Huber Continuous valuations

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Préliminaires

0.1 Valuation ring

Définition 0.1 cf [8, p.71] An integer ring \( R \) is a valuation ring if, noting \( K \) for its field of fraction, \( \forall x \in K, x \in R \) or \( x^{-1} \in R \).

cf aussi [12, p122]

Proposition 0.1 If \( R \) is a valuation ring, \( R \) is a local ring. The ideals of \( R \) are totally ordered by inclusion.

D : \( R \) is local if the non-invertible elements are stable by addition. If \( x \) and \( y \) are not invertible, if \( x \) or \( y \) is 0, then OK, otherwise, \( \frac{y}{x} \) say is in \( R \). Then \( x + y = y + y \frac{x}{y} = y(1 + \frac{x}{y}) \), and since \( y \) isn’t invertible, \( x + y \) neither.

Let \( I \) and \( J \) be two ideals of \( R \). Suppose \( I \nsubseteq J \) and \( J \nsubseteq I \), let then \( i \in I \setminus J \) and \( j \in J \setminus I \). By symmetry, suppose that \( \frac{i}{j} \in R \). Then \( i - j \frac{i}{j} \in J \), which is absurd. \( \Box \)
Actually, this also proves that \( R \) is a local ring: define \( m \) as the union of all the proper ideals of \( R \). It is a proper ideal because they are totally ordered by inclusion, and maximal.

**Proposition 0.2** Let \( K \) be a field, \( v: K \to \Gamma_0 \) such that
\[
\begin{align*}
v(x) &= 0 \iff x = 0 \\
v(xy) &= v(x)v(y).
\end{align*}
\]
Then \( v \) is a valuation iff \( \forall \ x \) such that \( v(x) \leq 1 \), \( v(1 + x) \leq 1 \).

\[
D: \Rightarrow v(1 + x) \leq \max(v(1), v(x)) - 1.
\]
\[
\Leftrightarrow: \text{let } x \text{ and } y \in K \text{ not zero (otherwise it is easy). Suppose } v(x) \geq v(y). \text{ Hence } v \left( \frac{x}{y} \right) \leq 1. \text{ Then } v(x + y) - v(x)(1 + \frac{y}{x}) < v(x).
\]

**Définition 0.2** A valuation on \( A \) is \( v: A \to \Gamma_0 \) such that \( v(ab) = v(a)v(b) \) and
\[
\begin{align*}
1. \ & v(x + y) \leq \max(v(x), v(y)) \quad \text{and the law on } \Gamma_0 \text{ is that } 0 < \infty, \text{i.e. greatest element} \quad \text{(we should note in fact } \Gamma.x \text{).}
2. \ & v(x + y) = \max(v(x), v(y)), \text{ and } 0 \text{ is the lowest element.}
\end{align*}
\]

Putting \( v : = \frac{1}{2} \), i.e. \( v(a) = v(a^2) \) if \( v(a) \neq 0 \), and \( v(1) = 0 \) (in fact \( 0 \Leftrightarrow \infty \)), gives a bijection between valuation of \( \theta \) types (i) and (ii). We will always take definition (ii).

### 0.2 product of valuation

If \( v_i: A \to \Gamma_i \) are two valuations \((i = 1, 2)\), then
\( v: A \to \Gamma_1 \times \Gamma_2 \) (with the lexicographic order) is not a valuation in general. Indeed if one can find \( a \) and \( b \) such that \( v_1(a) = v_2(b) = v_2(a) \), then we would have \( v(a + b) = v_1(b, v_2(a)) > \max(v(a), v(b)) = v_1(v_2(a, v_2(b)) \).

Exemples :

* \( A = \mathbb{Z}, v_2 \text{the } 2\text{-adic valuation, and } v_3 \text{the } 3\text{-adic one, then } v(2+3) = (0, 0), v(2) = (-1, 0), v(3) = (0, -1) \).

* \( A = k[T], v_1 = \eta_{B(0,1)} \) and \( v_2 = \eta_{B(0,\frac{1}{p})} \).

Then \( v(T^2 + p) = (1, \frac{1}{p}), v(T^2) = (1, \frac{1}{p^2}), v(p) = (\frac{1}{p}, p) \).

### 0.3 Completion of a topological ring

Let \( A \) be a topological ring. A sequence in \( A \) is said to be a Cauchy sequence if for every \( 0\)-neighborhood \( V \) there exists \( N \) such that \( n, m \geq N \) implies \( x_n - x_m \in V \).

If \( x_n \to l \), then \( x_n \) is Cauchy. Indeed let \( V \) be the \( 0 \)-neighborhood, then there exists a \( 0 \)-neighborhood \( W \) such that \( W \subseteq W \subseteq V \). Then for \( n \) big enough \( x_n \in W \) and \( x_n - x_m \in W \subseteq V \).

