac{1}{2}k_0 + E_q - i\epsilon$
— Only $q_0 = -\frac{1}{2}k_0 + E_q - i\epsilon$ can contribute to the imaginary part, neglect the second

Why:
Since $E_q > 0$, the only pair of poles that can pinch is

$$\frac{1}{2}k_0 - E_q + i\epsilon \;\leftrightarrow\; -\frac{1}{2}k_0 + E_q - i\epsilon\,, \qquad \Longrightarrow \text{ pinch at } E_q = 2k_0 \tag{5.21}$$

the rest always stay apart.

Picking the contribution of this pole corresponds to the following replacement in the integral:



$$\frac{1}{[(k/2+q)^2 - m^2 + i\epsilon]} \Longrightarrow -2\pi i \delta_+((k/2+q)^2 - m^2) \tag{5.22}$$

This gives:

$$\begin{aligned} iM &= -2\pi i \frac{\lambda^2}{2} \int \frac{d^3\vec{q}}{(2\pi)^4} \frac{1}{2E_q} \frac{1}{[(k^0 - E_q)^2 - E_q^2 + i\epsilon]} \\ &= -2\pi i \frac{\lambda^2}{2} \frac{4\pi}{(2\pi)^4} \int_m^\infty dE_q E_q |\vec{q}| \frac{1}{2E_q} \frac{1}{k_0[k_0 - 2E_q + i\epsilon]} \end{aligned} \tag{5.23}$$

The imaginary part appears because of the singularity at $k_0 = 2E_q$ which is exactly the pinching condition

$$\frac{1}{[k_0 - 2E_q + i\epsilon]} = \text{PV}\frac{1}{[k_0 - 2E_q]} - i\pi\delta(k_0 - 2E_q) \tag{5.24}$$

Taking ino account this delta-function contribution only, obtain

$$M \simeq \frac{1}{i}(-2\pi i)\frac{\lambda^2}{2}\frac{4\pi}{(2\pi)^4}\int_m^\infty dE_q\, E_q\,\sqrt{E_q^2-m^2}\,\frac{1}{2E_q}\frac{1}{k_0}(-i\pi)\delta(k_0-2E_q)$$

$$= \frac{i\lambda^2}{32\pi}\frac{1}{k_0}\sqrt{k_0^2-4m^2}\theta(k_0^2-4m^2) \tag{5.25}$$

or

$$\boxed{\mathrm{Im}\, M = \frac{\lambda^2}{32\pi}\sqrt{1-\frac{4m^2}{k^2}}\,\theta(k^2-4m^2)} \tag{5.26}$$

• note that the imaginary part must be positive (related to total cross section).
• This is consistent with $+i\epsilon$ prescription in propagators (and can be used to derive it)

Note also that picking up the pole is equivalent to replacing the original propagator by a delta-function

$$\frac{1}{[(k/2-q)^2-m^2+i\epsilon]} \Longrightarrow -2\pi i\delta_+((k/2-q)^2-m^2) \tag{5.27}$$

Going back to the original representation we can relabel the momenta in the loop as p_1 and p_2 and rewrite the momentum integration as

$$\int\frac{d^4q}{(2\pi)^4} = \int\frac{d^4p_1}{(2\pi)^4}\int\frac{d^4p_2}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p_1+p_2-k) \tag{5.28}$$

Then we can summarize our findings as

$$2i\,\mathrm{Im}M(k) = \frac{\lambda^2}{2}\int\frac{d^4p_1}{(2\pi)^4}\int\frac{d^4p_2}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p_1+p_2-k)(-2\pi i)^2\delta_+(p_1^2-m^2)\delta_+(p_2^2-m^2)$$

$$= \frac{i}{2}\int\frac{d^3p_1}{(2\pi)^3}\frac{1}{2E_1}\int\frac{d^3p_2}{(2\pi)^3}\frac{1}{2E_2}|M(k)^2|(2\pi)^4\delta^{(4)}(p_1+p_2-k) \tag{5.29}$$

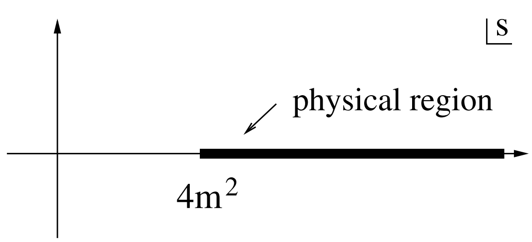
where to our accuracy on the r.h.s. $M(k)=\lambda$; factor 1/2 takes into account identity of particles in the final state.

— the first line above gives a representation as a cut diagram
— the second line is an optical theorem

Last but not least:
Scattering amplitudes can be viewed as analytic functions of invariant energies. In our case

$$M(s=k^2=(k_1+k_2)^2) = M(s+i\epsilon) \tag{5.30}$$

has a cut in the complex s-plane:



Two-point functions do not have other singularities apart from unitarity cuts (can be proven).

Landau has given a general classification for singularity structure of arbitrary Feynman diagrams, hence arbitrary three-point, four-point etc. functions. [important topic, but too much for these lectures] L14

6 Electron-Positron annihilation

6.1 Total cross section: Leading-order analysis

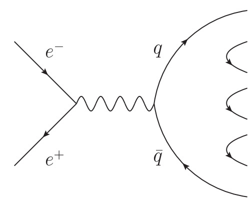
The best environment to study strong interactions: Electron-positron collisions at high energies

$$e^+(k_+) + e^-(k_-) \to \text{ hadrons} \tag{6.1}$$

pro: initial state exactly known, small backgrounds etc.
contra: electrons are more difficult to accelerate as protons because of radiation losses

The simplest quantity:

$$R(s) \;=\; \frac{\sigma_{tot}(e^+e^- \to \text{ hadrons})}{\sigma_{tot}(e^+e^- \to \mu + \mu^-)} \tag{6.2}$$



where

$$\sigma_{tot}(e^+e^- \to \mu + \mu^-) = \frac{4\pi\alpha^2}{3s}\,, \qquad \alpha = 1/137 \tag{6.3}$$

is used for normalization

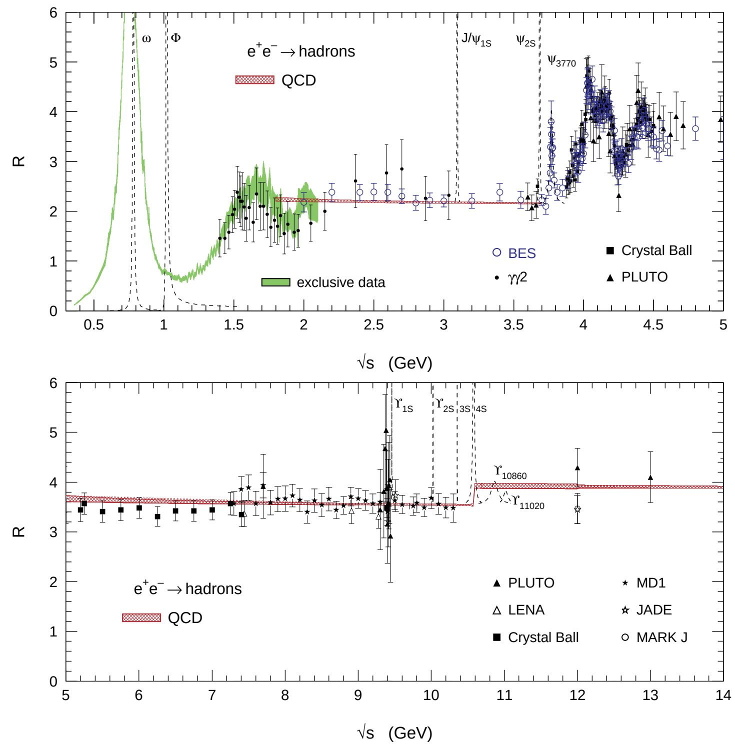
<u>Heuristic discussion:</u>

- The process involves two time scales (in CM system $q = k_+ + k_- = \{q_0, 0, 0, 0\}$, $s = q^2$):
  — quark-antiquark pair is produced at times $t \sim 1/q_0$ (uncertainty principle)
  — hadrons are produced when interaction becomes strong $T \sim 1/\Lambda_{\rm QCD} \sim 1$ fm
- If energy is large, $T \gg t$, these two (sub)processes cannot have any quantum interference; Probability to produce a given hadron state is a product of probabilities to produce a $q\bar{q}$ pair times the probability to produce a particular hadronic state
- For a total cross section it suffices to know that when q and $\bar{q}$ fly apart, *some* hadronic state will be produced with probability one
- Thus expect

$$\sigma_{tot}(e^+e^- \to \text{ hadrons}) \simeq \sigma_{tot}(e^+e^- \to q\bar{q})$$

or

$$R(s) \;\simeq\; N_c\Big[\underbrace{e_u^2 + e_d^2 + e_s^2 + \ldots}_{\text{quarks with mass } < \sqrt{s}/2}\Big]\,, \qquad e_u = 2/3\,, \quad e_d = -1/3\,, \text{ etc.} \tag{6.4}$$



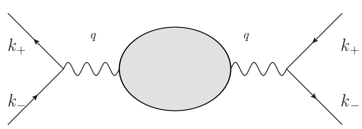
[Figure taken from: *Davier et al., Eur.Phys.J. C27:497-521,2003*]

- A strong argument for $N_c = 3$

We can make the two-time-scales argument more precise using optical theorem:

$$\sigma_{tot}(e^+e^- \to \text{ hadrons}) \;=\; \frac{1}{s}\,\mathrm{Im}\,M(e^+e^- \to e^+e^-) \tag{6.5}$$

[exact expression contains $1/(2\sqrt{s}p_{cm})$, reduces to $1/s$ for large energies]

$$M \;=$$  $$=\; e^4\bar{u}(k_-)\gamma^\mu v(k_+)\,\frac{1}{s}\,\Pi_{\mu\nu}(q)\,\frac{1}{s}\,\bar{v}(k_+)\gamma^\nu u(k_-) \tag{6.6}$$

where

$$\Pi_{\mu\nu}(q) = i\int d^4x\, e^{iqx}\langle\Omega|T\{j_\mu(x)j_\nu(0)\}|\Omega\rangle = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi(q^2)\,, \qquad s = q^2 \tag{6.7}$$

and

$$j_\mu = \sum_q e_q\bar{\psi}_q\gamma_\mu\psi_q\,, \qquad e_u = \frac{2}{3}\,, e_d = -\frac{1}{3}\,, \ldots \qquad e = \sqrt{4\pi\alpha} \tag{6.8}$$

Note that both momenta *and* spins of the e^+ and e^- must coincide in initial and final state.

Neglecting the electron mass and averaging over spin directions obtain

$$\frac{1}{2}\cdot\frac{1}{2}\sum_{s,s'}\bar{u}(k_-,s)\gamma^\mu v(k_+,s')\,\bar{v}(k_+,s')\gamma^\nu u(k_-,s)\big[q_\mu q_\nu - s g_{\mu\nu}\big]\Pi(s)$$

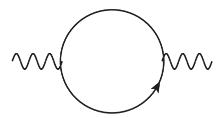
$$= \;-\frac{1}{4}s\Pi(s)\,\mathrm{Tr}\big[\not{k}_-\gamma^\mu\not{k}_+\gamma_\mu\big] = \frac{1}{2}s\Pi(s)\,4(k_-\cdot k_+) = s^2\Pi(s) \tag{6.9}$$

where I used $s = (k_- + k_+)^2 \simeq 2k_- \cdot k_+$.
Obtain therefore

$$\sigma_{tot}(e^+e^- \to \text{hadrons}) = \frac{1}{s}\, s^2 \,\text{Im}\,\Pi(s) \frac{e^4}{s^2} = \frac{1}{s}(4\pi\alpha)^2 \text{Im}\,\Pi(s) \tag{6.10}$$

and

$$\boxed{R(s) = 12\pi \,\text{Im}\,\Pi(s)} \tag{6.11}$$

• Let us check this relation in leading order 

If quark masses can be neglected, calculations often become easier in coordinate space.
I choose the present example to illustrate this technique

Massless quark propagator

$$\langle 0|T\{\psi^i(x)\overline{\psi}^j(0)\}|0\rangle = \delta^{ij} \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \frac{i\not{p}}{p^2 + i\epsilon} = \frac{i}{2\pi^2} \frac{\not{x}}{(-x^2 + i\epsilon)^2} \tag{6.12}$$

Then, ignoring electric charges

$$\begin{aligned} \Pi_{\mu\nu}(q) &= i \int d^4x \, e^{iqx} \langle 0|T\{\overline{\psi}(x)\gamma_\mu \psi(x)\, \overline{\psi}(0)\gamma_\nu \psi(0)\}|0\rangle \\ &= iN_c \left(\frac{i}{2\pi^2}\right)^2 \int d^4x \, e^{iqx} \,\text{Tr}\left[\frac{\not{x}}{x^4}\gamma_\mu \frac{\not{x}}{x^4}\gamma_\nu\right] \\ &= \frac{-iN_c}{4\pi^4} \int d^4x \frac{e^{iqx}}{x^8} \cdot 4\big[2x_\mu x_\nu - g_{\mu\nu}x^2\big] \end{aligned} \tag{6.13}$$

Fourier integral:

$$\int d^4x \, e^{ipx} \frac{x_\mu x_\nu}{(-x^2 + i\epsilon)^4} = \frac{i\pi^2}{48} \ln\left(\frac{\mu^2}{-p^2 - i\epsilon}\right) \left[2p_\mu p_\nu + p^2 g_{\mu\nu}\right] \tag{6.14}$$

Thus

$$\begin{aligned} \Pi_{\mu\nu}(q) &= \frac{-iN_c}{4\pi^4} \frac{i\pi^2}{48} \ln \frac{\mu^2}{-q^2}\, 4\left[2(2q_\mu q_\nu + q^2 g_{\mu\nu}) - 6g_{\mu\nu}q^2\right] \\ &= \frac{N_c}{12\pi^2} \ln \frac{\mu^2}{-q^2} \left[q_\mu q_\nu - q^2 g_{\mu\nu}\right] \end{aligned} \tag{6.15}$$

Finally

$$\text{Im}\ln \frac{\mu^2}{-q^2 - i\epsilon} = \pi\,\theta(q^2) \qquad \Longrightarrow \qquad \text{Im}\,\Pi(s) = \frac{N_c}{12\pi}\theta(s) \tag{6.16}$$

thus, up to quark electric charges

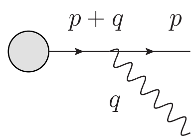
$$R(s) = N_c\Big[1 + \mathcal{O}(\alpha_s)\Big] \qquad \heartsuit \tag{6.17}$$

Corrections are due to radiation of "hard" gluons at small times $\sim 1/q_0$, so it should be possible to calculate them in perturbation theory

- the possibility to use perturbation theory at $q^2 \to +\infty$ is not obvious; we will develop a more rigorous approach later

6.2 Gluon Bremsstrahlung and jets

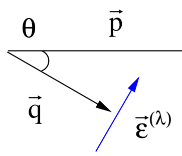
6.2.1 Soft and Collinear emission



$$\epsilon_\mu^{*(\lambda)} M^\mu \;\sim\; \bar{u}(p)\epsilon_\mu^{*(\lambda)}\gamma^\mu \frac{\not{p} + \not{q} + m}{(p+q)^2 - m^2} \tag{6.18}$$

Assume that a gluon with small energy is emitted at small angle:

$$\begin{aligned} p_\mu &= \{p_0, 0, 0, p\} \qquad p_0 = E \gg m \qquad p_\mu p^\mu = m^2 \\ q_\mu &= \{q_0, q\sin\theta, 0, q\cos\theta\} \qquad q_0 = \omega \ll E \qquad q_\mu q^\mu = \lambda^2 \quad \text{"gluon mass"} \end{aligned} \tag{6.19}$$



$\theta << 1$

$$\cos\theta \simeq 1 - \frac{\theta^2}{2}$$

$$|p| \simeq E\Big(1 - \frac{m^2}{2E^2}\Big)$$

$$|q| \simeq \omega\Big(1 - \frac{\lambda^2}{2\omega^2}\Big)$$

To this accuracy

$$\begin{aligned} 2pq &= 2E\omega - 2|p||q|\cos\theta \;=\; E\omega\left(\theta^2 + \frac{m^2}{E^2} + \frac{\lambda^2}{\omega^2}\right) \\ \epsilon^*\cdot p &= E\,\sin\theta \;=\; E\,\theta \end{aligned} \tag{6.20}$$

and therefore L15

$$\epsilon_\mu^{*(\lambda)} M^\mu \sim \frac{\epsilon^*\cdot p}{2pq + \lambda^2} \;\sim\; \frac{1}{\omega}\,\frac{\theta}{\theta^2 + \frac{m^2}{E^2} + \frac{\lambda^2}{\omega^2} + \cancel{\frac{\lambda^2}{E\omega}}} \tag{6.21}$$

The corresponding cross section is

$$d\sigma \sim |\epsilon^* M|^2 \frac{d^3q}{2q_0} \;\sim\; |\epsilon^* M|^2 \omega d\omega \theta d\theta \;\sim\; \frac{d\omega}{\omega}\cdot\frac{\theta^2 d\theta^2}{\left[\theta^2 + \frac{m^2}{E^2} + \frac{\lambda^2}{\omega^2}\right]^2} \tag{6.22}$$

One can consider two cases:
1) $\frac{\lambda}{\omega} \ll \frac{m}{E}$:

$$\sigma \sim \ln\frac{E}{\lambda}\ln\frac{E}{m} \tag{6.23}$$

2) $\frac{\lambda}{\omega} \gg \frac{m}{E}$:

$$\sigma \sim \int_{\lambda}^{E} \frac{d\omega}{\omega} \ln\frac{\omega}{\lambda} \;\sim\; \ln^2\frac{E}{\lambda} \tag{6.24}$$

The general pattern (common for QED and QCD)

$$\boxed{d\sigma \;\sim\; \frac{d\omega}{\omega}\frac{d\theta}{\theta}} \tag{6.25}$$

— soft and collinear emission

What is the meaning of the divergence at $\omega \to 0$ (called IR-divergence)?

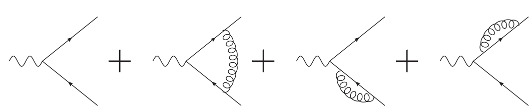
- Wrong question: — What is the total *number* of gluons emitted at a certain angle?
- Correct questions: — What is the total *energy* of gluons emitted at a certain angle?

  — What is the total *number* of gluons with energy $\omega > \omega_0$ (e.g. experimental resolution) emitted at a certain angle?

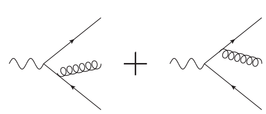
6.2.2 Total cross section to NLO

Continue our discussion of the e^+e^- annihilation

Our LO result can/will be modified by gluon emission at short distances $\sim 1/q_0$. To $\mathcal{O}(\alpha_s)$ accuracy there are two effects (cf. Section 4.1):
1) interaction between the outgoing quark and antiquark



2) gluon production



Explicit calculation gives (for $m_q = 0$), $Q = \sqrt{s}$:

$$\begin{aligned} \sigma_{e^+e^-\to\bar{q}q} &= \sigma_0 + \sigma_0 \frac{4}{3}\frac{\alpha_s}{\pi}\left[-2\ln^2\frac{Q}{\lambda} + 3\ln\frac{Q}{\lambda} - \frac{7}{4} + \frac{\pi^2}{6}\right] \\ \sigma_{e^+e^-\to\bar{q}qg} &= \sigma_0 \frac{4}{3}\frac{\alpha_s}{\pi}\left[2\ln^2\frac{Q}{\lambda} - 3\ln\frac{Q}{\lambda} + \frac{5}{2} - \frac{\pi^2}{6}\right] \end{aligned} \tag{6.26}$$

Both expressions do not makes sense:
— what is λ?
— gluons do not exist as free particles

. . . but the total cross section is well defined:

$$\sigma_{tot} = \sigma_{e^+e^-\to\bar{q}q} + \sigma_{e^+e^-\to\bar{q}qg} = \sigma_0 \left[1 + \frac{\alpha_s}{\pi} + \mathcal{O}\left(\frac{\alpha_s}{\pi}\right)^2\right] \tag{6.27}$$

The cancellation of IR divergencies in the sum of contributions of real and virtual emission is referred to as the "Bloch-Nordsieck cancellation"

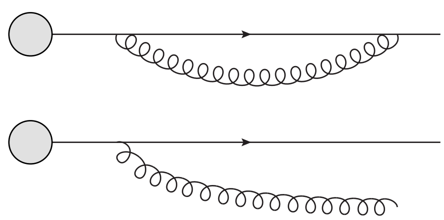
6.2.3 Sterman–Weinberg jets

— IR divergencies in QED signal that the question must be put in a more precise way
— IR divergencies in QCD signal that the quantity of interest recieves large contributions at long distances and cannot be calculated in perturbation theory at all; what to do?

1. Improve the theory. . .
2. Find a class of observables that are not sensitive to contributions of large distances and can be calculated within the theory that we have at present

Infrared-safe observables, that do not suffer from IR divergences, have a chance to be calculable.

Why did IR divergences cancel in the total cross section of e^+e^- annihilation?
— because for the total cross section it does not matter if the emitted gluon recombines with the quark at large distances or flies away:



"To be or not to be" . . .
(gluon)

Sterman and Weinberg formulated a simple criterium for such cancellations:

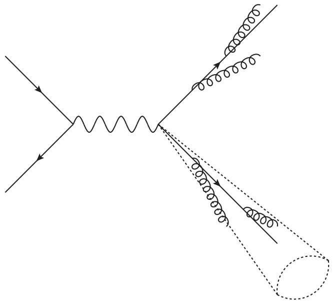
Large (potenitally divergent) contributions come from emission of soft gluons (small energy) or collinear gluons (small angles), therefore:

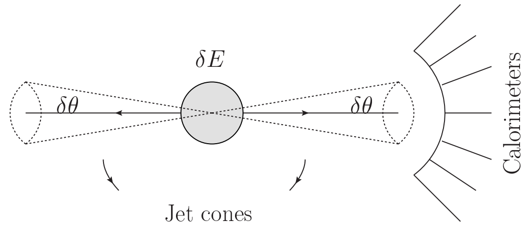
Observables that do not change if:
a) a "soft" gluon is emitted (with $\omega \to 0$) b) a "collinear" gluon is emitted (with $\theta \to 0$)
are IR safe and have a chance to be calculable

A jet: Collection of particles (spray, bundle,. . .) flying in more or less the same direction (inside a given solid angle $\delta\Omega$, called jet cone)

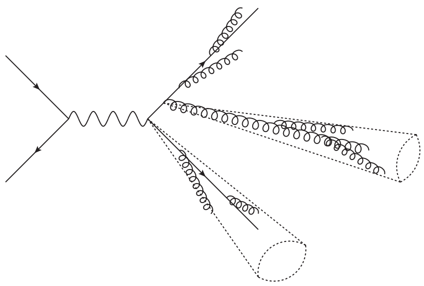
Jets are IR-stable because if a collinear particle is emitted it remains inside the jet cone, if a soft particle is emitted, the energy of the jet does not change.

