Universität Regensburg, Institut für Theoretische Physik WS23
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Exercises on Quantum Chromodynamics
problem sheet 1

Worksheet 5: Green's functions of the $\Box = \partial_\mu \partial^\mu$ operator

Consider the Green's functions of the d'Alembert operator $\Box = \partial_\mu \partial^\mu$

$$G_C(x) = i \int_C \frac{d^4k}{(2\pi)^4} e^{-ikx} \frac{1}{k^2} \tag{1}$$

corresponding to the four possible choices of the integration contour C in the complex k^0 plane:

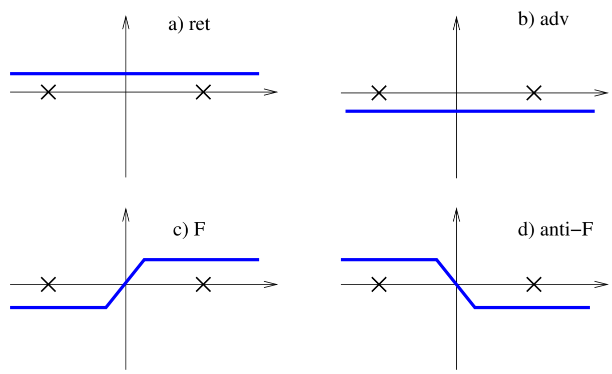

Figure 1: Contours in the complex energy plane: a) retarded; b) advanced; c) Feynman; d) anti-Feynman.

Problem 1: Use the Green functions to construct the solutions of the equation

$$\Box\, \phi(x) = j(x) \tag{2}$$

for a localized source $j(x)$ ($j(x)$ is nonzero only if $|x| < R$). Show that the different choices of the Green's function give rise to the solutions with the following boundary conditions:

a) vanishing fields in the past, $\phi(t, \vec{r}) = 0$ for $t < -T_0$ for some T_0 (radiation case)

b) vanishing fields in the future, $\phi(t, \vec{r}) = 0$ for $t > T_0$, (absorbtion case)

c) only positive frequencies exist in the far future $t > T_0$ and only negative frequencies in the distant past (Feynman), that is

$$\begin{aligned} \phi(t,\vec{r}) &= \phi_+(t,\vec{r}) = \int_0^\infty d\omega e^{-i\omega t}\tilde{\phi}(\omega,\vec{r}), \quad t > T_0 \\ \phi(t,\vec{x}) &= \phi_-(t,\vec{r}) = \int_0^\infty d\omega e^{+i\omega t}\tilde{\phi}(\omega,\vec{r}), \quad t < -T_0 \end{aligned}$$

Problem 2: Introduce

$$G_\pm(x) = i\int_{C_\pm} \frac{d^4k}{(2\pi)^4} e^{-ikx}\frac{1}{k^2}\,, \tag{3}$$

where the contours $C_\pm$ in the complex k^0 plane are

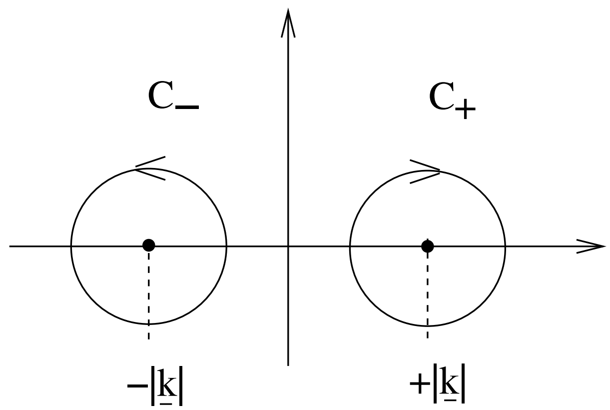

Figure 2: The C_+ and C_- contours

Show that $G_\pm(x)$ are the solutions of the homogeneous equation associated with the $\Box$ operator, and that the Green's functions are expressed in terms of $G_\pm(x)$ as

$$\begin{aligned} G_{ret}(x) =& \Theta(x_0)[G_+(x) - G_-(x)]\,, \\ G_{adv}(x) =& \Theta(-x_0)[G_+(x) - G_-(x)]\,, \\ G_F(x) =& \Theta(x_0)G_+(x) + \Theta(-x_0)G_-(x)\,. \end{aligned}$$

Problem 3: Evaluate $G_{ret}(x)$ explicitely and show that it is given by

$$G_{ret}(x) = \frac{1}{4\pi i}\frac{1}{|\vec{x}|}\delta(x_0 - |\vec{x}|)\,. \tag{4}$$