\( A_{\text{cauchy}} \) is a group for \( + \). If \( x_n \) and \( y_n \) are Cauchy, let \( V \) be a \( 0 \)-neighborhood. Let \( W \) be another one such that \( W \subseteq V \). Then for \( n \) big enough \( x_n - x_m \in W \) and the same thing for \( y_n \), so that \( (x_n + y_n) - (x_n + y_m) \in W \).

\( x_n \) Cauchy \( \Rightarrow \) \( x_n \) bounded. Let \( W \) be a \( 0 \)-neighborhood, and \( W \subseteq V \) (in particular \( W \subseteq V \)). Let \( X_1, X_2 \) two neighborhoods \( \forall \) \( x_1, x_2 \subseteq W \). \( \exists N \forall x, x_1, x_2 \subseteq W \). for \( i = 0 \ldots N - 1 \) \( \exists W_i \subseteq x_iW_i \subseteq W \).

Define \( Z = W_0 \cap \ldots \cap W_N \). Then for \( i = 0 \ldots N - 1 \) \( x_iZ \subseteq x_iW_i \subseteq W \).

For \( i > N \), \( x_iZ \subseteq \{ x_i - x_N \} z + \{ x_N \} X_2 \subseteq x_iX_2 + x_NW_N \subseteq W \subseteq V \).

\( x_n \) and \( y_n \) Cauchy \( \Rightarrow (x_n) \) are Cauchy. Let \( V \). \( \exists W \subseteq W \subseteq V \).

Then for \( n, m \geq 0 \) \( y_m - y_n \) and \( x_m - x_n \in X \) then \( x_m y_m - x_n y_n \in W \subseteq V \).

the \( 0 \)-sequence form an ideal of \( A_{\text{cauchy}} \). Let \( x_n \) be a \( 0 \)-sequence, and \( y_n \) a Cauchy one, so that it is bounded.

Let \( V \). As \( y_n \) is bounded there exists \( W \) such that \( y_n \subseteq V \). then for \( n \geq 0 \) \( x_n \in W \).

- Define \( \text{hat}A = A_{\text{cauchy}/0} \).
0.4 Ordered groups

\( \Gamma \) is an abelian ordered group.

**Proposition 0.3** \( a \geq b \iff a^{-1} \geq b^{-1} \).

\[ D : a \geq b \Rightarrow a(a^{-1}b^{-1}) \geq b(a^{-1}b^{-1}) \], i.e. \( b^{-1} \geq a^{-1} \). Applying this with \( a^{-1} \) and \( b^{-1} \) gives the other inequality. \( \Box \).

**Définition 0.3** \( A \) is a convex set if \( \forall x, y \in \Gamma \) with \( x \leq y \leq z \) and \( x, z \in X \) then \( y \in X \).

The convex sets are stable by intersection; if \( X_i \) are convex sets that all contain 1, then \( \bigcup X_i \) is convex.

If \( X \) is convex, \( X^{-1} = \{ x^{-1} \mid x \in X \} \) is convex.

**Définition 0.4** If \( A \subseteq \Gamma \), define \( A_{\text{conv}} = \{ x \in \Gamma \mid \exists a, b \in A \text{ with } a \leq x \leq b \} \). This is the smallest convex set that contains \( A \).

**Proposition 0.4** If \( H \subseteq \Gamma \) is a subgroup, \( H_{\text{conv}} \) is a subgroup.

Indeed if \( a \leq x \leq b \), and \( a' \leq x' \leq b' \), then \( ad \leq xd' \leq bb' \), and \( a^{-1} \geq x^{-1} \geq b^{-1} \).

**Proposition 0.5** If \( X \subseteq \Gamma \) is a convex subset that contains 1, then \( X > (\text{the subgroup generated by } X) \) is convex.

\[ D : \text{Let's show that } X \text{ is convex: let } a, b, c, d \in X \text{ and } ab \leq x \leq cd. \text{ Put } \{a, b, c, d\} = \{\alpha, \beta, \gamma, \delta\} \text{, so that we can assume that } a \leq b \leq c \leq d. \text{ Then } ab \leq cd. \]

If \( ab \leq x \leq cb \), \( a \leq xb^{-1} \leq c \), and \( xb^{-1} \in X \) since it is convex, so since \( b \in X \), \( x = x\gamma b^{-1}, b \in X.X \). Otherwise \( cb \leq x \leq cd \) and \( b \leq xe^{-1} \leq d \) and \( x = x\gamma b^{-1}c \in X.X \).

Define \( X' = X \cap X^{-1} \) which is convex since \( 1 \in X \) and \( X^{-1} \), then \( X > = \cap_{n>0} \gamma^n \) is then convex.

\( \Box \)

The hypothesis 1 \( \in X \) is necessary as shows the example \( X = \{2\} \subseteq (\mathbb{Z}, +, \leq) \), where \( < 2 > = 2\mathbb{Z} \) is not convex.