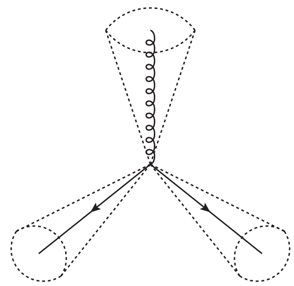
A two-jet event:
Almost all energy in the c.m. frame $q_\mu = (Q, 0, 0, 0)$ is deposited in two narrow cones





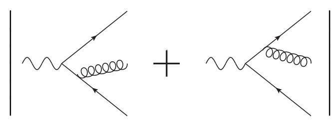
A three-jet event:
Almost all energy in the c.m. frame $q_\mu = (Q,0,0,0)$ is deposited in three narrow cones





Jet cross sections for given $\delta\theta$ and δE can be calculated in QCD:

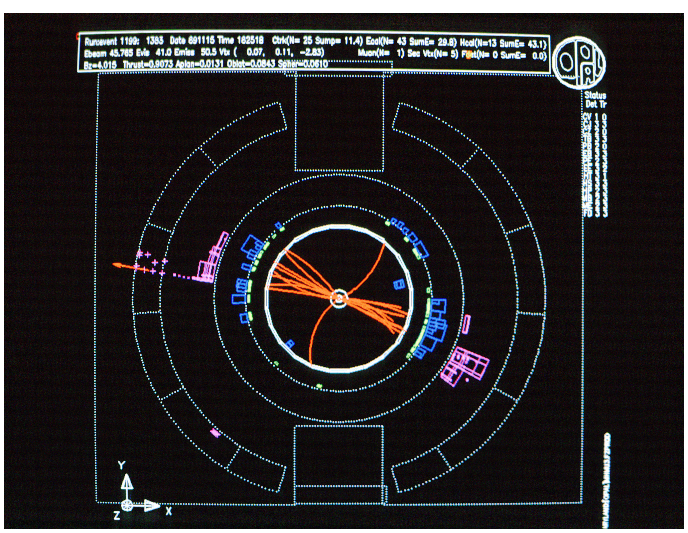
$\sigma_{e^+e^-\to 3\,\text{jets}} \sim$



$\Big|^2_{\theta>\theta_0,\ \delta E<\epsilon Q}$ (6.28)

One obtains

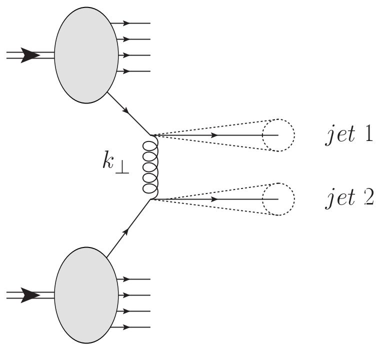
$$\begin{aligned}
\sigma_{e^+e^-\to 3\,\text{jets}} &= \sigma_0 \frac{4}{3}\frac{\alpha_s}{\pi}\left[4\ln\frac{1}{\theta_0}\,\ln\frac{1}{\epsilon} + 3\ln\frac{1}{\theta_0} - \frac{7}{4} + \frac{\pi^2}{3}\right] \\
\sigma_{e^+e^-\to 2\,\text{jets}} &= \sigma_{tot} - \sigma_{e^+e^-\to 3\,\text{jets}}
\end{aligned} \tag{6.29}$$

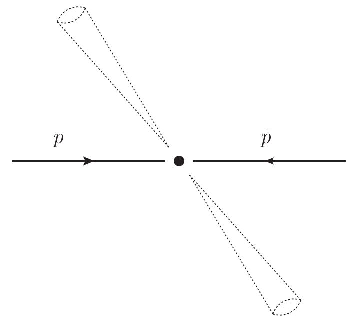


An example of real data collected from the OPAL detector on the Large Electron-Positron (LEP) collider at CERN, which ran between 1989 and 2000. Here a Z0 particle is produced in the collision between an electron and positron that then decays into a quark-antiquark pair. The quark pair is seen as a pair of hadron jets in the detector.

- Unphysical IR regulators (gluon mass) substituted by physical parameters of concrete experiment
- Observations of three-jet events (DESY, ca. 1983) gave direct evidence for existence of gluons

Similar:
Jet production in proton-antiproton collisions (Fermilab)





- Need to know quark distributions in protons — later

6.3 Unstable particles

L15

Most of the existing hadrons decay:

Weak decays	$n \to pe\bar{\nu}_e$	$\tau \sim 900s$	[life time]
	$\Lambda \to p\pi^-$	$\tau \sim 10^{-10}s$	
Strong decays	$\rho \to \pi\pi$	$\tau \sim 10^{-23}s$	(6.30)

The life time τ is defined through the decay rate [half-life $= \tau \ln 2$]

$$\tau = 1/\Gamma \tag{6.31}$$

where

$$\Gamma = \frac{\text{Number of decays per unit time}}{\text{Number of particles present}} \tag{6.32}$$

The decay rate can be calculated as (in the rest frame of the decaying particle A)

$$d\Gamma = \frac{1}{2m_A}|M(A \to \{p_f\})|^2 \left(\prod_f \frac{d^3p_f}{(2\pi)^3 2E_f}\right)(2\pi)^4\delta^{(4)}(P_A - \sum p_f) \tag{6.33}$$

For comparison:

$$d\sigma = \frac{1}{2E_A\,2E_B}\frac{1}{|v_A - v_B|}|M(A + B \to \{p_f\})|^2 \left(\prod_f \frac{d^3p_f}{(2\pi)^3 2E_f}\right)(2\pi)^4\delta^{(4)}(P_A + P_B - \sum p_f) \tag{6.34}$$

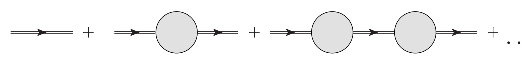
In the following I use a shorthand notation

$$d\Pi_f = \left(\prod_f \frac{d^3p_f}{(2\pi)^2 2E_f}\right)(2\pi)^4\delta^{(4)}(P_{\text{initial}} - P_{\text{final}}) \tag{6.35}$$

How can one describe unstable particles in quantum field theory?
— Consider a scalar particle as example

The propagator:

$$\frac{i}{p^2 - m_0^2 + \Sigma(p^2)} \;=$$



(6.36)

The pole (renormalized) mass:

$$m^2 - m_0^2 + \Sigma(m^2) = 0 \tag{6.37}$$

Renormalized propagator:

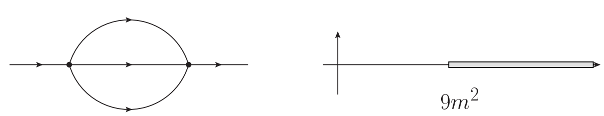
$$\begin{aligned}
\frac{i}{p^2 - m_0^2 + \Sigma(p^2)} &= \frac{i}{p^2 - m^2 + \Sigma(p^2) - \Sigma(m^2)} \\
&= \frac{i}{(p^2 - m^2)[1 + \Sigma'(m^2)] + [\Sigma(p^2) - \Sigma(m^2) - (p^2 - m^2)\Sigma'(m^2)]} \\
&= \frac{iZ}{p^2 - m^2 - \Sigma^{(r)}(p^2)}
\end{aligned} \tag{6.38}$$

Here we tacitly assumed that $\Sigma(m^2)$ is a real number. Is this always the case?

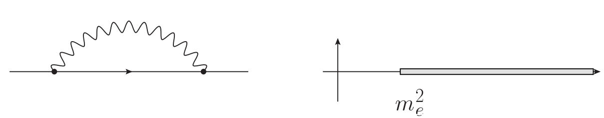
Optical theorem:

$$2\,\mathrm{Im}\,\Sigma(p^2) \;=\; \sum_f d\Pi_f\,|M(P \to \{p_f\})|^2 \tag{6.39}$$

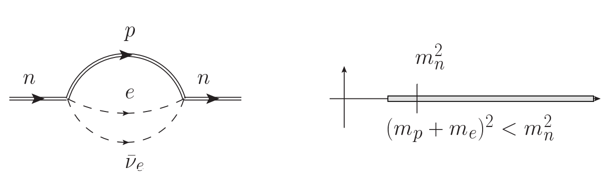
ϕ^4 − theory



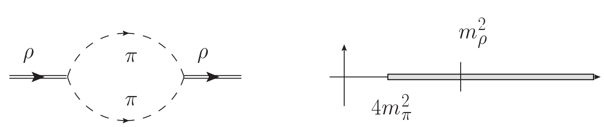
QED/QCD



β − decay



ρ − meson



(6.40)

For unstable particles

$$\Sigma(m^2) \;=\; \mathrm{Re}\,\Sigma(m^2) \;+\; i\,\mathrm{Im}\,\Sigma(m^2) \tag{6.41}$$

One defines the pole mass as

$$m^2 - m_0^2 + \mathrm{Re}\,\Sigma(m^2) \;=\; 0 \tag{6.42}$$

In this case

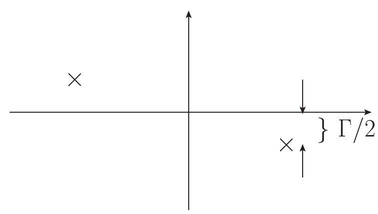
$$\frac{i}{p^2 - m_0^2 + \Sigma(p^2)} \;=\; \frac{i}{p^2 - m^2 + \,\mathrm{Re}\,\Sigma(p^2) - \,\mathrm{Re}\,\Sigma(m^2) + i\,\mathrm{Im}\,\Sigma(p^2)}$$
$$\overset{p^2\to m^2}{\simeq} \frac{iZ}{p^2 - m^2 + iZ\,\mathrm{Im}\,\Sigma(m^2)}\,, \qquad Z = 1/(1 + \,\mathrm{Re}\,\Sigma'(m)) \tag{6.43}$$

We define

$$Z\,\mathrm{Im}\,\Sigma(m^2) \;=\; m\Gamma \tag{6.44}$$

The renormalized propagator close to $p^2 = m^2$ is therefore

$$\frac{i}{p^2 - m^2 + im\Gamma}\,: \tag{6.45}$$



effectively

$$m \to m - i\Gamma/2$$

(mass acquires a *negative* imaginary part)

Note that our Γ is indeed the decay rate:

$$m\Gamma = Z\,\mathrm{Im}\,\Sigma(m^2) = \frac{1}{2}\sum_f \int d\Pi_f\,|M(P \to \{p_f\})|^2 \tag{6.46}$$

in agreement with the definition
[extra Z-factor corresponds to $\sqrt{Z}\cdot\sqrt{Z}$ for external legs (LSZ formula)]

- Why a negative imaginary part (in the mass) corresponds to a decay?

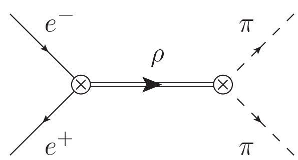
Wave function of a free particle is a plane wave

$$\phi_{\vec{p}}(\vec{x},t) \sim \frac{1}{\sqrt{2E}} e^{-ip_\mu x^\mu} \;\sim\; e^{-i(m - i\frac{1}{2}\Gamma)t} \tag{6.47}$$

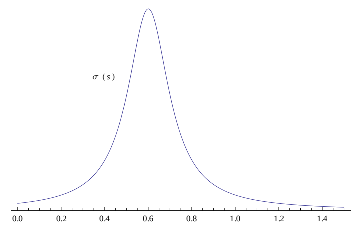
Total probability to find a particle is then

$$\int d^3\vec{x}\,\phi^*(\vec{x},t)\,i\,\overset{\leftrightarrow}{\frac{d}{dt}}\,\phi(\vec{x},t) \;\sim\; e^{-\Gamma t} \;=\; \text{decay} \tag{6.48}$$

Production cross section:



$$\sigma \;\sim\; \left|\frac{1}{s-m^2+im\Gamma}\right|^2$$



(6.49)

This shape is called a Breit-Wigner resonance

This is a good approximation if $\Gamma \ll m$; in general case $\Gamma \to \Gamma(s)$

In the review by the Particle Data Group one finds for $J^{PC} = 1^{--}$:

$$\begin{aligned}
\rho(770):&\quad m_\rho = 775\text{ MeV}, &\Gamma = 149\text{ MeV}&\quad \rho\to\pi\pi(\sim 100\%)\\
\omega(782):&\quad m_\omega = 783\text{ MeV}, &\Gamma = 8.49\text{ MeV}&\quad \omega\to\pi\pi\pi(\sim 90\%)\\
\phi(1020):&\quad m_\phi = 1019\text{ MeV}, &\Gamma = 4.26\text{ MeV}&\quad \phi\to K\,K(\sim 85\%)
\end{aligned} \tag{6.50}$$

ω does not decay in two pions because it has negative G-parity $I^G = 0^-$ compared to $I^G = 1^+$ for ρ-meson.

7 Operator Product Expansion

- **Motivation:**

Find a generalization of the Taylor series for operator products of the type

$$\mathrm{T}\{\phi(0)\phi(x)\} \stackrel{?}{=} \sum_n \frac{1}{n!} x_{\mu_1}\ldots x_{\mu_n}\phi(0)\partial_{\mu_1}\ldots\partial_{\mu_n}\phi(0) \tag{7.1}$$

or, more generally

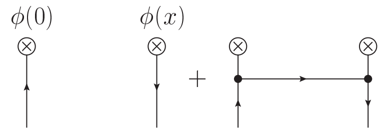
$$\mathrm{T}\{j(0)j(x)\} \stackrel{?}{=} \sum_n C^{(n)}(x)\mathcal{O}_n(0) \tag{7.2}$$

where $\mathcal{O}(0)$ are <u>local</u> operators built of fields and derivatives and $C(x)$ are some <u>functions</u>, i.e. for any states

$$\langle A|\mathrm{T}\{j(0)j(x)\}|B\rangle \stackrel{?}{=} \sum_n C^{(n)}(x)\langle A|\mathcal{O}_n(0)|B\rangle \tag{7.3}$$

This is useful if the sum can be truncated after a few terms:
— x-dependence is "universal" for all matrix elements
— can use RG methods (later)

A naive Taylor expansion does not work because the point $x = 0$ is singular:



$$\langle p|\mathrm{T}\{\phi(0)\phi(x)\}|p\rangle \sim 1 + ag^2 \ln x^2 p^2 + \dots , \qquad a \in \mathbb{R} \tag{7.4}$$

The trick is to introduce a factorization scale

$$\ln x^2 p^2 \;=\; \ln x^2 \mu_F^2 + \ln \frac{p^2}{\mu_F^2} , \qquad\qquad 1/x^2 \gg \mu_F^2 \gg p^2 \tag{7.5}$$

[Note obvious similarity with *renormalization scale*] and rewrite

$$1 + ag^2 \ln x^2 p^2 + \dots = \underbrace{\left(1 + ag^2 \ln x^2 \mu_F^2\right)}_{\text{coef. function}} \underbrace{\left(1 + ag^2 \ln p^2/\mu_F^2\right)}_{\text{operator matrix element}} + \dots \tag{7.6}$$

Thus, we envisage an expansion of the type

$$\mathrm{T}\{j(0)j(x)\} = \sum_n C^{(n)}(x, \mu_F)[\mathcal{O}_n]^{(\mu_F)}(0) \tag{7.7}$$

where the coefficient functions $C^{(n)}(x, \mu_F)$ contain all x-dependence and thus all singularities at $x \to 0$, and the operators' matrix elements have to be calculated with an UV cutoff μ_F (or other regulator)

CFs of composite operators have the generic structure

$$C(x, \mu_f) \sim 1 + \gamma g^2 \ln(-x^2 \mu_F^2) + \dots \tag{7.8}$$

If $|x| \to 0$, the logarithms eventually become large. If $g^2 \ll 1$ but $g^2 \ln(-x^2\mu_F^2) \sim \mathcal{O}(1)$ one has to account the whole series of contributions $\sim [g^2 \ln(-x^2\mu_F^2)]^k$ but can still neglect terms with $\sim g^2[g^2 \ln(-x^2\mu_F^2)]^k$.

This is the same situation as we had with the polarization operator or electron propagator in QED, so we can treat this problem using the same methods: The result must be independent on $\mu_F \Longrightarrow$ RG machinery: Callan-Symanzik equation etc.

- **Renormalized composite operators**

Composite operators are made of *renormalized* fields and derivatives, e.g.

$$\mathcal{O} = \phi_r(0)\partial_{\mu_1} \dots \partial_{\mu_n} \phi_r(0) \tag{7.9}$$

Despite the fact that the operator is written in terms of renormalized fields, its insertion in Green functions will produce additional divergences because the two fields stand at the same point. We can get rid of these divergences by introducing additional Z-factor:

$$[\mathcal{O}]_r = Z\mathcal{O} \tag{7.10}$$

Note that unrenormalized $\mathcal{O}$ is already scale-dependent because the renormalized fields are. Thus at first step we rewrite it in terms of bare fields

$$[\mathcal{O}]_r = \underbrace{Z\, Z_\phi^{-2}}_{Z_\mathcal{O}} \phi^{(0)}(0)\partial_{\mu_1}\ldots\partial_{\mu_n}\phi^{(0)}(0) \tag{7.11}$$

so that

$$\mu\frac{d}{d\mu}[\mathcal{O}]_r = \left(\mu\frac{d}{d\mu}Z_\mathcal{O}\right)\phi^{(0)}(0)\partial_{\mu_1}\ldots\partial_{\mu_n}\phi^{(0)}(0) = \left(\mu\frac{d}{d\mu}Z_\mathcal{O}\right)\frac{1}{Z_\mathcal{O}}[\mathcal{O}]_r \tag{7.12}$$

Define operator anomalous dimension

$$\frac{1}{Z_\mathcal{O}}\,\mu\frac{d}{d\mu}Z_\mathcal{O} = -\,\gamma^\mathcal{O}(\alpha) = -\gamma_0^\mathcal{O}\alpha + \ldots \tag{7.13}$$

$\Longrightarrow$ Callan-Symanzik equation

$$\boxed{\left\{\mu\frac{\partial}{\partial\mu} + \beta(\alpha)\frac{\partial}{\partial\alpha} + \gamma^O(\alpha)\right\}[\mathcal{O}]^{(\mu)} = 0} \tag{7.14}$$

The solution of this equation is, as we have seen

$$[\mathcal{O}]^{(\mu_1)} = \left(\frac{\alpha(\mu_1^2)}{\alpha(\mu_2^2)}\right)^{-\gamma_0^\mathcal{O}/\beta_0}[\mathcal{O}]^{(\mu_2)} \tag{7.15}$$

The operator product expansion can be formulated as the following

Theorem (K. Wilson)

Let $[\mathcal{O}_1]^{(\mu)}$ and $[\mathcal{O}_2]^{(\mu)}$ be renormalized operators. Then

$$[\mathcal{O}_1]^{(\mu)}(x)[\mathcal{O}_2]^{(\mu)}(0) \;=\; \sum_n C_{12}^n(x,\mu)[\mathcal{O}_n]^{(\mu)}(0) \tag{7.16}$$

where the sum goes over the complete set of renormalized operators $[\mathcal{O}_n]^{(\mu)}$ with suitable quantum numbers and $C_{12}^n(x,\mu)$ are (complex valued) functions.

Each of the operators in the OPE satisfies its own Callan-Symanzik equation:

$$\left\{\mu\frac{\partial}{\partial\mu} + \beta(\alpha)\frac{\partial}{\partial\alpha} + \gamma_{1,2}\right\}[\mathcal{O}]_{1,2}^{(\mu)} = 0$$

$$\left\{\mu\frac{\partial}{\partial\mu} + \beta(\alpha)\frac{\partial}{\partial\alpha} + \gamma_n\right\}[\mathcal{O}]_n^{(\mu)} = 0 \tag{7.17}$$

Since the μ-dependence of the two sides of the OPE must agree, the CFs have to satisfy a similar equation:

$$\left\{\mu\frac{\partial}{\partial\mu} + \beta(\alpha)\frac{\partial}{\partial\alpha} + \gamma_1 + \gamma_2 - \gamma_n\right\}C_{12}^n(x,\mu) = 0 \tag{7.18}$$

Let d_1, d_2 and d_n be the dimensions of the operators $[\mathcal{O}_{1,2,n}]^{(\mu)}$, respectively. Since dimensions in all terms in the OPE must be the same, the CFs have to be of the form

$$C_{12}^n(x,\mu) = \left(\frac{1}{|x|}\right)^{d_1+d_2-d_n} \tilde{C}(x\cdot\mu) \tag{7.19}$$

where $\tilde{C}$ is a dimensionless function that depends on the product $x\mu$ only.

The general solution (cf. electron propagator)

$$\begin{aligned} C_{12}^n(x,\mu) &= \left(\frac{1}{|x|}\right)^{d_1+d_2-d_n} \widetilde{C}_{12}^n(\alpha(1/|x|)) \exp\left\{ \int\limits_{\alpha(\mu)}^{\alpha(1/|x|)} d\alpha' \frac{1}{\beta(\alpha')}\Big[\gamma_n(\alpha')-\gamma_1(\alpha')-\gamma_2(\alpha')\Big]\right\} \\ &\simeq \left(\frac{1}{|x|}\right)^{d_1+d_2-d_n} \widetilde{C}_{12}^n\left(\frac{\alpha(1/|x|)}{\alpha(\mu)}\right)^{-(\gamma_0^n-\gamma_0^1-\gamma_0^2)/\beta_0} \end{aligned} \tag{7.20}$$

Special case: It can happen that $\beta(\alpha)=0$ or at least $\beta(\alpha^*)=0$ for a particularly chosen α^* (critical point, e.g. point of phase transition in condensed matter). In this case the solution is

$$C_{12}^n(x) = \widetilde{C}_{12}^n\left(\frac{1}{|x|}\right)^{D_1+D_2-D_n}, \qquad \begin{aligned} D_1 &= d_1+\gamma^1(\alpha^*) \\ D_2 &= d_2+\gamma^2(\alpha^*) \\ D_n &= d_n+\gamma^n(\alpha^*) \end{aligned} \tag{7.21}$$

This explains why γ^n are called anomalous dimensions.

