Bounded cohomology, cohomology with bounded values and d-bounded cohomology

(joint work with A. Sisto)

29th June 2020

Cohomology of groups

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- $V = \mathbb{R}$ with the usual absolute value, endowed with the trivial action by G
- $V = \ell^\infty(G, \mathbb{R})$, endowed with the action

$$(g_0 \cdot f)(g) = f(g_0^{-1} g)$$

Cohomology of groups

$$C^n(G, V) = \{\varphi\colon G^n \to V\}, \quad \delta\colon C^n(G, V) \to C^{n+1}(G, V)$$

$$\begin{aligned}\delta(\varphi)(g_1, \ldots, g_{n+1}) =& g_1 \cdot \varphi(g_2, \ldots, g_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^n \varphi(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) \\ &+ (-1)^{n+1} \varphi(g_1, \ldots, g_n)\end{aligned}$$

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The cohomology of G is

$$H^\bullet(G,V) = H^\bullet(C^\bullet(G,V))$$

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$$\text{For } \varphi \in C^n(G,V)\,, \quad \|\varphi\|_\infty = \sup_{\overline{g}\in G^n} \|\varphi(\overline{g})\|_V \;\in\; [0,\infty]$$

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The inclusion

$$C_b^\bullet(G,V) \hookrightarrow C^\bullet(G,V)$$

induces the comparison map

$$c^\bullet \colon H_b^\bullet(G,V) \to H^\bullet(G,V)$$

Bounded and weakly bounded classes

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It is weakly bounded if it admits a weakly bounded representative, i.e. a cocycle $\omega \in C^n(G, V)$ such that, for every $g_1 \in G$, the map

$$\omega(g_1, \cdot, \cdot, \dots, \cdot)\colon G^{n-1} \to V$$

is bounded.

Beware: weakly bounded chains do not define a complex, hence there is no "weakly bounded cohomology".

Cohomology with bounded values

The (equivariant) inclusion $\mathbb{R} \hookrightarrow \ell^\infty(G,\mathbb{R})$ into constant functions induces

$$H^\bullet(G,\mathbb{R}) \xrightarrow{\iota^\bullet} H^\bullet(G,\ell^\infty(G,\mathbb{R})) \quad := \; H^\bullet_{(\infty)}(G,\mathbb{R})$$

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Proposition

A class in $H^\bullet(G, \mathbb{R})$ is weakly bounded if and only if it lies in the kernel of $\iota^\bullet$.

Corollary (Gersten 92, Wienhard 12, Blank 15)

The composition

$$H_b^\bullet(G,\mathbb{R}) \xrightarrow{c^\bullet} H^\bullet(G,\mathbb{R}) \xrightarrow{\iota^\bullet} H_{(\infty)}^\bullet(G,\mathbb{R})$$

is null in every degree.

Corollary (Gersten 92, Wienhard 12, Blank 15)

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Question (Mineyev, Blank, Wienhard)

When is the sequence

$$H_b^n(G,\mathbb{R}) \to H^n(G,\mathbb{R}) \to H_{(\infty)}^n(G,\mathbb{R})$$

exact?

Proposition (F.–Sisto)

Let G be an n-dimensional non-amenable PD-group (e.g. the fundamental group of a negatively curved closed n-manifold). Then the sequence

$$H_b^{n+1}(G \times \mathbb{Z}, \mathbb{R}) \to H^{n+1}(G \times \mathbb{Z}, \mathbb{R}) \to H_{(\infty)}^{n+1}(G \times \mathbb{Z}, \mathbb{R})$$

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Corollary

For every $n \geq 3$, there exists a finitely presented group G such that the sequence

$$H_b^{n}(G \times \mathbb{Z}, \mathbb{R}) \to H^{n}(G \times \mathbb{Z}, \mathbb{R}) \to H_{(\infty)}^{n}(G \times \mathbb{Z}, \mathbb{R})$$

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Theorem (F.–Sisto)

There exists a finitely generated group G such that the sequence

$$H_b^2(G, \mathbb{R}) \to H^2(G, \mathbb{R}) \to H_{(\infty)}^2(G, \mathbb{R})$$

is not exact (i.e. G does not safisfy QITB).

Motivation(s)

Question

Why do we care?

Central extensions

G finitely generated. To any central extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow E \longrightarrow G \longrightarrow 1$$

there is associated its Euler class in $H^2(G, \mathbb{Z})$.

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Question (Neumann–Reeves)

Can quasi-isometrically trivial extensions be characterized in terms of bounded cohomology?

More GGT

Let G be Gromov hyperbolic. Then:

- The comparison map $c^n\colon H^n_b(G,\mathbb{R}) \to H^n(G,\mathbb{R})$ is surjective for every $n \geq 2$ [Mineyev 01].
- $H^n_{(\infty)}(G,\mathbb{R}) = 0$ for every $n \geq 2$ [Mineyev 00].

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Let G be amenable. Then:

- $H_b^n(G, \mathbb{R}) = 0$ for every $n \geq 2$.
- The map $\iota^n \colon H^n(G, \mathbb{R}) \to H^n_{(\infty)}(G, \mathbb{R})$ is injective for every $n \geq 2$ [Gersten 92].