**Corollaire 0.1** If \( X \subseteq \Gamma < (X \cup \{1\})_{\text{conv}} > = < X >_{\text{conv}} \)

Indeed \( X > < (\{1\})_{\text{conv}} > \subseteq < X >_{\text{conv}} \) and is convex so \( < (X \cup \{1\})_{\text{conv}} > \subseteq < X >_{\text{conv}} \). And \( < X >_{\text{conv}} \supseteq X \cup \{1\} \) and is a group, so \( < X >_{\text{conv}} \supseteq < (X \cup \{1\})_{\text{conv}} > \).

Define \( \text{Conv}(\Gamma) \) as the set of convex subgroups of \( \Gamma \). Then it is totally ordered for inclusion. It has the lower and upper bound properties (take \( \cap \) and \( \cup \)).

Call a convex subgroup \( < g >_{\text{conv}} \) a principal subgroup. Note that a convex subgroup isn’t necessarily principal.

For instance take \( \Gamma = (\mathbb{Z}, +, \leq)_{\mathbb{Q}} \), that is the sequences indexed by \( \mathbb{Q} \) almost everywhere zero, and ordered by the lexicographic order. Then \( H_{\mathbb{Q}} = \{ x \in \Gamma \mid \text{supp}(x) \subseteq [\alpha, \alpha+\sqrt{2}] \} \) is a convex subgroup, which is not principal, since any principal subgroup is of the form \( H_a = \{ x \in \Gamma \mid \text{supp}(x) \subseteq [\alpha, \alpha+\alpha] \} \) for \( a \in \mathbb{Q} \).

In fact, for any \( b \in \mathbb{R}, H^b = \{ x \in \Gamma \mid \text{supp}(x) \subseteq [\alpha, \alpha+\beta] \} \) is also a non principal subgroup.

**Définition 0.5** Let \( H \subseteq \Gamma \) a subgroup, then \( \gamma \in \Gamma \) is cofinal in \( H \) if \( \forall h \in H, \exists n \in \mathbb{N} \text{ s.t. } \gamma^n < h \).

**Proposition 0.6** \( g \) is cofinal in \( H \) iff \( < g >_{\text{conv}} \supseteq H \) and \( g < 1 \).

\[ \Rightarrow : \text{Since } \exists n \text{ s.t. } \gamma^n > g \text{, we have } 1 > g \text{, let } h \in H. \text{ If necessary, let's take } h^{-1} \text{, so that } h \leq 1 \text{. Then there exists a } n \geq 0 \text{ s.t. } h \geq g^n, \Rightarrow h \leq g_{\text{conv}}. \]

\[ \Leftrightarrow : \text{Let } h \in H, \text{ here again, taking } h^{-1} \text{ if necessary, we can assume that } 1 \geq h. \text{ Then there exists a } n \in \mathbb{N} \text{ s.t. } g^n \geq h \leq 1 \text{, and then } g^{n+1} < h \text{, so } g \text{ is cofinal in } H. \]
Proposition 0.7 Let \((H_i)\) be a increasing family of subgroups such that \(g \in H_i\). Then \(g\) is cofinal in \(H = \bigcup H_i\).

D : if \(h \in H\) then \(h \in H_i\) for some \(i\) and then \(g\) being cofinal for \(H_i\) \(\exists n \ni g^n < h\).

Corollaire 0.2 Let \(g < 1\). Then there exists a bigger convex subgroup \(H \ni g\) is cofinal in \(H\). In fact \(H = \langle g \rangle_{\text{conv}}\)

D : If \(\mathcal{F}\) is the family of convex subgroups \(G\) for which \(g\) is cofinal in \(H\), then \(H = \bigcup G\) is a convex subgroup, and \(g\) is cofinal in \(H\) according to the previous proposition.

Clearly it is maximal for this property.

Now, \(g\) is cofinal in \(\langle g \rangle_{\text{conv}}\).

Proposition 0.7 implies there exists \(n \geq 0\) such that \(g^n < x\) and \(g^{n+1} < x\). Then \(H \ni g < \langle g \rangle_{\text{conv}}\). Now if \(h \in H\), take \(h^{-1}\) if necessary we can assume \(h \leq 1\). Then \(\exists n > 0 \ni g^n < h < 1\).

Remark 1 Let \(X \ni \Gamma \ni \{ g \ni \Gamma \ni g < 1\}\). Define \(\text{Conv}(X) = \{ H \ni \text{convex subgroup such that } \forall x \ni X, x \text{ is cofinal in } H\}\). Since it stable by \(\ni\), and nonempty \(\{1\} \ni \text{Conv}(X)\) we can (taking its lower bound, i.e. intersection) see it has a subgroup: the smallest such that.

Then from what we have done, \(\text{Conv}(X) = \bigcap_{x \in X} \langle x \rangle_{\text{conv}}\).