- **Example: $[\phi^2]$ in the ϕ^4 theory**

We continue with the example from Section 4.3. The Lagrangian in terms of the renormalized field and coupling is

$$\mathcal{L} = \frac{1}{2}Z_\phi^2(\partial_\mu\phi_r)^2 - \mu^{2\epsilon}Z_\lambda\frac{\lambda_r}{24}Z_\phi^4\phi_r^4 \tag{7.22}$$

where

$$\begin{aligned} Z_\phi^2 &= 1+\frac{1}{\epsilon}\Big[-\frac{a^2}{24}+\frac{a^3}{48}\Big]+\frac{1}{\epsilon^2}\Big[-\frac{a^3}{24}\Big]+\mathcal{O}(a^4) \qquad\qquad a(\mu)=\frac{\lambda_r(\mu)}{(4\pi)^2} \\ Z_\lambda &= 1+\frac{1}{\epsilon}\Big[\frac{3a}{2}-\frac{17a^2}{12}\Big]+\frac{1}{\epsilon^2}\Big[-\frac{9a^2}{4}\Big]+\mathcal{O}(a^3) \end{aligned} \tag{7.23}$$

where from

$$\begin{aligned} \beta(a) &= \frac{da}{d\ln\mu} = -2\epsilon a+3a^2-\frac{17}{3}a^3+\mathcal{O}(a^4)\,, \\ \gamma_\phi &= \frac{1}{2}\frac{d\ln Z_\phi^2}{d\ln\mu} = \frac{1}{12}a-\frac{1}{16}a^2+\mathcal{O}(a^3) \end{aligned} \tag{7.24}$$

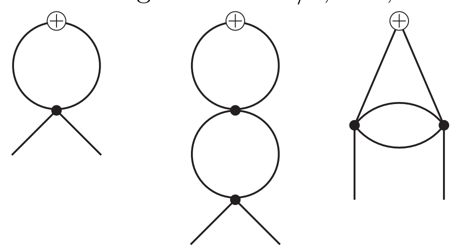
Now consider a *composite operator* built of two *renormalized* fields.

$$O(x) = \phi_r(x)\phi_r(x) \tag{7.25}$$

Its insertion in Green functions will produce additional divergences because the two fields stand at the same point. We can get rid of these divergences by introducing additional Z-factor

$$[\phi_r(0)\phi(0)_r]_r^{(\mu)} = Z_2\phi_r(0)\phi_r(0) \tag{7.26}$$

which can be calculated as a sum of divergent terms $1/\epsilon$, etc., in the sum of 1PI diagrams



Explicit calculation gives (*A. N. Vasil'ev, The field theoretic renormalization group in critical behavior theory and stochastic dynamics, Boca Raton, USA: Chapman & Hall/CRC (2004), p 277*)

$$Z_2 = 1 + \frac{1}{\epsilon}\left(\frac{1}{2}a - \frac{1}{4}a^2\right) + \frac{1}{\epsilon^2}\left(\frac{1}{2}a^2\right) + \mathcal{O}(a^3) \tag{7.27}$$

so that

$$Z_{\phi^2} = Z_2 Z_\phi^{-2} = 1 + \frac{1}{\epsilon}\left(\frac{1}{2}a - \frac{1}{4}a^2 + \frac{1}{24}a^2\right) + \frac{1}{\epsilon^2}\left(\frac{1}{2}a^2\right) + \mathcal{O}(a^3) \tag{7.28}$$

and

$$\begin{aligned}\gamma_{\phi^2} &= -\frac{d\ln Z_{\phi^2}}{d\ln\mu} = -\frac{d\ln Z_{\phi^2}}{da}\beta(a) \\ &= -\left\{\frac{1}{\epsilon}\left(\frac{1}{2} - \frac{5}{12}a\right) + \text{ higher poles } + \mathcal{O}(a^2)\right\}\left\{-2\epsilon\, a + 3a^2 - \frac{17}{3}a^3 + \mathcal{O}(a^4)\right\}\end{aligned} \tag{7.29}$$

The only way to obtain a finite contribution is from the product of the single pole terms in the Z_{ϕ^2}-factor and the $-2\epsilon a$ term in the β-function; the higher poles must cancel. Thus

$$\gamma_{\phi^2} = -(-2\epsilon a)\frac{1}{\epsilon}\left(\frac{1}{2} - \frac{5}{12}a\right) = a - \frac{5}{6}a^2 \tag{7.30}$$

- **Final remarks:**

- This scheme of calculations is the same for all theories; in QCD one of course has to substitute Z_ϕ by quark or gluon field renormalization constants for the operators built of quarks and gluons, respectively.

- If two (or more) operators have the same quantum numbers, they can *mix* with each other, meaning that counterterms to one of these operators can have contributions of all other operators. Hence the Z-factors become matrices instead of numbers.

- Operators with several open Lorentz indices should be decomposed in irreducible reps. of the Lorentz group which can be considered separately as they do not mix with each other. (Matrices of Z-factors are block-diagonal).

8 Electron-Positron annihilation II

8.1 OPE analysis and QCD sum rules

We have found that

$$R(s) \ = \ \frac{\sigma_{tot}(e^+e^- \to \text{ hadrons})}{\sigma_{tot}(e^+e^- \to \ \mu + \mu^-)} \ = \ 12\pi \, \text{Im} \, \Pi(s) \tag{8.1}$$

where

$$\Pi_{\mu\nu}(q) = i \int d^4x \, e^{iqx} \langle \Omega | T\{j_\mu(x) j_\nu(0)\} | \Omega \rangle = (q_\mu q_\nu - q^2 g_{\mu\nu}) \Pi(q^2) \, , \qquad s = q^2 \tag{8.2}$$

Our program is now:

1. Find a way to relate $\Pi(s)$ at large s to $T\{j_\mu(x) j_\nu(0)\}$ at small $|x|$
2. Study $T\{j_\mu(x) j_\nu(0)\}$ using OPE

8.1.1 Dispersion relations

Causality $\Longrightarrow$ $\Pi(q^2)$ is an analytic function of q^2 with a cut at positive real $q^2 = s > 4m_\pi^2$.

It is easy to see that the expansion of $T\{j_\mu(x) j_\nu(0)\}$ in powers of $|x|$ produces a series in $1/q$ after the Fourier transform, but convergence properties of this series can depend on the direction in the complex q^2 plane.

There are reasons to suspect that convergence becomes bad (no uniform convergence) for q^2 approaching the cut at real positive values.

Example 1: (extreme)
Assume $\Pi(q^2)$ has a contribution

$$\Pi(q^2) = \ldots + e^{q^2} \tag{8.3}$$

For $q^2 < 0$ it is exponentially suppressed, will not be seen in OPE in any finite order, but it explodes at $q^2 > 0$

Example 2: (more realistic)
Assume $\Pi(q^2)$ has a contribution

$$\Pi(q^2) = \ldots + K_0(\sqrt{-q^2 - i\epsilon})$$

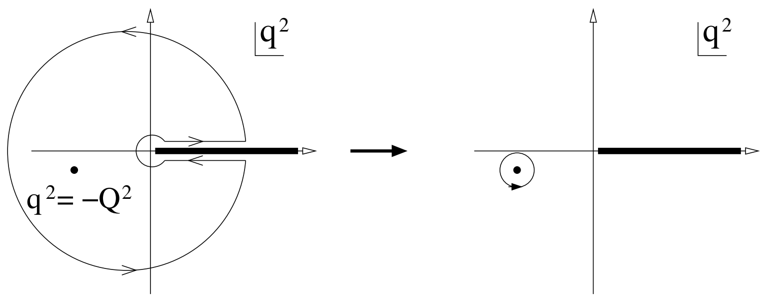
$$q^2 \to -\infty : \qquad K_0(\sqrt{-q^2}) \simeq \text{const.} \times \left(\frac{1}{-q^2}\right)^{-1/4} e^{-\sqrt{-q^2}}$$

$$q^2 \to +\infty : \qquad K_0(\sqrt{-q^2}) \simeq \text{const.} \times \left(\frac{1}{q^2}\right)^{-1/4} e^{i\sqrt{q^2} - i\pi/4} \tag{8.4}$$

For $q^2 < 0$ it is exponentially suppressed, will not be seen in OPE in any finite order, but produces an oscillating correction $\sim 1/\sqrt{q^2}$ at $q^2 > 0$

Thus, it is believed that the OPE has to work at $q^2 \to -\infty$ (called Euclidian region) but there might be subtleties for $q^2 \to +\infty$ which so far nobody was able to quantify.

The trick is to connect these two regions using dispersion relations L18



(8.5)

If the integral over the large circle can be neglected

$$\Pi(q^2) = \frac{1}{2\pi i}\oint ds \frac{\Pi(s)}{s-q^2} = \frac{1}{2\pi i}\int_0^\infty \frac{ds}{s-q^2} \underbrace{\Big[\Pi(s+i\epsilon) - \Pi(s-i\epsilon)\Big]}_{2i\,\mathrm{Im}\,\Pi(s)} \tag{8.6}$$

so that

$$\Pi(q^2) = \frac{1}{\pi}\int_0^\infty \frac{ds}{s-q^2}\,\mathrm{Im}\,\Pi(s) = \frac{1}{12\pi^2}\int_0^\infty \frac{ds}{s-q^2}\,R(s) \tag{8.7}$$

— a dispersion relation.

Unfortunately it does not work that way, because for large q^2 we derived $\Pi(q^2) \sim \ln(q^2)$. A simple estimate shows that in this case the large circle contribution does not vanish.

A possible way out: do the same for the derivative:

$$\frac{d}{dq^2}\Pi(q^2) = \frac{1}{12\pi^2}\int_0^\infty \frac{ds}{(s-q^2)^2}\,R(s) \tag{8.8}$$

or write a dispersion relation with a subtraction:

$$\Pi(q^2) = \Pi(0) + q^2\frac{1}{12\pi^2}\int_0^\infty \frac{ds}{s-q^2}\frac{1}{s}\,R(s) \tag{8.9}$$

How to use this:

- Choose large $Q^2 = -q^2$, much larger than hadron masses
- Calculate the r.h.s. inserting exp. data and integrating numerically
- Calculate the l.h.s. in QCD using PT and/or OPE
- Compare and draw conclusions

8.1.2 Operator product expansion

We want to write

$$\langle\Omega|T\{j_\mu(x)j_\nu(0)\}|\Omega\rangle = \sum_n C^n_{\mu\nu}\langle\Omega|O_n|\Omega\rangle \tag{8.10}$$

Which operators can contribute? $\Longleftarrow$ Gauge and Lorentz invariance

Only gauge-invariant scalar operators can have nonzero vacuum expectation value (VEV):

1)	Unity Operator	$\mathbb{I}$	$d_\mathbb{I}=0\,,\quad d_j=3$	$\Rightarrow\ C_\mathbb{I}\sim\frac{1}{x^6}$
2)	Quark Condensate	$\langle\Omega\vert\bar\psi\psi\vert\Omega\rangle$	$d_{\bar\psi\psi}=3\,,\quad d_j=3$	$\Rightarrow\ C_{\bar\psi\psi}\sim\frac{1}{x^3}$
3)	Gluon Condensate	$\langle\Omega\vert G^A_{\mu\nu}G^{\mu\nu;A}\vert\Omega\rangle$	$d_{GG}=4\,,\quad d_j=3$	$\Rightarrow\ C_{GG}\sim\frac{1}{x^2}$
4)	Mixed Condensate	$\langle\Omega\vert\bar\psi g\sigma_{\mu\nu}G^{\mu\nu}\psi\vert\Omega\rangle$	$d_{\bar\psi G\psi}=5$	$\Rightarrow\ C_{\bar\psi G\psi}\sim\frac{1}{x^1}$
5)		$\langle\Omega\vert(\bar\psi\Gamma\psi)^2\vert\Omega\rangle$	$d_{(\bar\psi\psi)^2}=6$	$\Rightarrow\ C_{(\bar\psi\psi)^2}\sim\ln x$

etc. (8.11)

After Fourier trafo

$$\int d^4x\, e^{iqx}\, C_\mathbb{I}(x)\ \sim q^2 \qquad = (q_\mu q_\nu - q^2 g_{\mu\nu})\cdot\mathcal{O}(q^0) \tag{8.12}$$

so that the corresponding contribution to $\Pi(q^2)$ is $\mathcal{O}(q^0)$ [In reality it is $\ln q^2/\mu^2$]

Similar

$$\int d^4x\, e^{iqx}\, C_{GG}(x) \sim \frac{1}{q^2}\,, \qquad \text{etc.} \tag{8.13}$$

Therefore OPE takes the form

$$\Pi(q^2) = \underbrace{\widetilde{C}_\mathbb{I}}_{\searrow\ \text{pert. theory}} + \frac{m_\psi}{q^4}\widetilde{C}_{\bar\psi\psi}\langle\bar\psi\psi\rangle + \frac{1}{q^4}\widetilde{C}_{GG}\langle G^2\rangle + \ldots \tag{8.14}$$

where $\widetilde{C}_i$ are dimensionless functions (may contain $\ln q^2/\mu^2$)

Explicit calculation (Shifman, Vainstein, Zakharov 1979)

$$\begin{aligned}\Pi(q^2) =\ & \frac{1}{4\pi^2}\ln\frac{\mu^2}{-q^2}\Big[1+\frac{\alpha_s(-q^2)}{\pi}+\ldots\Big] + \frac{m_u+m_d}{6q^4}\langle\bar\psi\psi\rangle \\ & + \frac{1}{12q^4}\Big\langle\frac{\alpha_s}{\pi}G^2\Big\rangle + \frac{224}{81q^6}\alpha_s\cdot\pi\langle\bar\psi\psi\rangle^2 + \ldots\end{aligned} \tag{8.15}$$

- in NLO, all coefficient functions receive corrections in $\alpha_s(|q^2|)$
- Perturbation theory is correct to $\mathcal{O}(1/q^4)$ accuracy: There exist no gauge-invariant operators

with dimension two.

- Leading corrections beyond perturbation theory involve two parameters

$$\langle\bar{\psi}\psi\rangle \simeq -(250\,\mathrm{MeV})^3$$
$$\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle \simeq 0.012 - 0.020\,\mathrm{GeV}^4 \quad (8.16)$$

- Interpretation: (background field)

Separate formally all field operators in "fast" and "slow" components:

$$\phi(x) = \left(\int\limits_{|k|>\mu}\frac{d^3k}{(2\pi)^3 2E_k} + \int\limits_{|k|<\mu}\frac{d^3k}{(2\pi)^3 2E_k}\right)\left[\hat{a}^\dagger(k)e^{ikx} + \hat{a}(k)e^{-ikx}\right] = \phi_{\text{fast}}(x) + \phi_{\text{slow}}(x) \quad (8.17)$$

so for quarks and gluons

$$\psi(x) = \psi_{\text{fast}}(x) + \psi_{\text{slow}}(x)\,, \qquad A^\mu(x) = A^\mu_{\text{fast}}(x) + A^\mu_{\text{slow}}(x) \quad (8.18)$$

Fast and slow components can be viewed as independent fields in the Lagrangian. One can, for example, consider "slow" fields as given external fields (think of a magnet in a lab) and develop a Feynman diagram technique for calculation of fast field propagation in the background of given slow fields (think of an electron propagating and emitting photons inside a magnet)

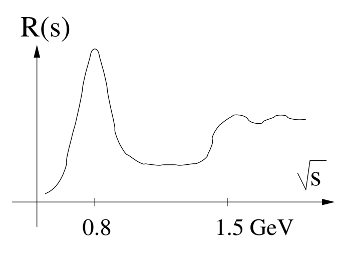
By this reason, fast and slow fields are often referred to as "quantum" and "classical"

Integrations over the fast ("quantum") fields in QCD can be done in perturbation theory; they correspond to the coefficient functions in the OPE (by construction, see scalar examples) and the result is expressed in terms of slow ("classical") fields in the vacuum, which cannot be described perturbatively.

Vacuum condensates are parametrizations of average properties of nonperturbative vacuum fields — a variant of the mean field aproach adapted for QFT.

8.1.3 QCD sum rules

Experiment:



where the peak corresponds to the ρ-meson. Let us calculate this contribution.

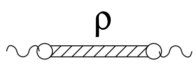
$$\begin{aligned}\Pi_{\mu\nu}(q) &= \int d^4x\, e^{iqx}\langle\Omega|\,\mathrm{T}\,\{j_\mu(x)j_\nu(0)\}|\Omega\rangle \\ &= \int d^4x\, e^{iqx}\Big[\theta(x_0)\langle\Omega|j_\mu(x)|\rho\rangle\langle\rho|j_\nu(0)\}|\Omega\rangle + \theta(-x_0)\langle\Omega|j_\nu(0)|\rho\rangle\langle\rho|j_\mu(x)\}|\Omega\rangle\Big] \quad (8.19)\end{aligned}$$

Define L19

$$\langle \Omega | j_\mu(0) | \rho^{(\lambda)}(p) \rangle \;=\; m_\rho^2 \frac{\sqrt{2}}{g_\rho} \epsilon_\mu^{(\lambda)}\,, \qquad \lambda = 1,2,3\,, \qquad \epsilon_\mu^{(\lambda)} p^\mu = 0 \tag{8.20}$$

Note that a ρ-meson can have three polarizations, two transverse ones and one longitudinal

The contribution of interest corresponds to a simple Feynman diagram



$$= \sum_\lambda m_\rho^2 \frac{\sqrt{2}}{g_\rho} \epsilon_\mu^{(\lambda)} \cdot \frac{1}{m_\rho^2 - q^2} \epsilon_\nu^{(\lambda)*} m_\rho^2 \frac{\sqrt{2}}{g_\rho} \tag{8.21}$$

The sum over polarizations can be done using

$$\sum_\lambda \epsilon_\mu^{(\lambda)} \epsilon_\nu^{(\lambda)*} = -g_{\mu\nu} + \frac{q_\mu q_\nu}{m_\rho^2} \tag{8.22}$$

Obtain

$$\begin{aligned} \ldots \;&=\; m_\rho^4 \frac{2}{g_\rho^2} \frac{1}{m_\rho^2 - q^2} \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{m_\rho^2} \right) = \frac{2m_\rho^2}{g_\rho^2} \frac{1}{m_\rho^2 - q^2} \left(-g_{\mu\nu} m_\rho^2 + q_\mu q_\nu \right) \\ &= \frac{2m_\rho^2}{g_\rho^2} \frac{1}{m_\rho^2 - q^2} \left(-g_{\mu\nu} q^2 + q_\mu q_\nu \right) \;+\; \text{term} \sim g_{\mu\nu} \text{ without the pole} \end{aligned} \tag{8.23}$$

? Is this extra term in contradiction with gauge invariance
— no, it will be cancelled by contributions of other states

Thus

$$\boxed{\Pi(q^2)\Big|_{\rho-\text{meson}} \;=\; \frac{2m_\rho^2}{g_\rho^2} \frac{1}{m_\rho^2 - q^2 - i\epsilon}} \tag{8.24}$$

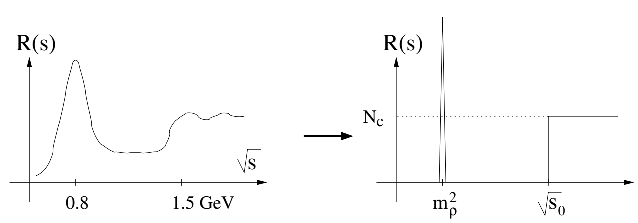
and therefore

$$R_\rho(s) \;=\; 12\pi \,\mathrm{Im}\,\Pi(s) \;=\; 12\pi^2 \frac{2m_\rho^2}{g_\rho^2} \delta(s - m_\rho^2) \tag{8.25}$$

In this derivation we assumed that ρ is a stable state (for simplicity);
taking into account decays $\rho \to \pi\pi$ will modify the δ-function to a Breit-Wigner resonance.

Idea: (Shifman, Vainstein, Zakharov)

Consider a simplified model of the spectrum



with free parameters m_ρ^2, g_ρ and s_0

Then on the one hand

$$\Pi^{\text{model}}(q^2) = \frac{1}{12\pi^2}\int^{\mu^2}\frac{ds}{s-q^2}R^{\text{model}}(s) = \frac{2m_\rho^2}{g_\rho^2}\frac{1}{m_\rho^2-q^2} + \frac{1}{12\pi^2}\int_{s_0}^{\mu^2}\frac{ds}{s-q^2}\cdot N_c \tag{8.26}$$

and on the other hand

$$\Pi^{\text{QCD}}(q^2) = \frac{N_c}{12\pi^2}\ln\frac{\mu^2}{-q^2} + \frac{1}{12q^4}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle + \frac{224}{81q^6}\alpha_s\cdot\pi\langle\bar\psi\psi\rangle^2 + \dots \tag{8.27}$$

The sum rule:

$$\boxed{\Pi^{\text{model}}(q^2) \simeq \Pi^{\text{QCD}}(q^2) \qquad \text{for moderately large negative}\, q^2} \tag{8.28}$$

Why moderatively large:
For small q^2 need many terms in the OPE, for large q^2 loose sensitivity to resonance

First trick: The pert. term in the OPE can be written as ($\mu^2 \gg q^2$)

$$\frac{N_c}{12\pi^2}\ln\frac{\mu^2}{-q^2} = \frac{N_c}{12\pi^2}\int_0^{\mu^2}\frac{ds}{s-q^2} = \int_0^{s_0} + \int_{s_0}^{\mu^2} \tag{8.29}$$

Subtracting the $\int_{s_0}^{\mu^2}$ part from the both sides obtain

$$\frac{2m_\rho^2}{g_\rho^2}\frac{1}{m_\rho^2-q^2} := \frac{N_c}{12\pi^2}\int_0^{s_0}\frac{ds}{s-q^2} + \frac{1}{12q^4}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle + \frac{224}{81q^6}\alpha_s\cdot\pi\langle\bar\psi\psi\rangle^2 \tag{8.30}$$

This must work for $q^2 \sim -(1-2)$ GeV^2, $\Longrightarrow$ three parameters m_ρ^2, g_ρ and s_0

Second trick: Borel transformation $q^2 \to M^2 =$ Borel parameter

$$B\left[\frac{1}{m^2-q^2}\right] \Rightarrow \frac{1}{M^2}e^{-m^2/M^2}$$

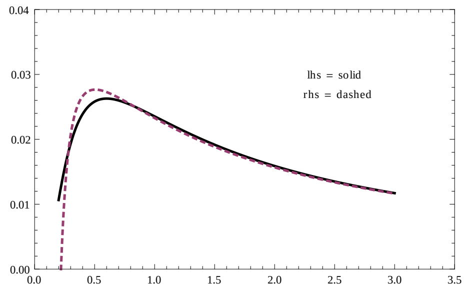
$$B\left[\frac{1}{(-q^2)^n}\right] \Rightarrow \frac{1}{(n-1)!}\frac{1}{M^{2n}} \tag{8.31}$$

Why:
— Contributions of higher mass states are suppressed exponentially e^{-s/M^2} instead of $1/(s-q^2)$
— Higher-order $1/q^{2n}$ terms in the OPE are suppressed by $1/(n-1)!$ factors

$\Longrightarrow$ SVZ sum rule

$$\frac{2m_\rho^2}{g_\rho^2}e^{-m_\rho^2/M^2}\frac{1}{M^2} := \frac{N_c}{12\pi^2M^2}\int_0^{s_0}ds\, e^{-s/M^2} + \frac{1}{12M^4}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle - \frac{112}{81M^6}\alpha_s\cdot\pi\langle\bar\psi\psi\rangle^2 \tag{8.32}$$

A numerical analysis:



$$
\begin{aligned}
m_\rho^2 &\simeq 0.5 - 0.6 \text{ GeV}^2 \\
g_\rho^2 &\simeq 28 \pm 2 \\
s_0 &\simeq 1.5 \pm 0.5 \text{ GeV}^2
\end{aligned}
\tag{8.33}
$$

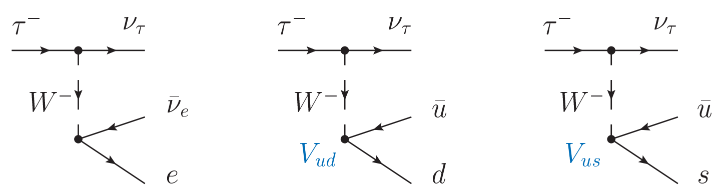
in fair agreement with experiment.

