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Exercises on Quantum Chromodynamics
problem sheet 9

Worksheet : QCD analysis of the tau hadronic width.

The τ lepton is the only lepton heavy enough to decay into hadrons. Our final goal will be to calculate the ratio of decay rates

$$R_\tau = \frac{\Gamma\left(\tau \to \nu_\tau X\right)}{\Gamma\left(\tau \to \nu_\tau e^- \overline{\nu}_e\right)} \tag{1}$$

where X is an arbitrary hadronic final state (that is, the sum over all states).

The decay rate (decay width) of an unstable particle (here: τ) into a certain final state can in general be calculated as an integral of the corresponding amplitude squared over the relevant phase space:

$$\Gamma = \frac{1}{2m_\tau} \int \left(\Pi_f \frac{d^3 p_f}{(2\pi)^3\, 2E_f} \right) (2\pi)^4\, \delta^{(4)} \left(p_\tau - \sum_f p_f \right) \left|\mathcal{M}\left(\tau \to \{p_f\}\right)\right|^2 \tag{2}$$

where $p_\tau^\mu = \left(m, \vec{0}\right)$ is the four-momentum in the rest frame of the τ lepton. In the calculation of three-particle phase space integrals that you will encounter in this exercise it is usually a good idea to reduce them to two-particle ones. This can be done by treating two of the final state particles (say the first and the second) as one effective "paricle" whith mass $s = (p_1 + p_2)^2$ and doing the s integration at a later step. In all calculations that follow you should neglect masses of quarks and leptons in the final state.

Problem 1

The contribution to the effective Lagrangian of weak interactions that is responsible for the leptonic decay $\tau \to \nu_\tau e^- \overline{\nu}_e$ has the form

$$L_{eff} = -\frac{G_F}{\sqrt{2}} \left[\overline{e}\gamma^\mu(1-\gamma_5)\nu_e\right] \left[\overline{\nu}_\tau \gamma_\mu (1-\gamma_5)\tau\right]. \tag{3}$$

Show that

$$\Gamma\left(\tau \to \nu_\tau e^- \overline{\nu}_e\right) = \frac{G_F^2 m_\tau^5}{192\pi^3}. \tag{4}$$

Problem 2

Consider, for simplicity, the contribution to the hadronic decays $\tau \to \nu_\tau X$ that is generated by the contribution to the effective Lagrangian involving u and d quarks and omitting the corresponding CKM angle:

$$L_{eff} = -\frac{G_F}{\sqrt{2}} \left[\overline{d}\gamma^\mu(1-\gamma_5)u\right] \left[\overline{\nu}_\tau \gamma_\mu (1-\gamma_5)\tau\right]. \tag{5}$$

Show that the ratio R_τ can be calculated (for this contribution) as

$$R_\tau = 12\pi \int_0^{m_\tau^2} \frac{ds}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left(1 + \frac{2s}{m_\tau^2}\right) \mathrm{Im}\Pi(s) \tag{6}$$

where (we assume massless quarks)

$$\Pi_{\mu\nu}(q) = i \int d^4x e^{iqx} \langle 0| T\left\{\overline{d}(x)\gamma_\mu\left(1-\gamma_5\right)u(x)\overline{u}(0)\gamma_\nu\left(1-\gamma_5\right)d(0)\right\}|0\rangle \tag{7}$$

$$= \left(q_\mu q_\nu - g_{\mu\nu}q^2\right)\Pi(q^2) \tag{8}$$

Comment: The QCD analysis of hadronic tau decays provides one with the most exact determination of the strong coupling constant.
At the scale of the Z - boson mass one obtains

$$\alpha_s\left(M_Z\right) = 0.11795 \pm 0.00076 \tag{9}$$

see *Beneke, Jamin JHEP 0809:044,2008* [$arXiv : 0806.3156$]