1 F-adic rings

Proposition 1.1 \(A^+ \ni A^+\)

D : Let \(a \ni A^+\), and \(V\) a 0 neighborhood. There exists \(W\) a 0 neighborhood such that \(W \ni V\). There exists \(N\) such that \(n \ni N\) implies \(a^n \in W\). For each \(i = 0 \ldots N - 1\) there exists \(W_i\) a \(a^n\) neighborhood such that \(a^n W_i \ni W\). Then if \(U = W_0 \cap \ldots \cap W_{N-1} \cap W\) then for each \(i \ni a^n U \ni V\).

An adic ring is bounded. Indeed, if \(I\) is an ideal of definition of \(A\), then if \(V\) is a 0 neighborhood, there exists a \(n \ni I^n \ni V\) and \(I^n A = I^n \ni V\).

For \(S\) and \(T\) two subsets of \(A\), let \(S \cdot T\) be subgroup of \((A, +)\) generated by the elements \(s \cdot t, s \in S\) and \(t \ni T\).

Définition 1.1 1. A topological ring \(A\) is \(f\)-adic if there exists a subset \(U\) and a finite subset of \(U\) such that \(\{ U^n \ni n \ni \mathbb{N}\}\) is a fundamental system of 0 neighborhood, and \(T \cdot U = \{ U \ni U\}\). 2. \(A\) is called a Tate ring if it is \(f\)-adic and has a topologically nilpotent unit.

A ring of definition of a \(f\)-adic ring is an open subring \(A_0\) of \(A\) which is adic.

Proposition 1.2 (Prop 1) Let \(A\) be a \(f\)-adic ring. Then

1. \(A\) has a ring of definition.
2. A subring \(A_0\) is a ring of definition iff it is open and bounded.
3. Every ring of definition of \(A\) has a finitely generated ideal of definition.

It is then clear that a topological ring \(A\) is \(f\)-adic if it has an open subring \(A_0\), which is adic for a finitely generated ideal \(I\) (since in this case \(A\) is clearly \(f\)-adic).

D : Let \(W\) be the subgroup of \(A\) generated by \(U\). Since \(U^2 \ni U\), we can conclude that \(W^2 \ni W\). Let \(B = Z + W\). Then \(B\) is a subgroup of \(A\) for its additive law. It is also stable by multiplication : \((n + w), (m + w) - nw + mw + w^2 \ni B\) \((W^2 \ni W)\). \(B\) is then a subring. It is open since it contains \(U\) which is an 0 neighborhood, and a subgroup of a topological group is open iff it contains a 0 neighborhood. For \(n \ni 2\), \(B \cdot U^n = U^n\), because \(Z \ni U^n\), and \(W \ni U^n = U^{n+1}\), and the fact that \(U^2 \ni U\) implies that \(U^{n+1} \ni U^n\). Hence the \(U^n\) being a fundamental system of neighborhoods of 0 implies that \(B\) is open.

Hence we can introduce \(A_0\) an open and bounded subring of \(A\).

For \(n \ni \mathbb{N}\) define the finite set \(T(n) = \{ t_1, t_2, \ldots, t_n \mid t_i \ni T\}\). Since \(T \ni U\) and \(T \cdot U = U^2\),
$T(n) \subseteq U^n$. In particular since the $U^n$ form a fundamental system of neighborhood and $A_0$ is open, $\exists k \ s.t \ T(k) \subseteq A_0$. Put then $I = t(k)A_0$. Let’s show that $I^n$ (seen here as an ideal of $A_0$) is a fundamental system of neighborhood of 0 (in $A$, or $A_0$, it is equivalent since $A_0$ is open).

First, there exists a $m \ s.t \ U^m \subseteq A_0$, and then for $n \in \mathbb{N}$, one easily sees that $I^n = T(nk)A_0 \supseteq T(nk)U^m = U^{nk+m}$, so $I^n$ is a 0 neighborhood.

Let $V$ be a 0 neighborhood. Then there exists $m \ s.t \ U^lA_0 \subseteq V$ because $A_0$ is bounded. But now $I^n - T(nk)A_0 \subseteq U^{nk}A_0 \subseteq U^mA_0 \subseteq V$. Hence $A_0$ is a ring of definition for $I$, and $I$ is of finite type, which proves (i) and (ii).

Now if $A_0$ is a ring of definition of $A$, as noted previously , $A_0$ is bounded (in $A_0$, so in $A$ too) , since it is adic. So by what we have done, it has a finitely generated ideal of definition. 

Now then, for $A$ a $f$-adic ring, we will consider it coming with a couple $(A_0, I)$, with $A_0$ a ring of definition and $I$ an ideal of definition. Then the $I^n$ form a fundamental system of neighborhood of 0.