- A large subfield of research inside QCD
- Original SVZ paper is one of the most cited in HEP, has 5607 citations as of 30.11.2022

8.2 τ–decay and duality

Heavy τ-lepton: (e, μ, τ) $m_\tau = 1.777$ GeV

The Standard Model:



$$
m_W = 80.4\,\text{GeV} \gg m_\tau\,, \qquad \text{hence for low energies} \qquad \frac{1}{m_W^2 - q^2} \simeq \frac{1}{m_W^2}
$$

$\Longrightarrow$ Effective four-fermion interaction (Fermi)

$$
\begin{aligned}
\tau \to e\nu_\tau\bar\nu_e : \qquad & \mathcal{L}_{eff} = -\frac{G_F}{\sqrt{2}}\left[\bar e\gamma^\mu(1-\gamma_5)\nu_e\right]\left[\bar\nu_\tau\gamma_\mu(1-\gamma_5)\tau\right] \\
\tau \to \text{hadrons} : \qquad & \mathcal{L}_{eff} = -V_{ud}\frac{G_F}{\sqrt{2}}\left[\bar d\gamma^\mu(1-\gamma_5)u\right]\left[\bar\nu_\tau\gamma_\mu(1-\gamma_5)\tau\right] + \ldots
\end{aligned}
\tag{8.34}
$$

Explicit calculation yields (exercises):

$$
\Gamma(\tau \to e\nu_\tau\bar\nu_e) = \frac{G_F^2 m_\tau^5}{192\pi^3} \tag{8.35}
$$

We expect, qualitatively

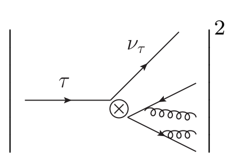
$$
R_\tau = \frac{\Gamma(\tau \to \text{hadrons})}{\Gamma(\tau \to e\nu_\tau\bar\nu_e)} \simeq \left(|V_{ud}|^2 + |V_{us}|^2\right)\cdot N_c \tag{8.36}
$$

This quantity can be measured very precisely, let us try to calculate it in QCD.

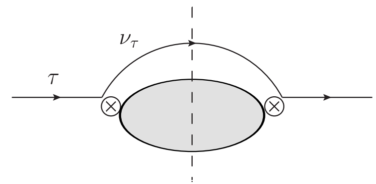
For simplicity set $V_{ud} \to 1$, $V_{us} \to 0$ and neglect quark masses $m_{u,d} \to 0$

Using unitarity (optical theorem)

$$\Gamma(\tau \to \text{ hadrons}) \quad \simeq$$



$$\simeq \quad 2\,\mathrm{Im}$$



(8.37)

where the shaded blob is given by

$$\begin{aligned}\Pi^\tau_{\mu\nu} &= i\int d^4x\, e^{iqx}\langle\Omega|\,\mathrm{T}\,\{\bar{d}(x)\gamma_\mu(1-\gamma_5)u(x)\,\bar{u}(0)\gamma_\nu(1-\gamma_5)d(0)\}|\Omega\rangle \\ &= (q_\mu q_\nu - g_{\mu\nu}q^2)\Pi^\tau(q^2) \end{aligned} \tag{8.38}$$

[transversality holds for $m_u = m_d = 0$, add an extra $\sim g_{\mu\nu}$ function otherwise]

Explicit calculation yields (exercises):

$$R_\tau \;=\; 12\pi \int_0^{m_\tau^2} \frac{ds}{m_\tau^2}\left(1-\frac{s}{m_\tau^2}\right)^2\left(1+\frac{2s}{m_\tau^2}\right)\mathrm{Im}\,\Pi^\tau(s) \tag{8.39}$$

which can be compared with e^+e^- annihilation cross section

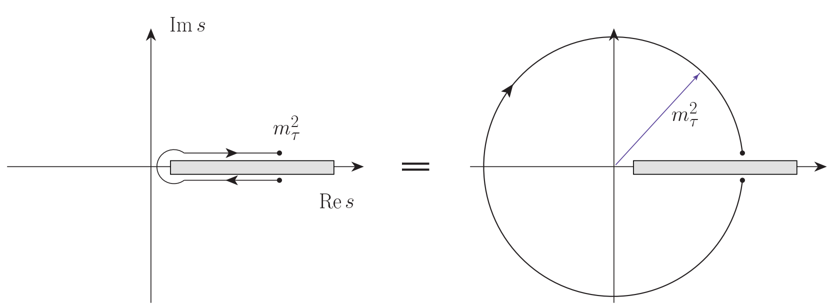
$$R_{e^+e^-}(s) = 12\pi\,\mathrm{Im}\,\Pi_{e^+e^-}(s) \tag{8.40}$$

The extra s-integration corresponds to neutrino energy in the final state L20

The trick:
write

$$2i\,\mathrm{Im}\,\Pi^\tau(s) = \Pi^\tau(s+i\epsilon) - \Pi^\tau(s-i\epsilon) \tag{8.41}$$

and transform the integration contour to a circle of radius m_τ^2 in the complex plane:



Thus

$$R_\tau \;=\; \frac{12\pi}{2i}\oint_C \frac{ds}{m_\tau^2}\left(1-\frac{s}{m_\tau^2}\right)^2\left(1+\frac{2s}{m_\tau^2}\right)\Pi^\tau(s) \tag{8.42}$$

In this form

- Integration contour is far away from $s=0$, $|s|=m_\tau^2 \gg \Lambda_{\rm QCD}^2$; thus perturbation theory (+ OPE corrections) can be used, apart from possibly a small region close to the real axis
- Luck: The contribution from the dangerous region $s \to m_\tau^2$ suppressed by the factor $\left(1-s/m_\tau^2\right)^2$
- Luck: The LO contributioon ofthe gluon condensate $\sim 1/s^2 = 1/q^4$ vanishes upon integration (accident)

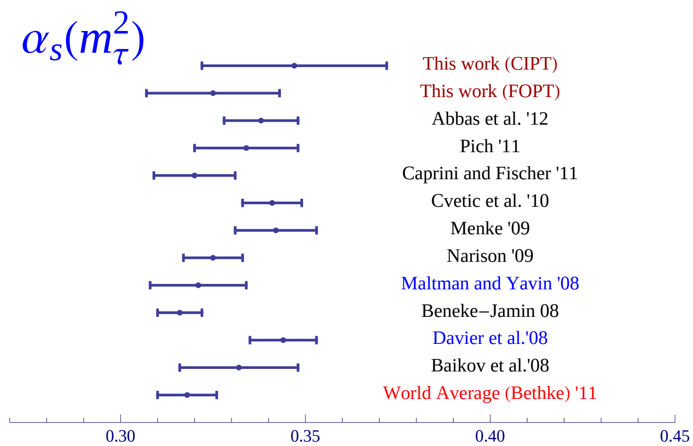
$\Longrightarrow$ a very accurate prediction possible

OPE:

$$\Pi(s) = \frac{1}{4\pi^2}\ln\frac{\mu^2}{-s}\left[1+a_1\alpha_s(s)+a_2\alpha_s^2(s)+a_3\alpha_s^3(s)\right]+\left[1+b_1\alpha_s(s)\right]\frac{1}{12s}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle + \left[c_0+c_1\alpha_s(s)\right]\pi\alpha_s\langle\bar\psi\psi\rangle^2 \quad (8.43)$$

Experiment:

$$R_\tau = 3.4771 \pm 0.0084 \qquad \text{HFAG, } \texttt{http://www.slac.stanford.edu/xorg/hfag/} \quad (8.44)$$



Recent α_s determinations from τ-decays using different hypothesises for the resummation of high orders in perturbation theory. Table taken from: D. Boito *et al.* [arXiv:1212.0091]

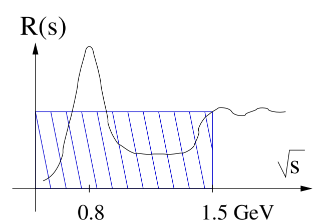
The trick that we used here is rather general.

Let $\Pi(s)$ be a certain two-point correlation function. Then $R(s) \sim \operatorname{Im}\Pi(s)$ for small $s \sim \Lambda_{QCD}^2$ cannot be calculated in perturbation theory, but the integral

$$\int_0^{s_0} ds\, w(s)\, \operatorname{Im}\Pi(s) \quad (8.45)$$

where $w(s)$ is a smooth function is often calulable (or at least can be estimated).

E.g. for the total cross section of e^+e^- annihilation



$\Leftarrow$ the area is the same

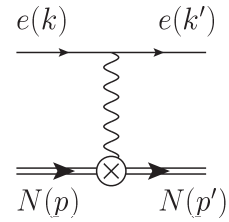
Accepted terminology: A resonance is “dual” to the $\bar{q}q$ pair in the “interval of duality” s_0

9 Deep–Inelastic Lepton–Nucleon Scattering

9.1 Elastic Lepton–Nucleon Scattering

Elastic scattering of electrons from protons (neutrons):

$$e(k) + N(p) \to e(k') + N(p')$$



Differential cross section for unpolarized particles (Rosenbluth)

$$\frac{d\sigma}{d\Omega} = \frac{\alpha_{\text{em}}^2}{4E^2 \sin^4 \frac{\theta}{2}} \frac{1}{1 + (2E/m)\sin^2 \frac{\theta}{2}} \left[\left(F_1^2 - \frac{q^2}{4m^2} F_2^2 \right) \cos^2 \frac{\theta}{2} - \frac{q^2}{2m^2} \left(F_1 + F_2 \right)^2 \sin^2 \frac{\theta}{2} \right] \quad (9.1)$$

where

E : electron energy in the lab frame
m : nucleon mass
θ : scattering angle in the lab frame
$q = k - k' = p' - p$: momentum transfer; $Q^2 \equiv -q^2 = 4E^2 sin^2 \frac{\theta}{2}$

and (definition):

$$\langle N(p')|j_\mu^{\text{em}}(0)|N(p)\rangle \;:=\; \bar{u}(p') \left[\gamma_\mu F_1(Q^2) + \frac{i}{2m} \sigma_{\mu\nu} q^\nu F_2(Q^2) \right] u(p) \quad (9.2)$$

The functions $F_1(Q^2)$ and $F_2(Q^2)$ are called Dirac and Pauli form factors, respectively

$F_1^p(0) = 1\,,$	$F_1^n(0) = 0\,,$	electric charge	
$F_2^p(0) = 2.792847\,\mu_N\,,$	$F_2^n(0) = -1.913043\,\mu_N$	magnetic moment	$\boxed{\mu_N = e\hbar/2m_N}$

(9.3)

In a non-relativistic theory form factor is a Fourier transform of the charge/current distribution:

$$F(\vec{q}^2) \;=\; \int d^3x\, e^{i\vec{q}\vec{x}}\, \rho(\vec{x}) \quad (9.4)$$

Hence a deviation from $F(Q^2) = $ const signals that a particle has internal structure

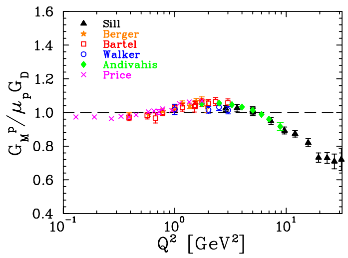
For $Q^2 \leq 1$ GeV^2 (Hofstadter)

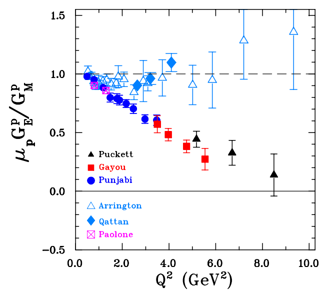
$$F_{1,2} \simeq \frac{1}{[1 + Q^2/Q_0^2]^2} \quad \text{with} \quad Q_0^2 \simeq 0.71\,\text{GeV}^2 \quad (9.5)$$

$\Longleftarrow$ corresponds to the proton radius ca. $R_p \simeq 0.85$ fm

Studies of form factors at large $Q^2 \gg 1$ GeV^2 is an active research topic (theory complicated)

Recent results from Jefferson Laboratory: [Arrington *et al.*, arXiv:1102.2463]

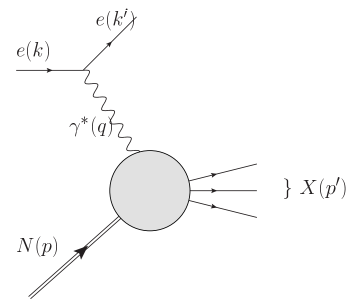




9.2 Deep-Inelastic Scattering and Bjorken scaling

In 1968 J. Bjorken proposed a different type of experiment: Sum over all hadronic final states

$$e(k) + N(p) \to e(k') + X(p')$$



$\Longleftarrow$ the Deep-Inelastic Lepton-Hadron Scattering (DIS)

- "Sum over all states" in practice means that hadrons are not identified (measured); the only detected particle is the scattered electron in the final state — one measures its scattering angle θ and energy E'

L21

Kinematic variables

- m: the nucleon mass
- $E = \frac{pk}{m}$: beam electron energy (in L.S. $p_\mu = \{m, 0, 0, 0\}$)
- $E' = \frac{pk'}{m}$: scattered electron energy (in L.S.)
- θ: electron scattering angle (in L.S.)

$$q = k - k' = p' - p \qquad\qquad Q^2 = -q^2 = 4EE' \sin^2\frac{\theta}{2}$$

- $\nu = \frac{pq}{m}$: energy transfer to the hadronic system (in L.S.)
- $y = \nu/E$: energy transfer/maximum energy transfer (in L.S.) $\boxed{0 < y < 1}$
- $W^2 = (p+q)^2$: invariant mass squared of the hadronic system

- $x_B = \frac{Q^2}{2pq} = \frac{Q^2}{2m\nu}$: Bjorken's scaling variable $\boxed{0 < x < 1}$

  Kinematic bound:

$$W^2 = m^2 + 2m\nu - Q^2 \geq m^2 \qquad \Longrightarrow \qquad x_B \leq 1$$

  Elastic scattering corresponds to $x_B = 1$ in which case $2m\nu = Q^2$

The Bjorken limit

$$\boxed{W \to \infty\,, \qquad Q^2 \to \infty \qquad \text{such that} \qquad x_B = \frac{Q^2}{2pq} = \text{const.}}$$

The transition amplitude for a given final state is

$$iM(eN \to eX) \;=\; (-ie)\bar{u}(k',\lambda')\gamma^\mu u(k,\lambda)\frac{-i}{q^2}(ie)\langle X(p')|j_\mu(0)|N(p)\rangle \tag{9.6}$$

where

$$j_\mu(x) = e_u\bar{u}(x)\gamma_\mu u(x) + e_d\bar{d}(x)\gamma_\mu d(x) + \ldots \tag{9.7}$$

The cross section summed over hadronic states is then

$$d\sigma \;=\; \frac{1}{2s}\frac{d^3k'}{(2\pi)^3 2|k'|}\frac{1}{2}\sum_{\lambda,\lambda'}\sum_X\int d\Pi_X|iM(eN \to eX(p')|^2(2\pi)^4\delta^4(p+q-p') \tag{9.8}$$

↖ As usual, we sum over spins of final state electrons and average over spins of initial electrons. This corresponds to the simplest experimental setup where spin is not measured

Thus

$$\frac{d\sigma}{d^3k'} \;=\; \frac{1}{2s}\frac{1}{(2\pi)^3 2|k'|}\frac{1}{2}\sum_{\lambda,\lambda'}\bar{u}(k,\lambda)\gamma^\mu u(k',\lambda')\bar{u}(k',\lambda')\gamma^\nu u(k,\lambda)\frac{e^4}{Q^4}(2\pi)W_{\mu\nu}(p,q) \tag{9.9}$$

where

$$(2\pi)W_{\mu\nu}(p,q) = \sum_X\int d\Pi_X\langle N(p)|j_\mu(0)|X(p')\rangle\langle X(p')|j_\nu(0)|N(p)\rangle(2\pi)^4\delta^4(p+q-p') \tag{9.10}$$

Using the optical theorem

$$W_{\mu\nu}(p,q) = \frac{1}{\pi}\,\mathrm{Im}\,T_{\mu\nu}(p,q) \tag{9.11}$$

where

$$T_{\mu\nu} = i\int d^4x\, e^{iqx}\langle N(p)|\,\mathrm{T}\,\{j_\mu(x)j_\nu(0)|\}|N(p)\rangle \tag{9.12}$$

is called *the forward Compton amplitude*:

$$iM(\gamma N \to \gamma N) = e^2\epsilon^*_\mu(q)\epsilon_\nu(q)T^{\mu\nu}\,, \qquad q^2 = 0 \tag{9.13}$$

We will use this representation later

Conservation of the electromagnetic current (Ward identity) implies

$$q^\mu W_{\mu\nu}(p,q) = q^\nu W_{\mu\nu}(p,q) = 0 \tag{9.14}$$

Therefore $W_{\mu\nu}(p,q)$ must have the form

$$\begin{aligned} W_{\mu\nu}(p,q) = & \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) W_1(pq,Q^2) + \left(p_\mu - q_\mu \frac{pq}{q^2}\right)\left(p_\nu - q_\nu \frac{pq}{q^2}\right) W_2(pq,Q^2) \\ & + i\epsilon_{\mu\nu\alpha\beta} q^\alpha p^\beta W_3(pq,Q^2) \end{aligned} \tag{9.15}$$

Parity conservation in strong and electromagnetic interactions $\Longrightarrow$ $W_3(\nu,Q^2)=0$

The scalar functions $W_{1,2}$ depend on the invariants $W(pq,Q^2)$ or equivalently $W(x_B,Q^2)$
They are called *structure functions*

Further we have

$$\frac{1}{2}\sum_{\lambda,\lambda'}[\bar{u}(k,\lambda)\gamma^\mu u(k',\lambda')][\bar{u}(k',\lambda')\gamma^\nu u(k,\lambda)] = \frac{1}{2}\,\mathrm{Tr}\,[\not{k}\gamma^\mu \not{k}'\gamma^\nu] = 2[k^\mu k'^\nu + k^\nu k'^\mu - g^{\mu\nu}(kk')] \tag{9.16}$$

and

$$d^3k' \;=\; |k'|^2 d|k'|\, d\phi\, d\cos\theta \;=\; (E')^2 dE' d\Omega' \tag{9.17}$$

so that after some algebra

$$\frac{d\sigma}{dE'd\cos\theta} = \frac{8\pi\alpha_{\rm em}^2 E'^2}{Q^4}\left\{\frac{1}{m}W_1(\nu,Q^2)\,\sin^2\frac{\theta}{2} + \frac{m}{2}W_2(\nu,Q^2)\,\cos^2\frac{\theta}{2}\right\} \tag{9.18}$$

Thus $W_1(\nu,Q^2)$ and $W_2(\nu,Q^2)$ can be measured experimentally from the E',θ of scattered electrons

A more convenient representation

$$(E',\cos\theta) \;\rightarrow\; (x_B, y) \tag{9.19}$$

Since

$$s = (p+k)^2 \simeq 2mE \qquad y = \frac{2pq}{2pk} = \frac{E-E'}{E} \qquad x_B = \frac{4EE'\sin^2\frac{\theta}{2}}{2m(E-E')} \tag{9.20}$$

obtain

$$\sin^2\frac{\theta}{2} = \frac{m^2 x_B}{s}\frac{y}{1-y} \qquad \frac{\partial(x_B,y)}{\partial(E',\cos\theta)} = \frac{2E'}{2m(E-E')} = \frac{2E'}{ys} \tag{9.21}$$

and neglecting terms $\mathcal{O}(m/E)$

$$\boxed{\frac{d\sigma}{dxdy} = \frac{\pi\alpha_{\rm em}^2 s}{Q^4}\Big[2x_B y^2 W_1(\nu,Q^2) + y(1-y)sW_2(\nu,Q^2)\Big]} \tag{9.22}$$

The prediction by Bjorken was that in the synchronous high-energy and high-Q^2 limit

$$
\begin{aligned}
W_1(\nu, Q^2) &= F_1(x_B) + \mathcal{O}(1/Q^2) \\
\frac{1}{4} ys W_2(\nu, Q^2) &= F_2(x_B) + \mathcal{O}(1/Q^2)
\end{aligned}
\tag{9.23}
$$

— the Bjorken scaling

confusing notation: Structure functions F_1 and F_2 are not Dirac F_1 and Pauli F_2 form factors !!