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Differential forms

Let M be a Riemannian manifold, and let $\omega \in \Omega^k(M)$. For every $x \in M$,

$$|\omega_x| = \sup |\omega_x(e_1 \wedge \cdots \wedge e_k)|\,, \quad e_1, \ldots, e_k \text{ orthonormal frame at } x$$

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$$\omega \in \Omega^k_\flat(M) \qquad \textit{if} \quad \sup_{x \in M} |\omega_x| < +\infty \,, \quad \sup_{x \in M} |(d\omega)_x| < +\infty$$

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If M is compact, for every $\omega \in \Omega^k(M)$ we can take its lift $\widetilde{\omega} \in \Omega^k_\flat(\widetilde{M})$, thus getting a map

$$H^\bullet_{DR}(M) \to H^\bullet_\flat(\widetilde{M})$$

Definition (Gromov 91)

A class $[\omega] \in H^k_{DR}(M)$ *is* $\widetilde{d}$*-bounded if it lies in the kernel of*

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Conjecture (Gromov 93)

Let M *be compact. A class* $\alpha \in H^2(M)$ *is bounded if and only if the corresponding class in* $H^2_{DR}(M)$ *is* $\widetilde{d}$*-bounded.*

Theorem (F.–Sisto)

Gromov's conjecture holds if and only if every finitely presented group satisfies QITB.

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Key ingredient: If M is compact and aspherical, there is a commutative diagram

$$\begin{array}{ccccc} H_b^2(M) & \longrightarrow & H^2(M) & \longrightarrow & H_\flat^2(\widetilde{M}) \\ \downarrow & & \downarrow & & \downarrow \\ H_b^2(\pi_1(M), \mathbb{R}) & \longrightarrow & H^2(\pi_1(M), \mathbb{R}) & \longrightarrow & H^2_{(\infty)}(\pi_1(M), \mathbb{R}) \end{array}$$

where the vertical arrows are isomorphisms (for the vertical arrow on the right, this is proved in [Mineyev99]).

The counterexample

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Let Σ be the closed oriented surface of genus 2. For every $n \in \mathbb{N}$,

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For every sequence $(\alpha_i)_{i \in \mathbb{N}}$ of real numbers, there exists a unique class $\alpha \in H^2(\widehat{G})$ such that

$$i_n^*(\alpha) = \alpha_n [\Sigma_n]^* \qquad \text{for every } n \in \mathbb{N}$$

where $[\Sigma_n]^* \in H_2(G_n)$ is the fundamental coclass of Σ_n.

The counterexample

If $\|\alpha\| \leq K$, then $\|i_n^*(\alpha)\| \leq K$ for every $n \in \mathbb{N}$ and

$$|\alpha_n| = |\langle i_n^*(\alpha), [\Sigma_n]\rangle| \leq \cdot K \|[\Sigma_n]\| \leq 4K$$

hence (α_n) is bounded.

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But $\widehat{G}$ is not finitely generated!

The counterexample

We thus look for a finitely generated quotient of $\widehat{G}$.

$$G_i = \langle a_i, b_i, c_i, d_i \mid [a_i, b_i] \cdot [c_i, d_i] \rangle$$

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We introduce on $\widehat{G} * \langle t_1, t_2, t_3, t_4 \rangle$ the relations

$$t_1 a_i t_1^{-1} = a_{i+1}\,, \qquad t_2 b_i t_2^{-1} = b_{i+1}$$
$$t_3 c_i t_3^{-1} = c_{i+1}\,, \qquad t_4 d_i t_4^{-1} = d_{i+1}$$

to get our desired finitely generated group G.

The counterexample

- We still have injections

$$G_n \to \widehat{G} \to \widehat{G} * \langle t_1, t_2, t_3, t_3 \rangle \to G$$

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- We still have that α is bounded if and only if (α_n) is bounded.
- If

$$\sup_{n \in \mathbb{N}} \frac{|\alpha_n|}{n} < +\infty$$

then α is weakly bounded.

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The implication

$$\sup_{n \in \mathbb{N}} \frac{|\alpha_n|}{n} < +\infty \quad \Longrightarrow \quad \alpha \text{ weakly bounded}$$

is pretty hard.

The counterexample

We have $G = F_8/N$, where

$$F_8 = \langle a, b, c, d, t_1, t_2, t_3, t_4 \rangle \, , \qquad N = \langle\langle r_0, r_1, \ldots, r_n, \ldots \rangle\rangle$$

$$r_n = [t_1^n a t_1^{-n}, t_2^n b t_2^{-n}] \cdot [t_3^n c t_3^{-n}, t_4^n d t_4^{-n}]$$

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$$r_n = [t_1^n a t_1^{-n}, t_2^n b t_2^{-n}] \cdot [t_3^n c t_3^{-n}, t_4^n d t_4^{-n}]$$

For every word $w \in N$ we denote by $|w|$ the length of w, and

$$A(w) = \min \left\{ \sum_{i=1}^{k} n_i \mid w = \prod_{i=1}^{k} w_i r_{n_i}^{\pm 1} w_i^{-1} \right\}$$

$A(w)$ is a *weigthed* area of w.

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Theorem

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$$1 \longrightarrow \mathbb{Z} \xrightarrow{\ j\ } E \xrightarrow{\ \pi\ } G \longrightarrow 1$$

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We look for a good section of π.

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Let

$$F_8 \to E, \qquad w \mapsto \overline{w}$$

be such that $\pi(\overline{w}) = [w]$ in G.

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For a fixed g, as $w \in F_8$ varies among its representatives

$$\overline{w} \cdot j(-|w|)$$

is bounded from above. We define $s(g)$ as its maximum.

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A representative of α is given by

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Exploiting the geometry of the section s we show that

$$|s(g_1)s(g_2)s(g_1g_2)^{-1}| \leq K \cdot |g_1|$$

hence the class α is weakly bounded.