**Lemme 1.1** Let $A$ be a $f$-adic ring, $S$ and $T$ bounded subsets. Then $S.T$ is bounded.

**D :** let $(A_0, I)$ a ring of definition, and $I^n$ a 0 neighborhood. $\exists m \ s.t \ S^m \subseteq I^n$. $\exists p \ s.t \ T^p \subseteq I^m$.

Then, if $s, t \in S \times T$, and $a \in I^p$, $a \in I^m$ so sta $\in I^n$. Since $I^n$ is a subgroup, one then conclude that $(S.T)^p \subseteq I^n$.

**Corollaire 1.1** Let $A$ be a $f$-adic ring.

1. If $A_0$ and $A_1$ are rings of definition, then $A_0, A_1$ and $A_0 \cup A_1$ also.
2. If $B$ is a bounded subring, and $C$ an open subring with $B \subseteq C$ subseteq $A$ , there exists $A_0$ a ring of definition with $B \subseteq A_0 \subseteq C$
3. $A^*$ is a subring , and it is the union of all rings of definition.

**D :**

(i) the second point of the previous proposition shows that $A_0$ and $A_1$ are open and bounded. Then $A_0 \cup A_1$ is also open, and bounded . Then $A_0, A_1$ is also open (it contains $A_0$) , and bounded according to the lemma. So the second point of the proposition shows that there are ring of definition.

(ii) Let $A_1$ be a ring of definition. Then $B, A_1$ is a subring, bounded (previous lemma) , and open (contains $A_0$) , so is a ring of definition. Then $A_0 - A_1 \cup C$ is an open bounded subring so is a ring of definition, and $B \subseteq A_0 \subseteq C$.

(iii) Let $(A_0, I)$ be a ring of definition for $A$. First 0 and 1 $\in A^*$.

Let now $a, b \in A^*$, and $I^n$ be a 0 neighborhood. There exists $m \ s.t \ \{a^k, k \in \mathbb{N}\}I^m \subseteq I^n$ and $\{b^k\}I^m \subseteq I^n$. Then for $r, s \in \mathbb{N}$, $a^rb^sI^{2m} = a^r(b^sI^m)I^m \subseteq a^rI^{n+2m} \subseteq a^rI^n \subseteq I^n$. Since $(a + b)^k - \sum \binom{k}{i}a^ib^{k-i}$, one has $(a + b)^kI^{2m} \subseteq I^n$, and $(ab)^kI^{2m} \subseteq I^n$, i.e. $a + b$ and $ab \in A^*$. So $A^*$ is a subring.

Now if $X$ is bounded, then $X \subseteq A^*$, in particular for any $A_0$ ring of definition, $A_0 \subseteq A^*$. On the other hand, $Z$ is bounded (this is a consequence of the fact that some ring of definition exist, that they are bounded and contain $Z$, more simply because $ZI^n \subseteq I^n$). Let now $x \in A^*$. Then by definition $\{x^n\}$ is bounded, so $B - Z, \{x^n\} - Z[x]$ is a bounded subring. $B \subseteq A$ which is open (1) so with (ii) there exists a ring of definition $A_0$ with $B \subseteq A_0$, and then $x \in A_0$ So $A^* = \cup A_0, A_0$.

**Proposition 1.3** If $A$ is $f$-adic $A^*$ is a subring (except if $\text{it doesn't contain 1}$) same proof

**Corollaire 1.2**

1. An adic ring is $f$-adic iff it has a finitely generated ideal of definition.

2. A $f$-adic ring is adic iff it is bounded

3. Let $A$ be a topological ring and $B$ an open subring. Then $A$ is $f$-adic iff $B$ is.
D : 
(i) \(\Rightarrow\) is a consequence of (ii) of the prop.
\(\Leftarrow\) : already seen.
(ii) Let \(A\) be \(f\)-adic. If \(A\) is adic it is bounded (this is true without the assumption \(f\)-adic). Conversely, if \(A\) is bounded, then (ii) of the proposition, since \(A\) is bounded and open in \(A\) \(f\)-adic, it is a ring of definition, hence is adic.
(iii) If \(B\) is \(f\)-adic, one can find \((B_0, I)\) ring of definition for \(B\), and since \(B\) is open, \((B_0, I)\) is also a ring of definition for \(A\) hence \(A\) is \(f\)-adic. Conversely, if \(A\) is \(f\)-adic, and \(B\) an open subring, then \(Z\) is a bounded subring, and \(Z \subseteq B\) which is open. Then accordingly to (ii) of the previous corollary, there exists a ring of definition \((A_0, I)\) for \(A\) such that \(A_0 \subseteq B\). This makes \(B\) a \(f\)-adic ring.

**Remarque 2** Let \(A\) a topological ring.
* \(A^*\) is not necessarily a subring. For instance, \(A = \mathbb{R}, \|\cdot\|\) then \(A^* = [-1, 1]\) is not a subring.
* \(A^*\) isn't necessarily open, take \(\mathbb{R}\) again, where \([-1, 1]\) isn't open. From what we've seen, these two properties are true for \(f\)-adic ring.
* \(A^*\) isn't bounded.