As we will see, this behaviour indicates the existence of free pointlike charged particles in hadrons

9.3 The Parton Model

The physical picture of DIS becomes more transparent in a special reference frame, the *Breit frame*:

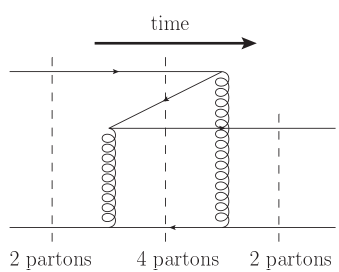
We choose

$$
q_\mu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -Q \end{pmatrix} \qquad\qquad p_\mu = \begin{pmatrix} \sqrt{p_z^2 + m^2} \\ 0 \\ 0 \\ p_z \end{pmatrix} \tag{9.24}
$$

- Since $q_0 = 0$, electron has the same energy before and after the collision (in this frame)
- Since $pq = Qp_z$ and $x_B = Q^2/(2pq)$ it follows that $p_z = Q/(2x_B) \to \infty$ in the Bjorken limit

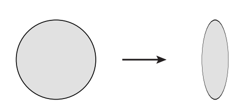
Assume that nucleon is a bound states of pointlike (small-size) constitients — *partons*
(e.g. quarks and gluons).
— In nonrelativistic QM such a state would be described by a wave function $\Psi(x_1, \ldots, x_n)$.
— In a relativistic theory this description is lost, in general, because the number of partons is not conserved

Z-graphs:



However, in a fast moving hadron all longitudinal distances are contracted and all processes are slowed down by the Lorentz factor $\gamma = 1/\sqrt{1 - v^2/c^2}$

- For an observer in the Breit frame the proton "looks" like a pancake



- An interaction between partons in the proton rest frame requires time $\tau_0 \sim R/c$;
  it becomes $t \simeq \tau_0 \frac{p_z}{m} \to \infty$ in Breit frame — partons do not have time to "talk" to each other

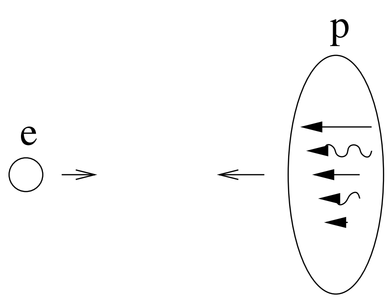
Thus: L22

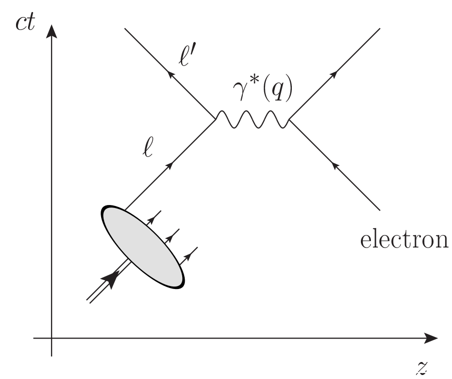
- A fast-moving hadron can be viewed as a collection of partons which fly in the same direction

- Since the partons have no time to interact, each of them can be considered as free and carrying a fixed momentum fraction of the proton
- A "hard" external probe (virtual photon) can only interact with one parton (unless parton density is very large)
- Scattering processes on different partons do not interfere quantum-mechanically; the cross section is given by a sum of cross sections on individual partons

The overall picture that arises in this way has become known as *the QCD parton model*

In our case (DIS)





parton (quark) momentum in the proton:

$$\ell_\mu = (\xi p_z, 0, 0, \xi p_z) \tag{9.25}$$

parton momentum after the collision:

$$\ell'_\mu = \ell_\mu + q = (\xi p_z, 0, 0, \xi p_z - Q) \tag{9.26}$$

parton remains to be on-shell:

$$0 = \ell'^2 = (\xi p_z)^2 - (\xi p_z - Q)^2 = 2\xi p_z Q - Q^2 \qquad \Longrightarrow \qquad \xi = \frac{Q}{2p_z} \tag{9.27}$$

since $p_z = Q/(2x_B)$ (see above) we obtain

$$\xi = \frac{Q}{2p_z} = \frac{Q}{2}\frac{2}{Q}x_B \qquad \Longrightarrow \qquad \boxed{\xi = x_B} \tag{9.28}$$

— <u>parton momentum fraction is equal to the Bjorken variable</u>

The parton content of the nucleon is described by *parton distributions.* Let

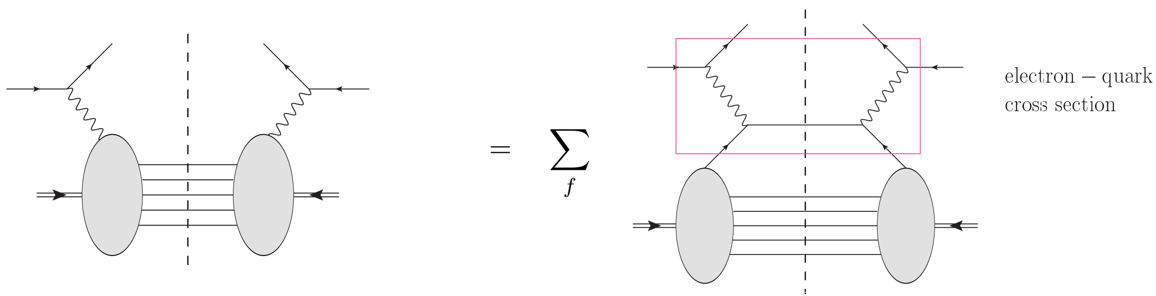
$$F_f(\xi)d\xi \qquad\qquad f = q, \bar{q}, g \tag{9.29}$$

be the number of quark, antiquark and gluon partons with momentum fractions in the interval between ξ and $\xi + d\xi$

The deep-inelastic cross section from a proton (in the parton model) is given by the incoherent sum of the DIS cross sections from the partons:

$$\boxed{\sigma_{e(k)N(p)\to e(k')X} = \sum_f \int_0^1 d\xi\, F_f(\xi)\, \sigma_{e(k)f(\xi p)\to e(k')f(\xi p+q)}} \tag{9.30}$$

This formula can be illustrated by the following picture:



Quantum-mechanical intepretation:

Transition amplitide between two states is proportional to the overlap between the wave functions:

$$F_{|1\rangle\to|2\rangle}(q) = \int d^3x\, e^{-iqx}\Psi_1(x)\Psi_2^*(x) = \int d^3x\, e^{-iqx}\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3p'}{(2\pi)^3}e^{ipx-ip'x}\Psi_1(p)\Psi_2^*(p')$$

$$= \int\frac{d^3p}{(2\pi)^3}\Psi_1(p)\Psi_2^*(p+q) \qquad \sim\ 1/|q|^k \quad \text{or} \quad \sim\ e^{-R|q|} \quad \text{for large q} \tag{9.31}$$

If we sum over the final states this suppression disappears:

$$\sum_{|2\rangle}|F_{|1\rangle\to|2\rangle}(q)|^2 = \sum_{|2\rangle}\int\frac{d^3p}{(2\pi)^3}\Psi_1(p)\Psi_2^*(p+q)\int\frac{d^3p'}{(2\pi)^3}\Psi_1^*(p')\Psi_2(p'+q)$$

$$= \int\frac{d^3p}{(2\pi)^3}\int\frac{d^3p'}{(2\pi)^3}\Psi_1(p)\Psi_1^*(p')\,(2\pi)^3\delta^3(p-p') = 1 \tag{9.32}$$

where I used the completeness condition

$$\sum_{|2\rangle}\Psi_2^*(p+q)\Psi_2(p'+q) = (2\pi)^3\delta^3(p-p') \tag{9.33}$$

Thus the cross section becomes much larger and, crucially, we can calculate it without any knowledge of the "true" eigenstates of the Hamiltonian:

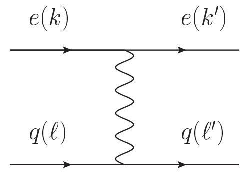
$$\sum_{|2\rangle:\text{eigenstates of H}}|F_{|1\rangle\to|2\rangle}(q)|^2 = \sum_{|2\rangle:\text{plane waves}}|F_{|1\rangle\to|2\rangle}(q)|^2 \tag{9.34}$$

Example: photodisintegration cross section for deuteron

In our context: we can neglect interaction of hit quark with the remnant partons in the final state

Electron-quark scattering

— a close analogue to electron-muon scattering (QED):



$$\frac{d\sigma}{d\Omega_{\rm CM}} = \frac{1}{64\pi^2\hat{s}}|M|^2 = \frac{\alpha^2 e_q^2}{2\hat{s}}\frac{\hat{s}^2+\hat{u}^2}{\hat{t}^2}$$

where

$$\ell_\mu \simeq \xi p_\mu \qquad \text{quark four-momentum}$$
$$\ell^2 = \ell'^2 = 0\,, \qquad k^2 = k'^2 = 0$$

Mandelstam variables for the electron-quark scattering:

$$\begin{aligned}\hat{s} &= (k+\ell)^2 = (k+\xi p)^2 = 2\xi(kp) + \not{p}^2 \simeq \xi(k+p)^2 = \xi s \\ \hat{t} &= (k'-k)^2 = -Q^2 \\ \hat{u} &= (k-\ell')^2 = -\hat{s}-\hat{t}\,, \qquad \Longleftarrow \qquad \hat{s}+\hat{t}+\hat{u} = 0\end{aligned} \tag{9.35}$$

In the physical region for the electron-quark scattering

$$\hat{s} \geq |\hat{t}| \qquad \Longrightarrow \qquad \xi s \geq Q^2 \tag{9.36}$$

In the CM system:

$$d\Omega_{\rm CM} = d\phi\, d\cos\theta_{\rm CM} \qquad\qquad \hat{t} = -\frac{1}{2}\hat{s}(1-\cos\theta_{\rm CM}) \tag{9.37}$$

Therefore

$$\frac{d\sigma}{dt} = \underbrace{(2\pi)}_{\int d\phi}\cdot\frac{2}{\hat{s}}\cdot\frac{\alpha^2 e_q^2}{2\hat{s}}\frac{\hat{s}^2+\hat{u}^2}{\hat{t}^2} = \frac{2\pi\alpha^2 e_q^2}{Q^4}\frac{\hat{s}^2+\hat{u}^2}{\hat{s}^2} \tag{9.38}$$

The last factor can be rewritten as $(\hat{s}^2+\hat{u}^2)/\hat{s}^2 = 1 + \left[(-\hat{s}-\hat{t})/\hat{s}\right]^2 = 1 + \left(1 - Q^2/(\xi s)\right)^2$
It follows that

$$\begin{aligned}\frac{d\sigma^{\rm DIS}}{dQ^2} &= \sum_f \int_0^1 d\xi\, F_f(\xi) e_f^2 \frac{2\pi\alpha^2}{Q^4}\left[1+\frac{\hat{u}^2}{\hat{s}^2}\right]\theta(\xi s - Q^2) \\ &:= \int_0^1 dx_B\, \frac{d\sigma^{\rm DIS}}{dx_B dQ^2}\end{aligned} \tag{9.39}$$

Since $\xi = x_B$ this means that

$$\frac{d\sigma^{\rm DIS}}{dx_B dQ^2} = \sum_f e_f^2 F_f(x_B)\frac{2\pi\alpha^2}{Q^4}\left[1+\frac{\hat{u}^2}{\hat{s}^2}\right]\theta(x_B s - Q^2) \tag{9.40}$$

It remains to express everything in terms of x_B, y variables:

$$y := \frac{2pq}{2pk} = \frac{2p(k-k')}{2pk} \cdot \frac{\xi}{\xi} = \frac{2\ell(k-k')}{2\ell k} = \frac{\hat{s}+\hat{u}}{\hat{s}}$$

$$\Longrightarrow \quad \frac{\hat{u}}{\hat{s}} = -(1-y)$$

$$Q^2 = \frac{Q^2}{2pq} \cdot \frac{2pq}{2pk} \cdot 2pk = x_B \cdot y \cdot s$$

$$\Longrightarrow \quad dx_B dQ^2 = \frac{dQ^2}{dy} dx_B dy = x_B s \, dx_B dy$$

$$\Longrightarrow \quad \frac{d\sigma^{\mathrm{DIS}}}{dx_B dy} = x_B s \frac{d\sigma^{\mathrm{DIS}}}{dx_B dQ^2} \tag{9.41}$$

We obtain

$$\frac{d\sigma^{\mathrm{DIS}}}{dx_B dy} = \left(\sum_f e_f^2 x_B F_f(x_B) \right) \frac{2\pi\alpha^2 s}{Q^4} \left[1 + (1-y)^2 \right] \tag{9.42}$$

On the other hand, we derived before

$$\frac{d\sigma^{\mathrm{DIS}}}{dx_B dy} = \frac{\pi\alpha_{\mathrm{em}}^2 s}{Q^4} \left[2x_B y^2 W_1(\nu, Q^2) + y(1-y) s W_2(\nu, Q^2) \right] \tag{9.43}$$

so that for $F_1 \equiv W_1$, $F_2 \equiv (ys/4) W_2$

$$F_2(x_B) = x_B F_1(x_B) = \sum_f e_f^2 x_B F_f(x_B) \tag{9.44}$$

- The structure functions F_1 and F_2 do not depend on y (or Q^2) (Bjorken scaling)
- Let

$$\begin{aligned} u(x_B) : \quad & u\text{-quark momentum fraction distribution} \\ d(x_B) : \quad & d\text{-quark momentum fraction distribution} \\ & \ldots \end{aligned} \tag{9.45}$$

  The structure function F_1 is given by the sum of quark distributions weighted with electric charges squared:

$$F_1(x_B) = \frac{4}{9} u(x_B) + \frac{1}{9} d(x_B) + \ldots \tag{9.46}$$

- The structure function F_2 is expressed in terms of F_1 through the Callan-Gross relation

$$F_2(x_B) = x_B F_1(x_B) \tag{9.47}$$

Let

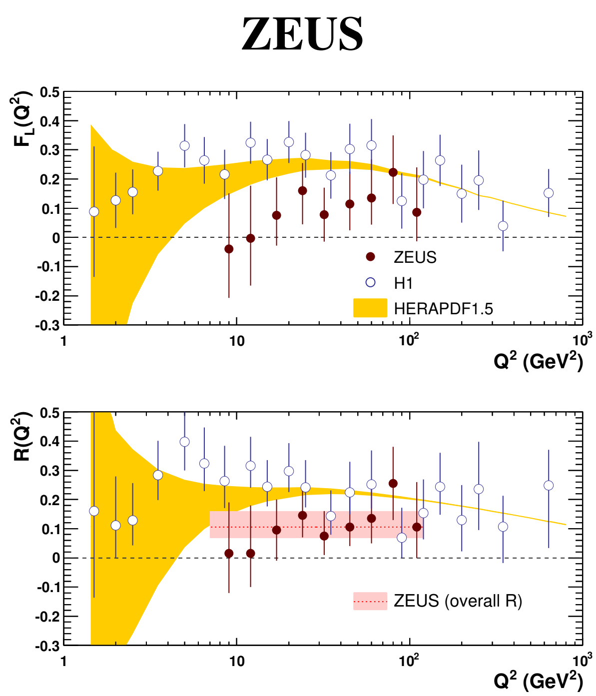
$$R(x_B) = \frac{F_2(x_B) - x_B F_1(x_B)}{F_2(x_B)} \tag{9.48}$$

One can show (exercise) that in the parton model

$$R(x) \;=\; \begin{cases} 0, & \text{if all charged partons have spin } 1/2 \\ 1, & \text{if all charged partons have spin zero} \end{cases} \tag{9.49}$$

$\Longrightarrow$ Experimental confirmation that partons have spin 1/2 (quarks)

Typical values: Shown: new ZEUS F_L (a) and R (b) measurements (solid points) in comparison with H1 measurements (open points) and NNLO HERAPDF



[Figure taken from: *Zhiqing Zhang, for the H1, ZEUS Collaborations, arXiv:1412.6328*]

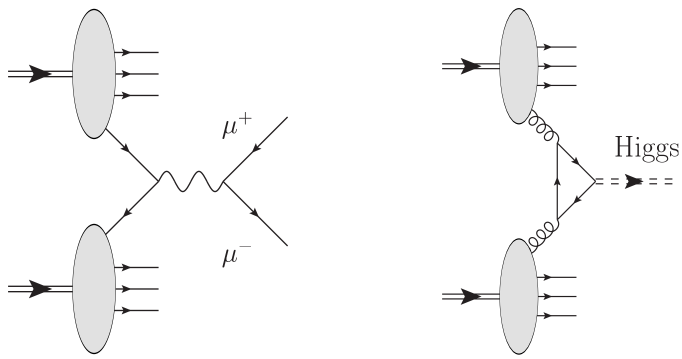
The power of the parton model is that it is applicable for many reactions. Example: L23

The Drell-Yan Process

$$N(p_1) + N(p_2) \;\to\; \mu^+(k_1) + \mu^-(k_2) + X$$

or

$$N(p_1) + N(p_2) \;\to\; \text{Higgs}(M) + X \tag{9.50}$$



Bjorken limit:

$$s = (p_1 + p_2)^2 \to \infty\,,$$
$$M^2 = (k_1 + k_2)^2 \to \infty\,,$$
$$M^2/s = \text{ const}$$

— Two "pancakes" approach each other at larger velocity and collide

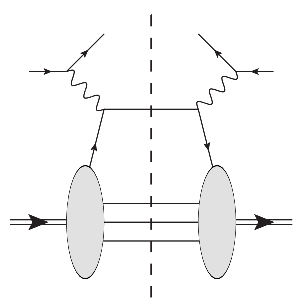
In the parton model

$$\begin{aligned}\frac{d\sigma}{dM^2}(NN \to \mu^+\mu^- + X) &= \sum_q \int_0^1 d\xi_1\, F_q^N(\xi_1) \int_0^1 d\xi_2\, F_{\bar q}^N(\xi_2) \frac{d\sigma}{dM^2}(q\bar q \to \mu^+\mu^- + X) \\ &= \frac{8\pi\alpha^2}{9M^4} \sum_q e_q^2 \int_0^1 d\xi_1\, F_q^N(\xi_1) \int_0^1 d\xi_2\, F_{\bar q}^N(\xi_2) \delta\left(\xi_1\xi_2 - \frac{M^2}{s}\right) \quad (9.51)\end{aligned}$$

Most importantly, $F_{q,\bar q}^N(\xi)$ are *the same* functions as in DIS, so that e.g. Higgs cross section can be predicted

10 Factorization and Parton Distributions

Aim of this section is to provide a QCD derivation of the parton model and go beyond it.
Let us summarize what we know about DIS



$$T_{\mu\nu} = i \int d^4x\, e^{iqx} \langle N(p)|\, \mathrm{T}\, \{j_\mu(x) j_\nu(0)|\}|N(p)\rangle$$

$$W_{\mu\nu}(p,q) = \frac{1}{\pi}\, \mathrm{Im}\, T_{\mu\nu}(p,q)$$

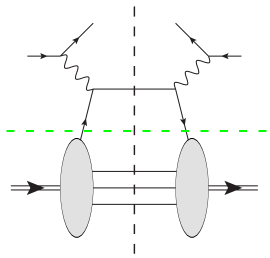
where

$$(2\pi) W_{\mu\nu}(p,q) = \sum_X \int d\Pi_X \langle N(p)|j_\mu(0)|X(p')\rangle\langle X(p')|j_\nu(0)|N(p)\rangle (2\pi)^4 \delta^4(p+q-p') \quad (10.1)$$

can be expressed in terms of the structure functions (observable quantities)

$$W_{\mu\nu}(p,q) = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) F_1(x_B, Q^2) + \left(p_\mu - q_\mu \frac{pq}{q^2}\right)\left(p_\nu - q_\nu \frac{pq}{q^2}\right) \frac{2}{pq} F_2(x_B, Q^2) \quad (10.2)$$

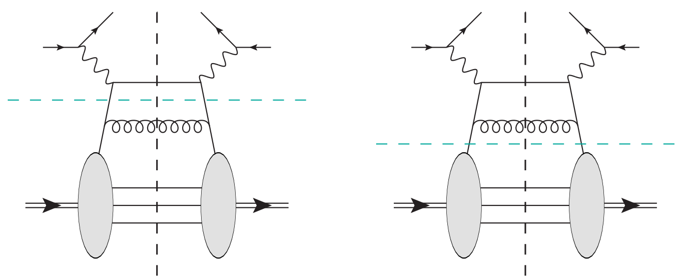
Parton model suggests that we can separate (factorize) the DIS cross section in two subprocesses:



$\nearrow$ electron-quark cross section

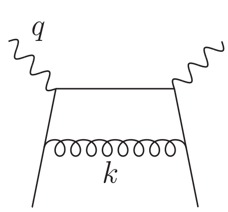
$\searrow$ quark momentum fraction distribution

QCD quarks interact by exchanging gluons; we will have more complicated Feynman diagrams like



Where to draw the line?
— The first possibility: gluon is just an "other" parton in the nucleon, part of "X"
— The second possiblity: gluon exchange is a correction to the electron-quark cross section

The guiding principle is whether the gluon is emitted at short distances or long time ago, as part of the preparation of the parton wave function. It is easy to see, however, that the integral over gluon momenta is logarithmic



$$\sim \alpha_s \int_m^Q \frac{d^4k}{k^4} \quad \sim \quad \alpha_s \ln Q^2/m^2$$

Logarithmic integrals are specific in that there is no dominant integration region — the answer comes from contributions of all momenta, both large ($\sim Q$) and small ($\sim m$).

This situation is similar to what we have seen in the operator product expansion.
Thus we need to make an explicit separation by introducing an intermediate *factorization scale*, schematically

$$\ln \frac{Q^2}{m^2} \;=\; \ln \frac{Q^2}{\mu_F^2} + \ln \frac{\mu_F^2}{m^2}\,, \qquad\qquad Q^2 \gg \mu_F^2 \gg m^2 \; (\Lambda^2_{\rm QCD}) \tag{10.3}$$

and rewrite, in the sum with the LO diagram

$$1 + a\alpha_s \ln \frac{Q^2}{m^2} + \ldots = \underbrace{\left(1 + a\alpha_s \ln \frac{Q^2}{\mu_F^2}\right)}_{e^-q \text{ cross section}} \underbrace{\left(1 + a\alpha_s \ln \frac{\mu_F^2}{m^2}\right)}_{\text{parton distribution}} + \ldots \tag{10.4}$$

In this way, the space-time picture of the parton model is recovered at the cost that the parton cross sections and parton distributions become *scale dependent*:

$$d\sigma_{eN} = \sum_f \int_0^1 d\xi\, F_f(\xi, \mu_F)\, d\sigma_{ef}(\mu_f) \tag{10.5}$$

or

$$F_{1,2}(x_B, Q^2) \;=\; \sum_f e_f^2 \int d\xi\, c_{1,2}(x_B, \xi, \frac{Q^2}{\mu_f^2}) F_f(\xi, \mu_f) \tag{10.6}$$

The scale-dependence of parton distributions can be studied using renormalization group.
After this is done, we can choose $\mu_f^2 = Q^2$ so that the coefficient functions will be calculable as a series in $\alpha_s(Q^2)$ without large logarithms.
To the leading order we will obtain e.g.

$$F_1(x_B, Q^2) = \frac{4}{9} u(x_B, Q^2) + \frac{1}{9} d(x_B, Q^2) + \dots \tag{10.7}$$

Thus we will develop a systematic approach:

- We predict that Bjorken scaling is violated by (small) logarithmic effects
- We can calculate the Q^2 dependence of the structure functions to arbitrary accuracy in perturbation theory (in principle)
- We can generalize this method to other processes, e.g. Drell-Yan or Higgs production

10.1 Leading-order calculation

As I discussed already in connection with vacuum condensates, the scale separation can be introduced at the level of quantum fields in the Lagrangian:

Separate formally all field operators in "fast" and "slow" components:

$$\phi(x) = \left(\int\limits_{|k|>\mu} \frac{d^3k}{(2\pi)^3 2E_k} + \int\limits_{|k|<\mu} \frac{d^3k}{(2\pi)^3 2E_k} \right) \left[\hat{a}^\dagger(k) e^{ikx} + \hat{a}(k) e^{-ikx} \right] \;=\; \phi_{\text{fast}}(x) + \phi_{\text{slow}}(x) \tag{10.8}$$

so for quarks and gluons

$$\psi(x) = \psi_{\text{fast}}(x) + \psi_{\text{slow}}(x)\,, \qquad A^\mu(x) = A^\mu_{\text{fast}}(x) + A^\mu_{\text{slow}}(x) \tag{10.9}$$

Fast and slow fields are often referred to as "quantum" and "classical" and can be viewed as independent fields in the Lagrangian:

$$\psi_q(x) \equiv \psi_{\text{fast}}(x)\,, \qquad \psi_c(x) \equiv \psi_{\text{slow}}(x) \tag{10.10}$$

For gauge fields one requires

$$\begin{aligned} A^\mu_q(x) &\longrightarrow V(x) A^\mu_q(x) V^\dagger(x) \\ A^\mu_c(x) &\longrightarrow V(x) \left(A^\mu_c(x) + \frac{i}{g} \partial_\mu \right) V^\dagger(x) \end{aligned} \tag{10.11}$$

under gauge transformations. This has two advantages:

- Renormalization is simplified because A^μ_q transforms homogeneously ($\to gA^\mu_q$ is not renormalized)
- If the integrals over all "quantum" fields are taken (loops), the result must be gauge-invariant under the transformations of "classical" fields. Hence one can use different gauge conditions for "quantum" and "classical" fields.