**exemple 1** Take \(C\) non reduced, and then a non zero \(s\) \(t a^n = 0\). Put \(B = C[X]\) and \(A = B_X = C[X, X^{-1}]\) with the induced structure of a Tate ring (cf example 1.1 (iv) of [5]) then \(\frac{X}{a_n} \in A^*\) for all \(m\), because \(\left(\frac{X}{a_n}\right)^n = 0\), but if there existed a \(p \in A^*(X^p) \subseteq B \subseteq C[X]\), we would have \(\frac{X}{a_n} = \frac{X}{a_n} X^p - \frac{X}{a_n} \in B\) which is absurd.
But here \(A\) is not reduced.

**exemple 2** Put \(B = k[X_1, X_2]_{\geq 0}/(X_1^2 - X)\), and \(A = B_X^*\) with the induced Tate structure. Put \(a_n = \frac{X_1}{X_{n+1}}\). Then \(\alpha_1 = 1\), so \(\alpha_n\) depends only on the parity of \(m\), and \(a_n \in A^*\). So for every \(m\), we have \(a_{m+1} X^m = \frac{X_1}{X_{m+1}} X^m \notin B\).
But here \(B\) is not noetherian, nor integral \((X_1 - X_2)(X_1 + X_2) = 0\).

**exemple 3 case with \(B\) noetherian and integral?**

**Proposition 1.4** Let \(A\) be a height 1 valuation ring. Then \(k = qf(A)\), with the topology induced by \(A\) is a Tate ring.

D : \(A - k^e\) is an open subring, and \(A\) is adic with a finitely generated ideal of definition. Indeed, let \(x \in k^e\) and \(x \neq 0\), i.e. such that \(x \in \mathfrak{m}_A\), i.e. \(0 < v(x) < 1\). Put \(I = (x) - A.x\). Then \(I\) is an ideal of definition of \(A\). Moreover, \(x\) is a nilpotent unit of 1.

**Définition 1.2** A ring homomorphism \(f : A \rightarrow B\) between \(f\)-adic rings is called adic if there exist \((A_0, I)\) and \((B_0, \mathcal{J})\) rings of definition such that \(f(A_0) \subseteq B_0\) and \(f(\mathcal{J}).B_0 = \mathcal{J}\).

**Lemme 1.2 (1.8(i))** If \(f : A \rightarrow B\) is an adic ring homomorphism and \(T \subseteq A\) is bounded, so is \(f(T)\).

D : let \(m\), and so \(J^m = B_0(f(I)^m)\) a 0 neighborhood. Then \(\exists p\) such that \(T.J^p \subseteq I^m \Rightarrow f(T).J^p \subseteq J^m\).

**Remarque 3** If \(f : A \rightarrow B\) is a ring homomorphism, then \(f(A^*) \subseteq f(B^*)\), and if \(f\) is adic, then from the lemma, \(f(A^*) \subseteq B^*\) because if \(\{a^n\}\) is bounded, so is \(f(a^n) - \{f(a^n)\}\).

**1.1 Microbial valuation**

**Proposition 1.5** Let \((K, v)\) be a valued field. Then the topologies of \((K, +)\) having \(U_g = \{x \in K \mid v(x) < g\}\), and \(V_g = \{x \in K \mid v(x) \leq g\}\) as fundamental system of neighborhood of 0 make \(K\) a topological field and are the same.

**Définition 1.3** Call the height of \(\Gamma\) the number of convex (called isolated in [23] ) subgroups of \(\Gamma\) (possibly \(\infty\)). If \(A\) is a valuation ring, call the height of \(A\), the height of its value group.

[23] prop 5 §4 , the height of \(A\) is the number of non-zero prime ideals of \(A\), i.e. its Krull dimension.
**Proposition 1.6** (prop 8 §4 [2]) \( \Gamma \) is of height 1 iff \( \Gamma \) is a subgroup of \((R, +, \leq)\).

**Définition 1.4** [6, p. 39] a non archimedean field is a topological field whose topology is defined by a rank 1 valuation.

**Proposition 1.7** Let \( K \) be a field, \( \nu, \nu' \) 2 valuations that are not unproper (unproper = trivial). According to [2] prop 3 §7 they define the same topology on \( K \) iff they are dependant, i.e. the ring genereted by \( \nu \) and \( \nu' \) is not \( K \).

Let \( A \) be microbial, \( \nu \) the valuation it induces on \( A \) and \( K = qf(A) \), then there exists \( w \) another valuation, which is of height one such that they define the same topology. Let \( B \) the subring of \( K \) generated by \( A - A_w \) and \( A_w \). Then we have seen that \( B \neq K \) and so [2] prop 1 §4 \( B \) is a valuation ring of \( K \), let’s call \( \nu \) its valuation. Then \( A_w \subseteq B \neq K \), and [2] prop 4 §4 the subrings containing \( A_w \) correspond bijectively with the convex subgroups of \( \Gamma_w \subseteq (R, +) \). In that case the only convex subgroups of \( \Gamma_w \) are \([0] \) and \( \Gamma \) itself, corresponding to the subrings \( A_w \) and \( K \). So \( B - A_w \) and \( A_w \subseteq A \), i.e. we have proved that if \( A \) and \( B \) are dependant valuation ring and \( B \) is of height 1, then \( A \subseteq B \).

If \( A \) is a valuation ring, \( \Gamma \) its value group , \( K = qf(A) \) there are correspondans:

\[
\begin{array}{c|c|c}
\{p \text{ prime ideals of } A\} & \leftrightarrow & \{B \mid A \subseteq B \subseteq K, B \text{ subring}\} \\
\{\nu\} & \leftrightarrow & \{[H \subseteq \Gamma, \text{convex subgroup}]\}
\end{array}
\]

These correspondances are [2] 3.3 and 1 §4.

Hence \( A \) is of height 1 iff \( A \) is maximal for the subrings of \( K \) such that \( A \subseteq B \subseteq K \) iff \( \Gamma_A \) doesn’t have any convex subgroups except \([0]\) and \( \Gamma_A \) iff \( A \) is of Krull dimension 1, i.e. its only prime ideals are \([0]\) and \( m_A \).

**Proposition 1.8** Let \( \Gamma \) be an ordered group. Then it has a convex subgroup \( G \neq \Gamma \) maximal iff \( \exists x \in \Gamma \) such that \( < x >_{\text{conv}} \nsupseteq \Gamma \).

Let \( G \subseteq \Gamma \) with \( G \) convex and maximal. Let \( x \in \Gamma \setminus G \). The convex subgroups being totally ordered, and since \( x \notin G \), \( G \nsubseteq < x >_{\text{conv}} \) so \( < x >_{\text{conv}} \nsupseteq \Gamma \) because of the maximality of \( G \).

\( \Leftrightarrow \) Let \( G = \cup_{H \subseteq \Gamma} \text{ convex. Since convex subgroups are stable by union (for instance because they are totally ordered)}, G \) is convex. Since \( x \notin H \forall H \) in the union, \( x \notin G \) hence \( G \nsubseteq \Gamma \), and is maximal for this property.

Hence a valuation ring \( A \) is microbial
\( \Leftrightarrow \exists A \subseteq B \subseteq K \) with \( B \) of height 1
\( \Leftrightarrow \exists A \subseteq B \subseteq K \) with \( B \) maximal
\( \Leftrightarrow A \) contains a prime ideal \( p \neq 0 \) minimal
\( \Leftrightarrow \Gamma \) contains a convex subgroup maximal \( \neq \Gamma \).
\( \Leftrightarrow \exists g \in \Gamma \) such \( G = < x >_{\text{conv}} \)

**Définition 1.5** [6, p. 40] A valuation ring \( A \) is microbial if it satisfies one of the following equivalent property :

1. \( qf(A) \) ( with the topology induced by \( A \) ) is a non archimedean field.
2. \( qf(A) \) is a Tate ring.
3. \( qf(A) \) has a topologically nilpotent unit.
4. \( A \) is non-discrete and adic
5. \( A \) has a prime ideal of height 1.
Hence

Let $X$

2.1 compactness, filters

$\Rightarrow 3$ is in the definition of being a Tate ring.

3 $\Rightarrow 1$: let $x$ be a nilpotent unit. Then $x^n \to 0$, and it is easy to see that $< x >_{\text{conv}} \subseteq \Gamma$ and we are done with the preceding remark.

$1 \Rightarrow 4$: Since $A$ is of height 1, it is not discrete (for 0 is not open), and if $B$ is a valuation ring of height 1 of qf(A) that induces the same topology that $A$ we can pick $x \in m_B$ small enough such that $x \in A$ (since $A$ is a neighborhood of 0), and then we see that if $I = A \cdot x$, then $A$ is I-adic.

$4 \Rightarrow 1$: if $A$ isn’t discrete and adic. Let $i \in \Gamma \setminus \{0\}$ (this is possible precisely because $A$ is not discrete so $I \neq \{0\}$. Then $\forall g \in \Gamma$, there exists a $n$ such that $I^n \subseteq \{a \in A \mid v(a) < g\}$ hence $v(i)^n < g$ and using the fact that $v(A) \leq 1$ we have that $< i >_{\text{conv}} = \Gamma$.

$1 \Rightarrow 5$ was in the previous remark.

example: Let $K = k(x,y)$, and $v_1 : K \to \mathbb{Z}_\text{lex}$ $P = \sum a_{n,m} x^m y^n \mapsto - \min((n,m) \mid a_{n,m} \neq 0)$.