Propagators of quantum fields are formally defined with an IR cutoff, e.g. in Feynman gauge

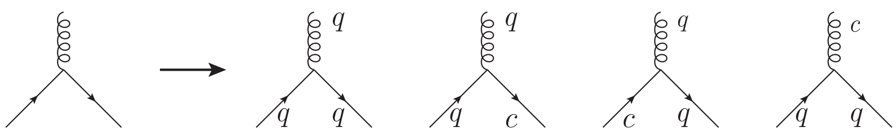
$$\langle 0|\,\mathrm{T}\,\{A^\mu_q(x)A^\nu_q(0)\}|0\rangle = \int\limits_{|k|>\mu} \frac{d^4k}{(2\pi)^4}e^{-ikx}\frac{ig^{\mu\nu}}{k^2+i\epsilon} \tag{10.12}$$

but the cutoff dependence only matters if the integrals are IR divergent (and in this case can be regularized dimensionally)

Classical and quantum fields are orthogonal:

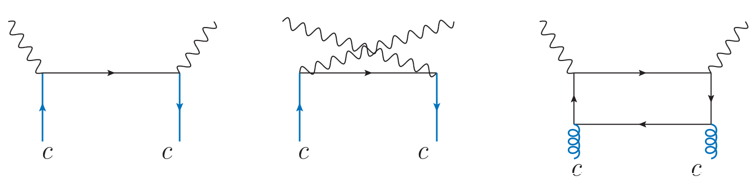
$$\langle 0|\,\mathrm{T}\,\{A^\mu_q(x)A^\nu_c(0)\}|0\rangle = 0 \tag{10.13}$$

and all interaction vertices must contain at least two quantum fields



Let us do the leading-order calculation using this logic (we expect to reproduce the parton model). L24

$$\begin{aligned}
T_{\mu\nu} &= i\int d^4x\, e^{iqx}\langle N|\,\mathrm{T}\,\{\overline{\psi}(x)\gamma_\mu\psi(x)\overline{\psi}(0)\gamma_\nu\psi(0))|\}|N\rangle \\
&= i\int d^4x\, e^{iqx}\langle N|\,\mathrm{T}\,\{\overline{\psi}_c(x)\gamma_\mu\psi_q(x)\overline{\psi}_q(0)\gamma_\nu\psi_c(0))|\}|N\rangle \\
&\quad + i\int d^4x\, e^{iqx}\langle N|\,\mathrm{T}\,\{\overline{\psi}_q(x)\gamma_\mu\psi_c(x)\overline{\psi}_c(0)\gamma_\nu\psi_q(0))|\}|N\rangle \\
&\quad + i\int d^4x\, e^{iqx}\langle N|\,\mathrm{T}\,\{\overline{\psi}_q(x)\gamma_\mu\psi_q(x)\overline{\psi}_q(0)\gamma_\nu\psi_q(0))|\}|N\rangle + \ldots
\end{aligned} \tag{10.14}$$



(massless) quark propagator in coordinate space:

$$\langle 0|\,\mathrm{T}\,\{\psi_q(x)\overline{\psi}_q(0)\}|0\rangle = \int\limits_{|k|>\mu} \frac{d^4p}{(2\pi)^4}e^{-ipx}\frac{i\not{p}}{p^2+i\epsilon} \simeq \frac{i}{2\pi^2}\frac{\not{x}}{[-x^2+i\epsilon]^2} \tag{10.15}$$

Thus to LO

$$T_{\mu\nu} = i\frac{i}{2\pi^2}\int d^4x\,\frac{e^{iqx}}{x^4}\langle N|\Big[\bar\psi_c(x)\gamma_\mu \not{x}\gamma_\nu\psi_c(0) - \bar\psi_c(0)\gamma_\nu\not{x}\gamma_\mu\psi_c(x)\Big]|N\rangle \tag{10.16}$$

In what follows we will omit the subscript "classical"; just remember that all matrix elements include small frequencies, less than μ.

- First, use

$$\gamma_\mu\not{x}\gamma_\nu = x_\mu\gamma_\nu + x_\nu\gamma_\mu - g_{\mu\nu}\not{x} + i\epsilon_{\mu\rho\nu\sigma}x^\rho\gamma_5\gamma^\sigma \tag{10.17}$$

The matrix element of $\gamma_5\gamma^\sigma$ must be a pseudovector

$$\langle N|\bar\psi(x)\gamma_5\gamma^\sigma\psi(0)|N\rangle \;\sim\; S^\rho \qquad \text{[nucleon spin vector]}$$

For unpolarized nucleons

$$\frac{1}{2}\sum_s\langle N(p,s)|\ldots|N(p,s)\rangle$$

such matrix elements vanish

- Second, the relevant matrix elements have the following general structure:

$$\langle N|\bar\psi(x)\gamma^\sigma\psi(0)|N\rangle \;\sim\; p^\sigma f_1(px,x^2) + x^\sigma f_2(px,x^2) \tag{10.18}$$

If $Q^2\to\infty$ (Bjorken limit) the second structure and also terms in the expansion of $f_1(px,x^2)$ in powers of x^2 produce corrections $\sim 1/Q^2$ and can be neglected

Indeed, one can easily derive

$$\int d^4x\,\frac{e^{iqx}}{x^4}x_\alpha = 2\pi^2\frac{q_\alpha}{q^2}$$

$$\int d^4x\,\frac{e^{iqx}}{x^2}x_\alpha = 8\pi^2\frac{q_\alpha}{q^4} \qquad \longleftarrow \quad \text{note extra } 1/q^2 \tag{10.19}$$

If $x^2=0$ the MEs are functions of one variable (px) and can be written in the form of a Fourier integral

$$\langle N|\bar\psi(x)\gamma^\sigma\psi(0)|N\rangle\Big|_{x^2=0} = 2p^\sigma\int_{-1}^{1}du\,e^{iupx}F(u)$$

$$\langle N|\bar\psi(0)\gamma^\sigma\psi(x)|N\rangle\Big|_{x^2=0} = \langle N|\bar\psi(-x)\gamma^\sigma\psi(0)|N\rangle\Big|_{x^2=0} \;=\; 2p^\sigma\int_{-1}^{1}du\,e^{-iupx}F(u) \tag{10.20}$$

[why $\int_{-1}^1 du$: the function $F(u)$ vanishes outside of $u\in(-1,1)$; will motivate later; formal proof complicated]

We obtain:

$$
\begin{aligned}
T_{\mu\nu} &= -\frac{1}{2\pi^2}2\int\frac{d^4x}{x^4}\int_{-1}^{1}du\,F(u)\Big\{e^{iupx+iqx}\Big[x_\mu p_\nu+x_\nu p_\mu-g_{\mu\nu}px\Big]-e^{-iupx+iqx}\Big[x_\mu p_\nu+x_\nu p_\mu-g_{\mu\nu}px\Big]\Big\} \\
&= -2\int_{-1}^{1}du\,F(u)\Big\{\frac{1}{(q+up)^2+i\epsilon}\Big[(q+up)_\mu p_\nu+(q+up)_\nu p_\mu-g_{\mu\nu}p\cdot(q+up)\Big] \\
&\qquad -\frac{1}{(q-up)^2+i\epsilon}\Big[(q-up)_\mu p_\nu+(q-up)_\nu p_\mu-g_{\mu\nu}p\cdot(q-up)\Big]\Big\} \qquad (10.21)
\end{aligned}
$$

Let $p^2=m_N^2\to 0$, then

$$
\begin{aligned}
(q+up)^2 &= q^2+2uqp=-Q^2+2uqp=2qp(u-x_B) \\
(q-up)^2 &= 2qp(-u-x_B) \qquad (10.22)
\end{aligned}
$$

so that

$$
\begin{aligned}
T_{\mu\nu} &= -2\frac{1}{2qp}\int_{-1}^{1}du\,F(u)\Big\{\frac{1}{u-x_B+i\epsilon}\Big[(q+up)_\mu p_\nu+(q+up)_\nu p_\mu-g_{\mu\nu}pq\Big] \\
&\qquad -\frac{1}{-u-x_B+i\epsilon}\Big[(q-up)_\mu p_\nu+(q-up)_\nu p_\mu-g_{\mu\nu}pq\Big]\Big\} \qquad (10.23)
\end{aligned}
$$

Next

$$
\mathrm{Im}\,\frac{1}{u-x_B+i\epsilon}=-\pi\delta(u-x_B)\,,\qquad \mathrm{Im}\,\frac{1}{-u-x_B+i\epsilon}=-\pi\delta(u+x_B) \qquad (10.24)
$$

so that

$$
\begin{aligned}
\frac{1}{\pi}\mathrm{Im}\,T_{\mu\nu} &= \frac{1}{qp}\Big\{F(x_B)\Big[(q+x_Bp)_\mu p_\nu+(q+x_Bp)_\nu p_\mu-g_{\mu\nu}qp\Big] \\
&\qquad -F(-x_B)\Big[(q+x_Bp)_\mu p_\nu+(q+x_Bp)_\nu p_\mu-g_{\mu\nu}qp\Big]\Big\} \qquad (10.25)
\end{aligned}
$$

Finally, we can rewrite

$$
(q+x_Bp)_\mu p_\nu+(q+x_Bp)_\nu p_\mu-g_{\mu\nu}qp=pq\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)+2x_B\left(p_\mu+q_\mu\frac{1}{2x_B}\right)\left(p_\nu+q_\nu\frac{1}{2x_B}\right) \qquad (10.26)
$$

and thus

$$
\boxed{\frac{1}{\pi}\mathrm{Im}\,T_{\mu\nu} = \Big[F(x_B)-F(-x_B)\Big]\Big\{\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)+\frac{2x_B}{pq}\left(p_\mu+q_\mu\frac{1}{2x_B}\right)\left(p_\nu+q_\nu\frac{1}{2x_B}\right)\Big\}} \qquad (10.27)
$$

or

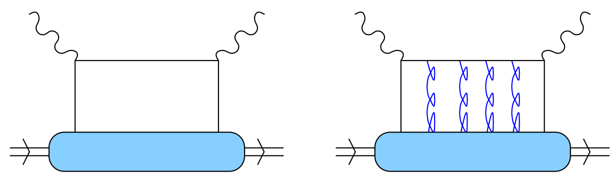
$$\begin{aligned} F_1(x_B, Q^2) &= F(x_B) - F(-x_B) \\ F_2(x_B, Q^2) &= x_B F_1(x_B, Q^2) \end{aligned} \tag{10.28}$$

thus reproducing the parton model, with the identification

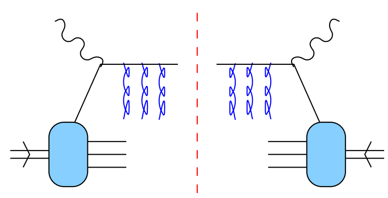
$$\begin{aligned} F(x_B) &= F_q(x_B): \qquad \text{quark distribution} \\ -F(-x_B) &= F_{\bar{q}}(x_B): \qquad \text{antiquark distribution} \end{aligned} \tag{10.29}$$

Thus:

- We are on the right track, but:
- Our definition of parton distributions is not gauge invariant. Result gauge dependent?
- We used free quark propagator in this calculation. In reality the quark propagates inside the nucleon and could interact with "clasical" fields theerein. Can we neglect this interaction or not?

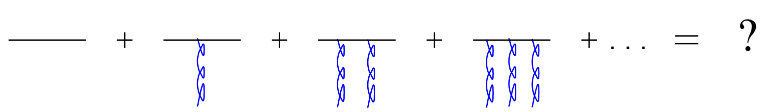


Note that this is a reformulation of the question whether we can neglect the final state interaction



10.2 Light-cone expansion of the quark propagator in the background field

Quark propagator in the background field

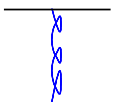


Light-cone expansion:

$$S(x) = \frac{1}{x^4} S_1(x, A_\mu) + \frac{1}{x^2} S_2(x, A_\mu) + \ln x^2 S_3(x, A_\mu) + \dots \tag{10.30}$$

Only the most singular term $\sim 1/x^4$ is relevant in the present context

$$= \psi(x)\bar{\psi}(0) \;=\; \frac{i}{2\pi^2}\frac{\not{x}}{x^4}$$



$$= \psi(x) ig \int d^4z\, \bar{\psi}(z) A_\mu(z)\gamma^\mu \psi(z)\bar{\psi}(0)$$

$$= ig\left(\frac{i}{2\pi^2}\right)^2 \int d^4z\, \frac{\not{x}-\not{z}}{(x-z)^4} A_\mu(z)\gamma^\mu \frac{\not{z}}{z^4} \qquad \boxed{\bar{u} = 1-u}$$

$$= -\frac{ig}{4\pi^4}\int_0^1 du\, u(1-u) \int d^4z\, A_\mu(z)(\not{x}-\not{z})\gamma^\mu \not{z}\frac{\Gamma(4)}{[(z-ux)^2+u\bar{u}x^2]^4}$$

$$= -\frac{ig}{4\pi^4}\int_0^1 du\, u\bar{u} \int d^4z\, A_\mu(z+ux)(\bar{u}\not{x}-\not{z})\gamma^\mu(\not{z}+u\not{x})\frac{\Gamma(4)}{[z^2+u\bar{u}x^2]^4} \tag{10.31}$$

Since there is no dependence on $z \cdot x$ in the denominator, the expansion of the field L25

$$A_\mu(z+ux) \;=\; A_\mu(ux) + z^\nu \partial_\nu A_\mu(ux) + \ldots \tag{10.32}$$

will produce subleading terms that are less singular at $x^2 \to 0$.

We need

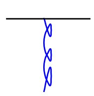
$$\int d^4z \frac{\Gamma(4)}{[z^2+A]^4} \;=\; -i\pi^2\frac{1}{A^2}$$

$$\int d^4z \frac{\Gamma(4)}{[z^2+A]^4} z_\alpha z_\beta \;=\; -i\pi^2\left(\frac{1}{2}g_{\alpha\beta}\right)\frac{1}{A} \tag{10.33}$$

After some algebra one obtains

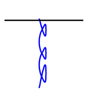
$$\ldots = -\frac{g}{4\pi^2}\int_0^1 du\left\{\frac{2x_\mu}{x^4}A_\mu(ux)\not{x} - \frac{1}{2x^2}\Big[\partial^\alpha A^\beta(ux)\Big(u\gamma_\alpha\gamma_\beta\not{x} - \bar{u}\not{x}\gamma_\beta\gamma_\alpha\Big) + x_\xi u\bar{u}\not{x}\partial^2 A^\xi(ux)\Big]\right\} + \mathcal{O}(\ln x^2) \tag{10.34}$$

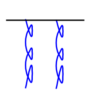
Here we only need the first term:



$$+ \;=\; \frac{i\not{x}}{2\pi^2 x^4}\left[1 + ig\int_0^1 du\, x_\mu A^\mu(ux)\right] \tag{10.35}$$

The leading contributions are easy to calculate to all orders in the field. One obtains





$$+ \;+\; +\; \ldots \;=\; \frac{i\not{x}}{2\pi^2 x^4}\mathrm{Pexp}\left[ig\int_0^1 du\, x_\mu A^\mu(ux)\right] \tag{10.36}$$

where

$$\mathrm{Pexp}\Big[**\Big] = 1 + ig\int_0^1 du\, x_\mu A^\mu(ux) + (ig)^2\int_0^1 du \int_0^u dv\, x_\mu A^\mu(ux) x_\nu A^\nu(vx) + \ldots \tag{10.37}$$

The fields cannot be interchanged because

$$A_\mu \equiv A_\mu^a t^a = A_\mu^a \frac{\lambda^a}{2} \qquad 3\times 3 \,\text{matrix in color space} \tag{10.38}$$

[full analogy with time-ordered exponent in QM time-dependent perturbation theory]

A convenient shorthand notation:

$$[x,y] = \text{Pexp}\left[ig \int_0^1 du\,(x-y)_\mu A^\mu(ux+\bar{u}y)\right] \qquad \bar{u}=1-u \tag{10.39}$$

Thus we obtain the quark propagator in the background gluon field (Gross, Treiman '71)

$$\boxed{S(x) = \psi_q(x)\bar{\psi}_q(0) = \frac{i\not{x}}{2\pi^2 x^4}[x,0]_c + \mathcal{O}\left(\frac{1}{x^2}\right)} \tag{10.40}$$

10.3 Parton model revisited

Now go back to the calculation of the DIS cross section. The first contribution becomes

$$\begin{aligned} T_{\mu\nu} &= i\int d^4x\, e^{iqx}\langle N|\,\mathrm{T}\,\{\bar{\psi}_c(x)\gamma_\mu\psi_q(x)\bar{\psi}_q(0)\gamma_\nu\psi_c(0))|\}|N\rangle + \ldots \\ &= i\frac{i}{2\pi^2}\int d^4x\,\frac{e^{iqx}}{x^4}\langle N|\bar{\psi}_c(x)\gamma_\mu\not{x}\gamma_\nu[x,0]_c\psi_c(0)|N\rangle \end{aligned} \tag{10.41}$$

The calculation remains the same, but the operators become decorated with a gauge-link factor (called Wilson line) connecting the quark fields. Thus a better definition of the parton distribution is

$$\boxed{\langle N(p)|\bar{\psi}(x)\gamma^\sigma[x,0]\psi(0)|N(p)\rangle\Big|_{x^2=0} = 2p^\sigma\int_{-1}^{1} du\, e^{iupx}F(u)} \tag{10.42}$$

Let us check that the operator on the l.h.s. is gauge invariant. For simplicity I consider abelian gauge trafos (like in QED):

$$\bar{\psi}(x)\gamma^\sigma e^{ig\int_0^1 du\, x_\mu A^\mu(ux)}\psi(0) \longrightarrow \bar{\psi}(x)\gamma^\sigma e^{-i\alpha(x)}e^{ig\int_0^1 du\, x_\mu(A^\mu(ux)+\frac{1}{g}\partial^\mu\alpha(ux))}e^{i\alpha(0)}\psi(0) \tag{10.43}$$

Thus the extra term under the exponent is

$$-i\alpha(x) + i\alpha(0) + i\int_0^1 du\, x^\mu\partial_\mu\alpha(ux) \tag{10.44}$$

Euler's Homogeneous Function Theorem implies

$$x^\mu\frac{\partial}{\partial x^\mu}\alpha(ux) = u\frac{d}{du}\alpha(ux) \tag{10.45}$$

Therefore

$$\int_0^1 du\, x^\mu \partial_\mu \alpha(ux) = \int_0^1 du\, x^\mu \frac{\partial}{\partial(ux^\mu)}\alpha(ux) = \int_0^1 du\, \frac{1}{u} u \frac{d}{du}\alpha(ux) = \alpha(x) - \alpha(0) \tag{10.46}$$

and all α-dependent terms cancel as they should

Another possibility to see gauge invariance: Let $x \to 0$ (all components); then

$$\begin{aligned} \bar\psi(x) &= \bar\psi(0)[1 + x^\mu \overleftarrow{\partial}_\mu + \ldots] \\ [x,0] &= 1 + ig\int_0^1 du\, x_\mu A^\mu(ux) + \ldots = 1 + igx_\mu A^\mu(0) + \ldots \end{aligned} \tag{10.47}$$

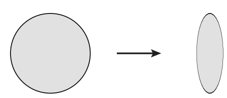
Thus to first order

$$\bar\psi(x)\Gamma[x,0]\psi(0) = \bar\psi(0)\Gamma\psi(0) + x^\mu\bar\psi(0)\Gamma[\overleftarrow{\partial}_\mu + igA_\mu]\psi(0) = \bar\psi(0)\Gamma\psi(0) + x^\mu\bar\psi(0)\Gamma \overleftarrow{D}_\mu \psi(0) \tag{10.48}$$

— the expansion of nonlocal operators with Wilson lines goes over covariant derivatives. This is true to all orders:

$$\bar\psi(x)\Gamma[x,0]\psi(0) = \sum_n \frac{1}{n!} x_{\mu_1}\ldots x_{\mu_n}\bar\psi(0)\Gamma \overleftarrow{D}_{\mu_1}\ldots \overleftarrow{D}_{\mu_n}\psi(0) \tag{10.49}$$

The parton model is largerly based on intuition that fast moving hadrons are Lorentz-contracted, similar to what we know about rigid bodies. But is it really obvious?



Example I: Assume that a proton can be modelled as a point-like charge which is a source of a scalar potential (Yukawa with mass zero)

$$\phi(\vec x) = \frac{q}{|\vec x|} \qquad \longleftarrow \quad \text{for proton at rest} \tag{10.50}$$

The field is certainly spherically symmetric in this frame.
Making a Lorentz transformation (boost along the x_3-direction) we obtain

$$\phi'(x') = \frac{q}{[x_\perp^2 + \gamma^2(vt' - x_3')^2]^{1/2}} \qquad x_\perp^2 = x_1^2 + x_2^2 \tag{10.51}$$

where $\gamma = 1/\sqrt{1-v^2}$. As $\gamma \to \infty$ the field vanishes as $1/\gamma$ except for a narrow strip (pancake) around $x_3' \simeq vt'$, as expected.

Example II: A Coulomb potential in Eigenkoordinatensystem (QED)

$$A_\mu(\vec x) = \frac{q\delta_{\mu 0}}{|\vec x|} \qquad \longleftarrow \quad \text{for proton at rest} \tag{10.52}$$

Then

$$
\begin{aligned}
A'_0(x') &= \frac{q\gamma}{[x_\perp^2 + \gamma^2(vt' - x'_3)^2]^{1/2}} \\
A'_3(x') &= \frac{-qv\gamma}{[x_\perp^2 + \gamma^2(vt' - x'_3)^2]^{1/2}} \\
A'_\perp(x') &= 0
\end{aligned}
\tag{10.53}
$$

In this case for $\gamma \to \infty$ both $A'_0(x')$ and $A'_3(x') \sim \mathit{const}(?!)$ Where is my pancake???

The resolution of this paradox is that

$$
\text{for} \quad \gamma \to \infty \qquad\qquad A'_\mu(x') \sim q\partial_\mu \ln(vt' - x'_3)
$$

— can be removed by a gauge transformation. Thus, although the four-potential is *not* Lorentz-contracted, the electric and magnetic fields are contracted to a pancake, e.g.

$$
E'_3(x') = \frac{-q\gamma(vt' - x'_3)}{[x_\perp^2 + \gamma^2(vt' - x'_3)^2]^{3/2}} \tag{10.54}
$$

The parton model is recovered in a certain gauge where the Wilson line can be neglected

$$
x^\mu A_\mu(ux) = 0 \tag{10.55}
$$

where x^μ is the 4-vector in the direction of the outgoing (struck) quark.

$$
\begin{aligned}
&\text{light-cone gauge} \qquad && n^\mu A_\mu(x) = 0\,, \qquad && n^2 = 0 \\
&\text{Fock-Schwinger gauge} \qquad && x^\mu A_\mu(x) = 0\,, \qquad && A_\mu(0) = 0
\end{aligned}
\tag{10.56}
$$

10.4 The DGLAP Evolution equation

10.4.1 Preliminary remarks

Let us look at our definition of the parton distribution more closely. We have

$$
\langle N(p)|\bar\psi_c(x)\gamma^\sigma[x,0]\psi_c(0)|N(p)\rangle\Big|_{x^2=0} \;=\; 2p^\sigma \int_{-1}^{1} du\, e^{iupx} F(u,\mu) \tag{10.57}
$$

where I restored the subscript "classical" and added the argument μ to $F(u,\mu)$ to remind that only low-frequency part of the fields is included. The role of the cutoff is that it allows the limit $x^2 \to 0$ to be taken.

Consider the similar matrix element with "full" fields and finite x^2: L26

$$
\langle N(p)|\bar\psi(x)\gamma^\sigma[x,0]\psi(0)|N(p)\rangle\Big|_{x^2p^2\ll 1} \;=\; 2p^\sigma \int_{-1}^{1} du\, e^{iupx} \tilde F(u,x^2) \tag{10.58}
$$

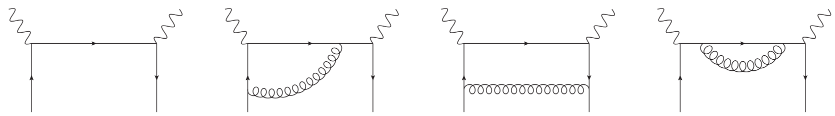
The only UV divergences in this ME are due to field and coupling renormalization: adding $\sqrt{Z_2}$ for each quark field and expressing the result in terms of the renormalized coupling the result is finite. However, the limit $x^2 \to 0$ cannot be taken. Schematically

$$
\langle N(p)|\bar\psi(x)\gamma^\sigma[x,0]\psi(0)|N(p)\rangle\Big|_{x^2p^2\ll 1} \sim 1 + c\alpha_s \ln\frac{1}{x^2p^2} + \dots \tag{10.59}
$$

The role of the cutoff in the fields is that it regularizes additional UV divergences that appear in the $x^2 \to 0$ limit:

$$\langle N(p)|\bar{\psi}_c(x)\gamma^\sigma[x,0]\psi_c(0)|N(p)\rangle\Big|_{x^2=0} \sim 1 + c\alpha_s \ln\frac{\mu^2}{p^2} + \ldots \tag{10.60}$$

The dependence on μ will be compensated by the similar dependence of the Feynman diagrams for $T_{\mu\nu}$ in higher orders:



These diagrams (for quark legs on-shell) would be IR divergent if we forget that only frequencies above μ are included in "quantum" fields. The sum is schematically

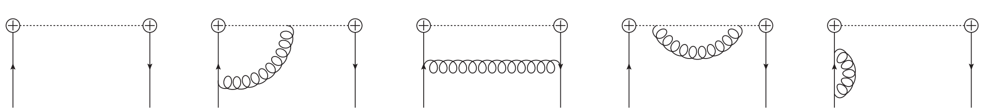
$$1 + c\alpha_s \ln\frac{1}{\mu^2 x^2} \qquad \stackrel{\text{Fourier}}{\longrightarrow} \qquad 1 + c\alpha_s \ln\frac{Q^2}{\mu^2} \tag{10.61}$$

In order to avoid large logarithms in the coefficient functions we should set the scale $\mu \sim Q$. Therefore our leading-order result to the structure functions actually involves parton distributions at the scale $\sim Q$

$$F_1(x_B, Q^2) = F(x_B, \mu = Q) - F(-x_B, \mu = Q) = \frac{4}{9}u(x,Q) + \frac{4}{9}\bar{u}(x,Q) + \frac{1}{9}d(x,Q) + \frac{1}{9}\bar{d}(x,Q) + \ldots \tag{10.62}$$

and we expect that Bjorken scaling will be violated by logarithmic corrections corresponding to the scale-dependence of parton distributions.

To calculate this dependence, split "slow" fields that appear in the operator matrix elements in
— "moderately slow" , with frequencies $\mu_0 < |k| < \mu$
— "very slow", with frequencies $|k| < \mu_0$
We can treat "moderately slow" fields as "quantum", that is involved the loops and "very slow" as "classical" external fields



Integrating over the "moderately slow" fields we obtain operator matrix elements of "very slow" operators corresponding to parton distributions at the scale μ_0:

$$[\text{operator at the scale}\,\mu] \sim (1 + c\alpha_s \ln\frac{\mu}{\mu_0})\,[\text{operator at the scale}\,\mu_0] \tag{10.63}$$

and obtain the finite-difference equation

$$[\text{operator at the scale}\,\mu] - [\text{operator at the scale}\,\mu_0] \sim c\alpha_s \ln\frac{\mu}{\mu_0}\,[\text{operator at the scale}\,\mu_0] \tag{10.64}$$

which we will rewrite as a differential renormalization group equation which is our goal.

A technicality:
as we know, a calculation with momentum cutoffs is very awkward. One scale can be introduced via the dimensional regularization but for the second scale we need another regulator.

A convenient choice: use
— finite x^2 as a substitute for $\mu^2 \sim 1/|x^2|$ (larger scale);
— dimensional regularization for μ_0^2 (smaller scale, thus an IR divergence) and, simultaneously, the remaining UV divergences (coupling and field renormalization)

[One can prove that both IR and UV divergences can be regularized dimensionally]

10.4.2 One-loop calculation

To avoid an open vector index consider the operator multiplied by extra x^σ so that

$$\langle N(p)|\bar{\psi}(x)\not{x}[x,0]\psi(0)|N(p)\rangle \;=\; 2(px)\int_{-1}^{1} du\, e^{iupx} F(u) + \mathcal{O}(x^2) \tag{10.65}$$

Let us calculate the diagram corresponding to gluon emission from the Pexp.
We write

$$A^\mu(z) = A^\mu_q(z) + A^\mu_c(z) \tag{10.66}$$

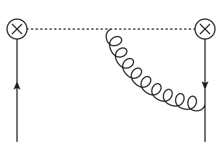
and consider the first term in the expansion of Pexp[...] in the "quantum" gluon field

$$[x,0] = [x,0]_c + ig\int_0^1 du\,[x,ux]_c x_\mu A^\mu_q(ux)[ux,0]_c \tag{10.67}$$

We choose

$$\begin{array}{lll} A^\mu_q: & \text{Feynman gauge} & \\ A^\mu_c: & \text{Fock-Schwinger gauge} & [x,0]_c = [x,ux]_c = [ux,0]_c = 1 \end{array} \tag{10.68}$$

Then



$$\longleftrightarrow \quad \overline{\psi}_c(x) ig \int_0^1 du\; x_\mu \overbrace{A^\mu_q(ux)\not{x}\overbrace{\psi_q(0)\Big[ig\int d^dz\,\overline{\psi}_q}(z)\not{A}_q}(z)\psi_c(z)\Big] \tag{10.69}$$

Use

$$\overbrace{\psi_a(z)\overline{\psi}_b}(0) \;=\; \delta_{ab}\frac{i}{2\pi^{d/2}}\not{z}\frac{\Gamma(d/2)}{[-z^2+i\epsilon]^{d/2}}$$

$$\overbrace{A^A_\mu(ux)A^B_\nu}(z) \;=\; -\delta_{AB}g_{\mu\nu}\frac{1}{4\pi^{d/2}}\frac{\Gamma(d/2-1)}{[-(ux-z)^2+i\epsilon]^{d/2-1}} \tag{10.70}$$

leading to

$$\ldots = (ig)^2 \int_0^1 du \int d^d z \bar{\psi}_c(x) \not{x} \frac{i}{2\pi^{d/2}} \frac{-1}{4\pi^{d/2}} t^A (-\not{z}) \frac{\Gamma(d/2)}{[-z^2]^{d/2}} \frac{\Gamma(d/2-1)}{[-(ux-z)^2]^{d/2-1}} \not{x} t^A \psi_c(z) \quad (10.71)$$

Next, combine the propagators

$$\frac{\Gamma(d/2)}{[-z^2]^{d/2}} \frac{\Gamma(d/2-1)}{[-(ux-z)^2]^{d/2-1}} = \int_0^1 dv\, v^{d/2-2} \bar{v}^{d/2-1} \frac{\Gamma(d-1)}{[-(z-uvx)^2 - x^2u^2\bar{v}v]^{d-1}} \quad (10.72)$$

and shift the integration variable $z \to z + uvx$ to obtain

$$\ldots = \frac{-ig^2}{8\pi^d} C_F \int_0^1 du \int_0^1 dv \int d^d z \frac{v^{d/2-2}\bar{v}^{d/2-1}\Gamma(d-1)}{[-z^2 - x^2u^2\bar{v}v]^{d-1}} \bar{\psi}_c(x) \not{x} (\not{z} + uv\not{x}) \not{x} \psi_c(z+uvx) \quad (10.73)$$

We are looking for the contributions of the type

$$\ln \frac{1}{x^2\mu^2} \quad \longleftarrow \quad \frac{\Gamma(d/2-2)}{[x^2\mu^2]^{d/2-2}}, \qquad g^2 \to g^2 \mu^{4-d} \quad (10.74)$$

Note:

- do not need terms with positive powers of x^2 $\Rightarrow$ can expand the field

$$\psi_c(z+uvx) = \psi(uvx) + z_\alpha \partial^\alpha \psi(uvx) + \ldots$$

- argument of the Gamma-function: $\Gamma(d/2-2)$ stands for IR divergence, $\Gamma(2-d/2)$ stands for UV divergence.

Why:

$$\int \frac{d^d k}{k^4} = ? \qquad \leftarrow \begin{cases} \text{choose d} < 4 \text{ to suppress the large-k region (UV)} \\ \text{choose d} > 4 \text{ to suppress the small-k region (IR)} \end{cases} \quad (10.75)$$

One can regularize IR or UV divergence "by hand" so that only the other one remains:

$$\begin{aligned} \int \frac{d^d k}{(k^2+a^2)^2} &= i\pi^{d/2} \frac{\Gamma(2-d/2)}{[-a^2]^{2-d/2}} \qquad \text{finite for } d < 4 \\ \int \frac{d^d k}{k^4} e^{2ika} &= i\pi^{d/2} \frac{\Gamma(d/2-2)}{[-a^2]^{d/2-2}} \qquad \text{finite for } d > 4 \end{aligned} \quad (10.76)$$

In general situation the identification — UV or IR — is not simple, one needs to consider each case separately.

Using generic integrals

$$\begin{aligned} \int dz \frac{\Gamma(\alpha)}{[-z^2-a^2]^\alpha} &= -i\pi^{d/2} \frac{\Gamma(\alpha - d/2)}{[-a^2]^{\alpha-d/2}} \\ \int dz \frac{\Gamma(\alpha)}{[-z^2-a^2]^\alpha} z_\mu z_\nu &= -i\pi^{d/2} \left(-\frac{g_{\mu\nu}}{2}\right) \frac{\Gamma(\alpha-d/2-1)}{[-a^2]^{\alpha-d/2-1}} \end{aligned} \quad (10.77)$$

obtain L27

$$\ldots = \frac{-ig^2}{8\pi^d} C_F(-i\pi^{d/2}) \int_0^1 du \int_0^1 dv\, v^{d/2-2} \bar{v}^{d/2-1} \Big\{ x^2 uv \bar{\psi}_c(x) \not{x} \psi_c(uvx) \frac{\Gamma(d/2-1)}{[-x^2u^2v\bar{v}]^{d/2-1}}$$

$$+ \bar{\psi}_c(x) \not{x} \gamma_\mu \not{x} \partial_\nu \psi_c(uvx) \left(-\frac{g_{\mu\nu}}{2}\right) \frac{\Gamma(d/2-2)}{[-x^2u^2v\bar{v}]^{d/2-2}} \Big\}$$

$$= \frac{g^2 C_F}{8\pi^{d/2}} \int_0^1 du \int_0^1 dv \bar{\psi}_c(x) \Big\{ \frac{\Gamma(d/2-1)}{[-x^2]^{d/2-2}} u^{3-d} \not{x} + \frac{1}{2} \frac{\Gamma(d/2-2)}{[-x^2u^2]^{d/2-2}} \not{x} \gamma_\mu \not{x} \bar{v} \partial^\mu \Big\} \psi_c(uvx) \quad (10.78)$$

The first term:

$$\int_0^1 \frac{du}{u^{d-3}} : \qquad \text{finite for}\, d < 4 \qquad \Rightarrow \qquad \text{UV divergence at}\, u \to 0 \quad (10.79)$$

$\Longrightarrow$ contributes to coupling/field renormalization, irrelevant for us

In the second term can replace

$$\frac{1}{2} \not{x} \gamma_\mu \not{x} \partial^\mu \to \not{x}(x\partial) + \mathcal{O}(x^2)$$

so it becomes

$$\frac{g^2 C_F}{8\pi^{d/2}} \int_0^1 du \int_0^1 dv\, \bar{v} \frac{\Gamma(d/2-2)}{[-x^2u^2]^{d/2-2}} \bar{\psi}_c(x) \not{x}(x\partial) \psi_c(uvx) \quad (10.80)$$

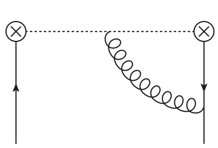
The familiar trick:

$$(x\partial)\psi_c(uvx) = x^\mu \frac{\partial}{\partial(uvx^\mu)} \psi_c(uvx) = \frac{1}{uv} u \frac{d}{du} \psi_c(uvx) = \frac{1}{v} \frac{d}{du} \psi_c(uvx) \quad (10.81)$$

The dependence on u in $[-x^2u^2]^{d/2-2}$ can be neglected to logarithmic accuracy ($\ln 1/(x^2\mu^2)+$ const) leaving us with

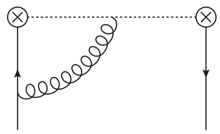
$$\int_0^1 du \frac{d}{du} \psi_c(uvx) = \psi_c(vx) - \psi_c(0) \quad (10.82)$$

Thus the result is

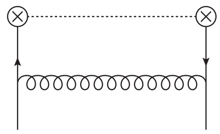


$$\longleftrightarrow \qquad \frac{g^2 C_F}{8\pi^{d/2}} \frac{\Gamma(d/2-2)}{[-x^2\mu_{IR}^2]^{d/2-2}} \int_0^1 dv \frac{\bar{v}}{v} \Big[\bar{\psi}_c(x) \not{x} \psi_c(vx) - \bar{\psi}_c(x) \not{x} \psi_c(0) \Big] \quad (10.83)$$

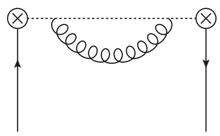
where I restored the IR scale μ_{IR} dependence coming from $g^2 \to g^2\mu^{4-d}$.
Other contributions:



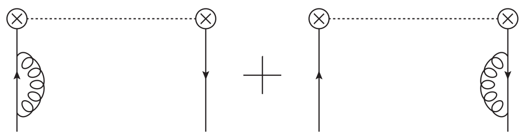
$$\longleftrightarrow \quad \frac{g^2 C_F}{8\pi^{d/2}}\frac{\Gamma(d/2-2)}{[-x^2\mu_{IR}^2]^{d/2-2}}\int_0^1 dv\,\frac{v}{\bar v}\Big[\bar\psi_c(vx)\not x\psi_c(0)-\bar\psi_c(x)\not x\psi_c(0)\Big]$$



$$\longleftrightarrow \quad \frac{g^2 C_F}{8\pi^{d/2}}\frac{\Gamma(d/2-2)}{[-x^2\mu_{IR}^2]^{d/2-2}}\int_0^1 du\int_0^u dv\,\bar\psi_c(ux)\not x\psi_c(vx)$$



$\longleftrightarrow$ only UV divergences, coupling renormalization in Pexp



$$\longleftrightarrow \quad \sqrt{Z_2}\cdot\sqrt{Z_2} \quad \text{field renormalization } |x^2|\to\mu_{IR}^2 \tag{10.84}$$

The final result:

$$\bar\psi(x)\not x\psi(0) = \bar\psi_c(x)\not x\psi_c(0)+\frac{g^2 C_F}{8\pi^{d/2}}\frac{\Gamma(d/2-2)}{[-x^2\mu_{IR}^2]^{d/2-2}}\int_0^1 du\int_0^u dv\,K(u,v)\,\bar\psi_c(ux)\not x\psi_c(vx) \tag{10.85}$$

where

$$K(u,v) = \delta(\bar u)\left[\frac{\bar v}{v}\right]_+ + \delta(v)\left[\frac{u}{\bar u}\right]_+ + 1 - \frac{1}{2}\delta(\bar u)\delta(v) \tag{10.86}$$

The "plus-distribution":

$$\int_0^1 dv\,\left[\frac{\bar v}{v}\right]_+ f(v) := \int_0^1 dv\,\frac{\bar v}{v}[f(v)-f(0)]$$
$$\int_0^1 du\,\left[\frac{u}{\bar u}\right]_+ f(u) := \int_0^1 du\,\frac{u}{\bar u}[f(u)-f(1)] \tag{10.87}$$

It remains a little bit of cosmetics.

- We are only interested in matrix elements between the states with the same momenta. In this case

$$\begin{aligned}\langle N(p)|\bar\psi(ux)\not x\psi(vx)|N(p)\rangle &= \langle N(p)|e^{+i\hat P vx}\bar\psi((u-v)x)\not x\psi(0)e^{-i\hat P vx}|N(p)\rangle\\ &= \langle N(p)|\bar\psi((u-v)x)\not x\psi(0)|N(p)\rangle \end{aligned}\tag{10.88}$$

where $\hat{P}$ is the momentum operator: $\hat{P}_\mu|N(p)\rangle = p_\mu|N(p)\rangle$
Thus e.g.

$$\int_0^1 du \int_0^u dv\, \bar{\psi}((u-v)x)\not{x}\psi(0) = \int_0^1 du \int_0^u dt\, \bar{\psi}(tx)\not{x}\psi(0) = \int_0^1 dt\,(1-t)\bar{\psi}(tx)\not{x}\psi(0) \quad (10.89)$$

- Expanding at $d = 4 - \epsilon$

$$\frac{g^2}{8\pi^{d/2}} C_F \left(-\frac{2}{\epsilon}\right)\left[1 - \frac{2}{\epsilon}\ln\frac{1}{-x^2\mu_{IR}^2}\right] \quad \longrightarrow \quad \frac{\alpha_s}{2\pi} C_F \ln\frac{1}{-x^2\mu_{IR}^2} \quad + \text{const.} \quad (10.90)$$

Remember that the goal of the calculation was to change the definition of the "classical" field from the cutoff $\mu_1^2 \equiv 1/-x^2 \sim Q^2$ to a lower cutoff $\mu_2^2 \equiv \mu_{IR}^2 < \mu_1^2$. Thus we get, for $x^2 = 0$

$$[\bar{\psi}(x)\not{x}\psi(0)]_{\mu_1^2} = [\bar{\psi}(x)\not{x}\psi(0)]_{\mu_2^2} + \frac{\alpha_s C_F}{2\pi}\ln\frac{\mu_1^2}{\mu_2^2}\int_0^1 du\, K(u)\,[\bar{\psi}(ux)\not{x}\psi(0)]_{\mu_2^2} \quad (10.91)$$

with

$$K(u) = 2\left[\frac{u}{\bar{u}}\right]_+ + \bar{u} - \frac{1}{2}\delta(\bar{u}) \quad (10.92)$$

This relation is valid if

$$\alpha_s(\mu_2)\ln\frac{\mu_1^2}{\mu_2^2} \ll 1$$

- The last step, we have to take the matrix element $\langle N|\dots|N\rangle$. Consider quark contribution:

$$\langle N(p)|[\bar{\psi}(x)\not{x}[x,0]\psi(0)]_{\mu^2}|N(p)\rangle \;=\; 2(px)\int_0^1 d\xi\, e^{i\xi px}F(\xi,\mu^2) \quad (10.93)$$

We obtain

$$\cancel{2(px)}\int_0^1 d\xi\, e^{i\xi px}\Big[F(\xi,\mu_1^2) - F(\xi,\mu_2^2)\Big] = \frac{\alpha_s C_F}{2\pi}\ln\frac{\mu_1^2}{\mu_2^2}\int_0^1 du\, K(u)\,\cancel{2(px)}\int_0^1 d\xi\, e^{iu\xi px}F(\xi,\mu_2^2) \quad (10.94)$$

Next, insert

$$1 = \int_0^1 dv\, \delta(v - u\xi)$$

in the integral on the r.h.s.:

$$\int_0^1 du\, K(u)\int_0^1 d\xi\, e^{iu\xi px}F(\xi)\cdot\int_0^1 dv\,\delta(v-u\xi) = \int_0^1 dv\, e^{ivpx}\int_0^1 du\, K(u)\underbrace{\int_0^1 d\xi\, F(\xi)\,\delta(v-u\xi)}_{\frac{1}{u}F\left(\frac{v}{u}\right)\theta(u-v)}$$

$$= \int_0^1 dv\, e^{ivpx}\int_v^1 \frac{du}{u}K(u)\,F\left(\frac{v}{u}\right) \quad (10.95)$$

and finally

$$F_q(v,\mu_1^2) - F_q(v,\mu_2^2) \;=\; \frac{\alpha_s}{2\pi} C_F \ln\frac{\mu_1^2}{\mu_2^2} \int\limits_v^1 \frac{du}{u}\, K(u)\, F_q\Big(\frac{v}{u},\mu_2^2\Big) \tag{10.96}$$

or

$$\boxed{\mu^2 \frac{d}{d\mu^2} F_q(v,\mu^2) \;=\; \frac{\alpha_s(\mu)}{2\pi} C_F \int\limits_v^1 \frac{du}{u}\, K(u)\, F_q\Big(\frac{v}{u},\mu^2\Big)} \tag{10.97}$$

— the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equation

Accepted notation:

$$P_{q\to q}(u) \equiv C_F K(u) \;=\; C_F \left[\frac{1+u^2}{[1-u]_+} + \frac{3}{2}\delta(1-u)\right] \tag{10.98}$$

- We could consider antiquarks instead of quarks, replacing $\int_0^1 d\xi e^{i\xi px} F(\xi) \to \int_{-1}^0 d\xi e^{i\xi px} F(\xi)$. To this order in perturbation theory there will be no difference:

$$P_{\bar q\to\bar q}(u) \;=\; P_{q\to q}(u) \tag{10.99}$$

— quarks and antiquarks have autonomous (and the same) scale dependence to leading order.

10.5 Solution of the DGLAP equation

The DGLAP equation can be solved by numerical integration, but a better approach is to consider *moments*:

$$M_q^N(Q^2) \;=\; \int_0^1 dx_B\, x_B^{N-1} F_q(x_B,Q^2) \tag{10.100}$$

Taking moments

$$\begin{aligned}
\mu^2 \frac{d}{d\mu^2} \int\limits_0^1 dv\, v^{N-1} F_q(v,\mu^2) &= \frac{\alpha_s(\mu)}{2\pi} \int\limits_0^1 dv\, v^{N-1} \int\limits_v^1 \frac{du}{u}\, P_{qq}(u)\, F_q\Big(\frac{v}{u},\mu^2\Big) \\
&= \frac{\alpha_s(\mu)}{2\pi} \int\limits_0^1 du\, P_{qq}(u)\, u^{N-1} \int\limits_0^u \frac{dv}{u} \,\Big(\frac{v}{u}\Big)^{N-1} F_q\Big(\frac{v}{u},\mu^2\Big) \\
&= \frac{\alpha_s(\mu)}{2\pi} \int\limits_0^1 du\, P_{qq}(u)\, u^{N-1} \int\limits_0^1 dt\, t^{N-1} F_q\Big(t,\mu^2\Big)\,; \qquad \boxed{t = \frac{v}{u}}
\end{aligned} \tag{10.101}$$

Thus we obtain

$$\mu^2 \frac{d}{d\mu^2} M_q^N(\mu^2) \;=\; -\frac{\alpha_s(\mu)}{2\pi} \gamma_{qq}^N\, M_q^N(\mu^2) \tag{10.102}$$

where

L28

$$\gamma_{qq}^N = -\int_0^1 du\, P_{qq}(u)\, u^{N-1} = -C_F \int_0^1 du\, u^{N-1} \left\{ 2\left[\frac{u}{\bar{u}}\right]_+ + \bar{u} - \frac{1}{2}\delta(\bar{u}) \right\}$$
$$= -C_F \left\{ 2\int_0^1 du\, \frac{u}{\bar{u}}(u^{N-1}-1) + \left(\frac{1}{N} - \frac{1}{N+1}\right) - \frac{1}{2} \right\} \qquad (10.103)$$

The remaining integral:

$$\int_0^1 du\, \frac{1}{1-u}(u^N - u) = \int_0^1 du\, \frac{1}{1-u}(u^N - 1 + 1 - u) = -\int_0^1 du\, \frac{1-u^N}{1-u} + 1$$
$$= -\int_0^1 du\, (1 + u + \ldots + u^{N-1}) + 1 = -\sum_{j=1}^{N} \frac{1}{j} + 1 = -\sum_{j=2}^{N} \frac{1}{j} \qquad (10.104)$$

We obtain

$$\boxed{\gamma_{qq}^N = \frac{4}{3}\left\{ 2\sum_{j=2}^{N} \frac{1}{j} - \frac{1}{N(N+1)} + \frac{1}{2} \right\} \qquad \text{Gross, Wilczek '73}} \qquad (10.105)$$

The solution of Eq. (10.102) is then

$$M_q^N(Q^2) = \left(\frac{\alpha_s(Q^2)}{\alpha_s(\mu_0^2)}\right)^{2\gamma_{qq}^N/b} M_q^N(\mu_0^2)\,; \qquad b = \frac{11}{3}N_c - \frac{2}{3}n_f \qquad (10.106)$$

Check:

$$Q^2 \frac{d}{dQ^2} M_q^N(Q^2) = \frac{1}{[\alpha_s(\mu_0^2)]^{2\gamma_{qq}^N/b}} M_q^N(\mu_0^2) \frac{2\gamma_{qq}^N}{b} [\alpha_s(Q^2)]^{2\gamma_{qq}^N/b-1} Q^2 \frac{d}{dQ^2}\alpha_s(Q^2) \qquad (10.107)$$

The β-function:

$$\mu \frac{d}{d\mu}\alpha_s(\mu) = \beta(\alpha_s) = -\frac{1}{2\pi} b\alpha_s^2 + \mathcal{O}(\alpha_s^3)$$
$$Q^2 \frac{d}{dQ^2} = \frac{1}{2} Q \frac{d}{dQ} \qquad \longleftarrow \quad \text{extra factor } 1/2 \qquad (10.108)$$

Thus we obtain

$$Q^2 \frac{d}{dQ^2} M_q^N(Q^2) = -\frac{1}{2\pi}\gamma_{qq}^N \alpha_s(Q^2) \cdot \underbrace{\left(\frac{\alpha_s(Q^2)}{\alpha_s(\mu_0^2)}\right)^{2\gamma_{qq}^N/b} M_q^N(\mu_0^2)}_{M_q^N(Q^2)} \qquad \boxed{\text{OK}} \qquad (10.109)$$

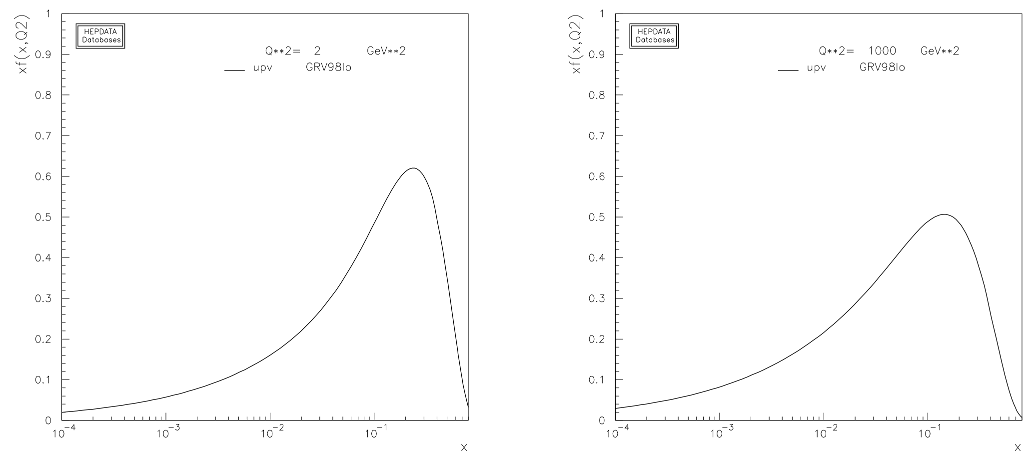
Since $\alpha_s(Q^2)$ decreases with Q^2, the moments $M_q^N(Q^2)$ decrease as well: the quark looses momentum because of gluon radiation

The parton distributions can be reconstructed from the moments using *inverse Mellin transform*

$$F_q(x) = \frac{1}{2\pi i}\int\limits_{1/2-i\infty}^{1/2+i\infty} dN\, x^{-N} M_q^N \tag{10.110}$$

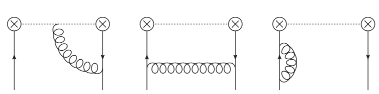
The example below shows the so-called valence u-quark distribution

$$xu_v(x,Q^2) = x\big[u(x,Q^2) - \bar{u}(x,Q^2)\big] \tag{10.111}$$

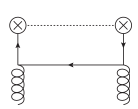


This is, however, not the end of the story. . .

$$[\psi(x)\not{x}\psi(0)]_{\mu_1^2} = [\psi(x)\not{x}\psi(0)]_{\mu_2^2}$$



$$+\frac{\alpha_s}{2\pi}C_F \ln\frac{\mu_1^2}{\mu_2^2}\int\limits_0^1 du\, K_{qq}(u)\, [\bar{\psi}(ux)\not{x}\psi(0)]_{\mu_2^2}$$



$$+\frac{\alpha_s}{2\pi}C_F \ln\frac{\mu_1^2}{\mu_2^2}\int\limits_0^1 du\, K_{qg}(u)\, x^\nu x^\alpha [F_{\mu\nu}(ux)F^\mu{}_\alpha(0)]_{\mu_2^2} \tag{10.112}$$

— our calculation is not complete

10.6 Gluon parton distribution

The definition:

$$x^\mu x^\nu \langle N(p)|[G^A_{\mu\xi}(x)[x,0]_{AB}G^B_{\nu\xi}(0)]_{\mu^2}|N(p)\rangle \overset{x^2=0}{=} 2(px)^2\int\limits_{-1}^{1} du\, e^{iupx} u F_g(u,\mu^2) \tag{10.113}$$

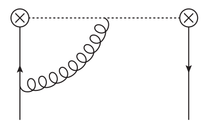
- There exist no "antigluons":

$$F_g(u) = F_g(-u) \tag{10.114}$$

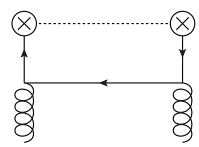
- The gauge link is in adjoint representation:

$$
\begin{aligned}
[x,0] &= \mathrm{Pexp}\left\{ig\int_0^1 du\, x^\mu A_\mu(ux)\right\} \\
A_\mu &= A_\mu^A (T^A)_{BC}\,, \qquad (T^A)_{BC} = -if_{ABC} \qquad \longleftarrow \qquad 8\times 8\,\text{matrix}
\end{aligned}
\tag{10.115}
$$

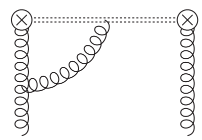
Explicit calculation:



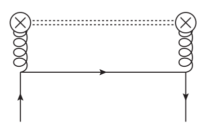
$+\ldots \qquad \longrightarrow \qquad P_{qq}$



$\longrightarrow \qquad P_{qg}$



$+\ldots \qquad \longrightarrow \qquad P_{gg}$



$\longrightarrow \qquad P_{gq}$ (10.116)

Obtain the system of coupled equations:

$$
\begin{aligned}
Q^2\frac{d}{dQ^2}F_g(x,Q^2) &= \frac{\alpha_s(Q)}{2\pi}\int_x^1 \frac{dy}{y}\left\{P_{gq}(y)\sum_q\left[F_q(\frac{x}{y},Q)+F_{\bar{q}}(\frac{x}{y},Q)\right]+P_{gg}(y)F_g(\frac{x}{y},Q)\right\} \\
Q^2\frac{d}{dQ^2}F_q(x,Q^2) &= \frac{\alpha_s(Q)}{2\pi}\int_x^1 \frac{dy}{y}\left\{P_{qq}(y)F_q(\frac{x}{y},Q)+P_{qg}(y)F_g(\frac{x}{y},Q)\right\} \\
Q^2\frac{d}{dQ^2}F_{\bar{q}}(x,Q^2) &= \frac{\alpha_s(Q)}{2\pi}\int_x^1 \frac{dy}{y}\left\{P_{qq}(y)F_{\bar{q}}(\frac{x}{y},Q)+P_{qg}(y)F_g(\frac{x}{y},Q)\right\}
\end{aligned}
\tag{10.117}
$$

where

$$
\begin{aligned}
P_{qq}(y) &= C_F\left[\frac{1+y^2}{[1-y]_+}+\frac{3}{2}\delta(1-y)\right] \\
P_{gq}(y) &= C_F\left[\frac{1+(1-y)^2}{y}\right] \\
P_{qg}(y) &= \frac{1}{2}\left[y^2+(1-y)^2\right] \\
P_{gg}(y) &= 2N_c\left[\frac{1-y}{y}+\frac{y}{(1-y)_+}+y(1-y)+\frac{1}{12}\left(\frac{11}{3}N_c-\frac{2}{3}n_f\right)\delta(1-y)\right]
\end{aligned}
\tag{10.118}
$$

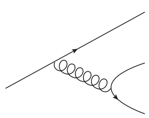
• Gluons only couple to $F_q(x) + F_{\bar{q}}(x)$ (sum of quarks and antiquarks). Thus we obtain

— a separate equation for $F_q(x) - F_{\bar{q}}(x)$ which is the same as we had before, and
— a coupled system of two equations for $F_q(x) + F_{\bar{q}}(x)$ and $F_g(x)$

For example for the moments

$$Q^2 \frac{d}{dQ^2}\left(M_q^N - M_{\bar{q}}^N\right) = -\frac{\alpha_s(Q)}{2\pi}\gamma_{qq}^N \left(M_q^N - M_{\bar{q}}^N\right)$$
$$Q^2 \frac{d}{dQ^2}\begin{pmatrix} M_q^N + M_{\bar{q}}^N \\ 2M_g^N \end{pmatrix} = -\frac{\alpha_s(Q)}{2\pi}\begin{pmatrix} \gamma_{qq}^N & \gamma_{qg}^N \\ \gamma_{gq}^N & \gamma_{gg}^N \end{pmatrix}\begin{pmatrix} M_q^N + M_{\bar{q}}^N \\ 2M_g^N \end{pmatrix} \tag{10.119}$$

• A proton can be viewed as a collection of three "valence quarks" that carry the quantum numbers, extra quark-antiquark pairs that are called "sea quarks(antiquarks)", and gluons



$$q(x) = q_v(x) + q_s(x)\,, \quad \bar{q}(x) = \bar{q}_s(x) \tag{10.120}$$

Then

$$F_{q_v}(x) = F_q(x) - F_{\bar{q}}(x) \tag{10.121}$$

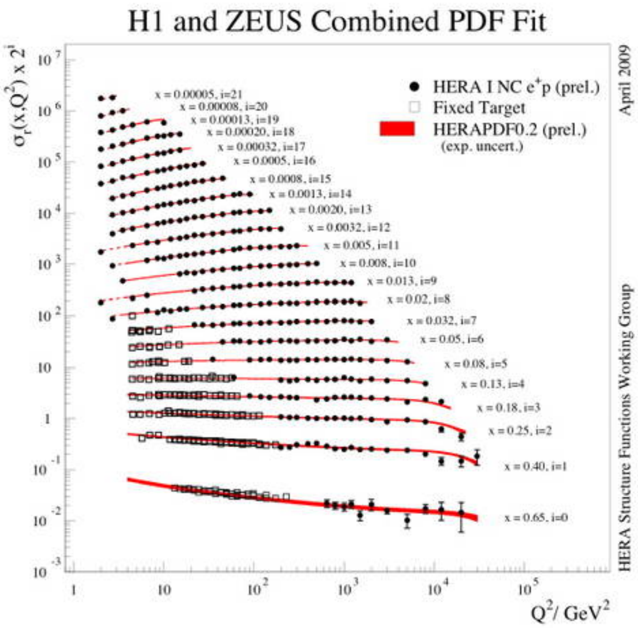
can be interpreted as the valence quark distribution.

(The number of valence quarks)/3 = baryon charge, so we expect that for any Q^2

$$\int_0^1 dx\,[F_q(x,Q^2) - F_{\bar{q}}(x,Q^2)] = 3 \tag{10.122}$$

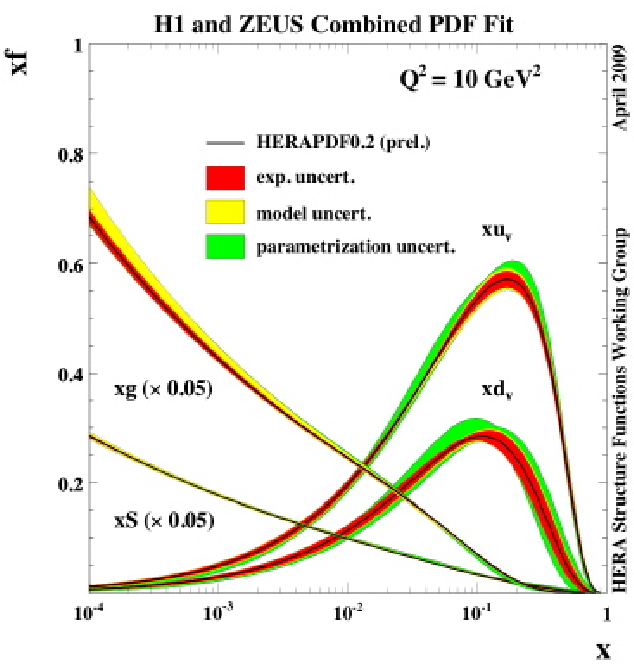
This is consistent with the evolution equation since

$$Q^2\frac{d}{dQ^2}\int_0^1 dx\,[F_q(x,Q^2) - F_{\bar{q}}(x,Q^2)] = -\frac{\alpha_s(Q)}{2\pi}\underbrace{\gamma_{qq}^{N=1}}_{\uparrow = 0!}\int_0^1 dx\,[F_q(x,Q^2) - F_{\bar{q}}(x,Q^2)] \tag{10.123}$$



Measured reaction rates (reduced cross sections) for neutral current processes, as a function of the scaling variable Q2 for different values of x, the proton momentum fraction carried by the partons. These results represent the combined analysis of H1 and ZEUS. Scaling violations, i.e. a Q2 - dependence of the rates, are clearly observed.

Presented by V. Radescu at Int. Conference DIS2009, April 2009, Madrid, Spain



Parton distribution functions of the proton as obtained by the HERA Structure Functions Working Group, at a scale of $Q^2 = 10$ GeV^2. The PDFs (solid lines) are shown separately for the gluon (xg), the sea quarks (xS), (both scaled down for visibility by a factor of 20), and the valence quarks up xuv and down xdv. The uncertainties are indicated by the coloured bands.

Presented by V. Radescu at Int. Conference DIS2009, April 2009, Madrid, Spain

Why is the gluon distribution so large for $x \to 0$?

$$P_{gg}(y) \simeq 2N_c \frac{1}{y} \tag{10.124}$$

The DGLAP equation is not in danger since it involves a cutoff $y > x$, but this behavior implies that

$$F_g(x) \overset{x\to 0}{\geq} \frac{1}{x} \tag{10.125}$$

so that the number of gluons in the proton is infinite:

$$\int_0^1 dx\, F_g(x) = \infty \tag{10.126}$$

Let us see why this happens. For $Q_1^2 \sim Q_0^2$ and small x

$$F_g(x, Q_1^2) - F_g(x, Q_0^2) \simeq \frac{3\alpha_s}{\pi} \ln \frac{Q_1^2}{Q_0^2} \int_x^1 \frac{dy}{y} \frac{1}{y} F_g(\frac{x}{y}, Q_0^2) \tag{10.127}$$

Try an ansatz

$$F_g(x, Q_0^2) = \text{const} \tag{10.128}$$

Then

$$F_g(x, Q_1^2) = \text{const} \left(1 + \frac{3\alpha_s}{\pi} \frac{1}{x} \ln \frac{Q_1^2}{Q_0^2}\right) \tag{10.129}$$

$\longleftarrow$ the gluon distribution cannot be "flat" at $x \to 0$, our ansatz was bad.

Second try:

$$F_g(x, Q_0^2) = \text{const} \cdot \frac{1}{x}, \qquad \text{or} \qquad xF_g(x, Q_0^2) = \text{const} \tag{10.130}$$

In this case obtain

$$xF_g(x, Q_1^2) = \text{const} \left(1 + \frac{3\alpha_s}{\pi} \ln \frac{1}{x} \ln \frac{Q_1^2}{Q_0^2}\right) = xF_g(x, Q_0^2) \left(1 + \frac{3\alpha_s}{\pi} \ln \frac{1}{x} \ln \frac{Q_1^2}{Q_0^2}\right) \tag{10.131}$$

One can show that this structure is general: In the limit $x \to 0$ and $Q^2 \to \infty$ each power of α_s is accompanied by two logarithms:

$$1 + c_1 \alpha_s \ln \frac{1}{x} \ln \frac{Q_1^2}{Q_0^2} + c_2 \left(\alpha_s \ln \frac{1}{x} \ln \frac{Q_1^2}{Q_0^2}\right)^2 + \dots \tag{10.132}$$

It is possible to sum this series to all orders (the so-called double-log approximation)

$$xF_g(x, Q^2)\Big|_{\substack{x\to 0\\ Q^2\infty}} \sim \exp \sqrt{\frac{48}{b} \ln \frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)} \ln \frac{1}{x}} \tag{10.133}$$

What to do if $x \to 0$ but $Q^2/Q_0^2 \sim 1$?

In higher orders

$$\frac{\alpha_s}{2\pi} P_{gg}(x)\Big|_{x\to 0} = \frac{3\alpha_s}{\pi x} \left[1 + 9\zeta(3) \left(\frac{\alpha_s}{\pi} \ln \frac{1}{x}\right)^3 + \frac{81}{20} \zeta(5) \left(\frac{\alpha_s}{\pi} \ln \frac{1}{x}\right)^5 + \mathcal{O}\left[\left(\frac{\alpha_s}{\pi} \ln \frac{1}{x}\right)^6\right]\right] \tag{10.134}$$

$\hookleftarrow$ this series is known to all orders $\longleftarrow$ BFKL equation

Resummation

$$\sum_k (\alpha_s \ln 1/x)^k \qquad\qquad xF_g(x, Q^2) \sim x^{-12 \ln 2 \frac{\alpha_s}{\pi}} \tag{10.135}$$

Too strong! What stops this rise? — Active field of research

11 To be continued