It is a valuation (\cite{15} §3, ex. 6), with the general case $v : k[\Gamma^+] \to \Gamma$, $\sum a_g x^g \mapsto - \min\{n \mid \exists msuch that a_{n,m} \neq 0\}$.

Let $v_2 : k(x,y) \to \mathbb{Z}$ $\sum a_{n,m} x^m y^n \mapsto - \min\{n \mid \exists msuch that a_{n,m} \neq 0\}$.

Let $\pi : \mathbb{Z}^2 \to \mathbb{Z}$, $(n,m) \mapsto n$. Then $v_2 = \pi \circ v_1$. Let’s call $\mathcal{T}_i$ the topologies generated by $v_i$.

$\mathcal{T}_1 \neq \mathcal{T}_2$

Let $V_{n,m} = \{f \in K \mid v_1(f) < (n,m)\}$ and $U_p = \{f \mid v_2(f) < p\}$.

Then $\forall (n,m), U_{n-1} \subseteq V_{n,m}$ so $\mathcal{T}_1 \subseteq \mathcal{T}_2$. Conversely, $\forall (n,m), V_{n,m} \subseteq U_n$ so $\mathcal{T}_2 \subseteq \mathcal{T}_1$.

In fact since $v_2 = \pi \circ v_1$. $A_{\mathcal{T}_1} \subseteq A_{\mathcal{T}_2}$ and they are proper valuation, so cf prop \cite{15}, they define the same topology. In this example, $A_{\mathcal{T}_2}$ is not a valuation of height 1, but it is microbial.

a valuation not microbial

Let $\Gamma = \mathbb{Z}^\mathbb{N}$ (the sequences in $\mathbb{Z}$ with finite support), with the (reverse) lexicographic order, i.e. $x = (x_0, \ldots, x_n, 0, \ldots)$ with $\text{supp}(x) \subseteq \{0, \ldots, n\}$, and the same thing for $y$, if $x_n > y_n$ then $x > y$.

More generally, if $x_i = y_i$ for $i > n$ and $x_n > y_n$, $x > y$. We can then define $v : k[x]_{\mathbb{N}} \to \mathbb{Z}^\mathbb{N}$ by $v(\sum a_i x^i) = - \min\{v \mid a_i \neq 0\}$. It is easy to see that the convex subgroups of $\Gamma = \mathbb{Z}^\mathbb{N}$ are the $\Gamma_n = \{x \mid \text{supp}(x) \subseteq \{0, \ldots, n\}\} = < x >_{\text{conv}}$ for any $x$ of the form $x = (x_0, \ldots, x_n, 0, \ldots)$ with $x_n \neq 0$, and hence there doesn’t exist a proper maximal convex subgroup.

Hence $A_{\mathcal{T}_1}$ is not microbial. (we could also have seen, that for any $x$, $< x >_{\text{conv}} \subseteq \Gamma$)

Remarque 4 We can extend the definition of being microbial to fields (this is actually nothing since a field is a valuation ring), to valued ring $v : A \to \Gamma$, univ, by saying if $B = A/\text{supp}(v)$ and $K = qf(B)$, $K$ is microbial. All the preceding properties work as well.

Here is an example with

$A \xrightarrow{v} \Gamma_0$

$\phi$

$\downarrow$

$\downarrow i$

$B \xrightarrow{w} H_0$

with $v \sim w \circ \phi$, $w$ microbial but $v$ not microbial.

Indeed take $v$ not microbial on $A - k$ a field. Then put $B = k[x]/H = \mathbb{Z} \times \Gamma$ and $\alpha(\sum a_i X^i) = \max\{-i, v(a_i)\} \mid a_i \neq 0\). It is a microbial valuation on $B$ for instance $< (1, 0) >_{\text{conv}} = H$, or $0 \times \Gamma$ is a maximal proper convex subgroup).

2 Valuation Spectrum

2.1 compactness, filters

\cite{15} [6.7.9]

Let $X$ be a topological space, $F$ a filter on $X$, $x \in X$, $B(x)$ the filter of neighborhood of $x$. We say that $G$ is finer than $F$ if $G \supseteq F$.
Définition 2.1 $x$ is a limit point of $F$ if it is finer than $B(x)$, i.e. every neighborhood contains an element of $F$.

$B$ is said to be a base of filter if it is stable by finite intersection, and doesn’t contains $\emptyset$.

Définition 2.2 Let $B$ be a base of filter, $x$ is adherent to $B$ if for every $B \in B$, $x \in B$.

If $F$ is finer than $G$ and $x$ adherent to $F$, then it is also adherent to $G$.

Proposition 2.1 ($\S 6$, cor 2) Let $\Phi - \{F_i\}$ a set of filter. There exists a filter finer that all the $F_i$ iff for all $F_1, \ldots, F_n \in \Phi$ and $F_i \in F_i$, $F_1 \cap \ldots \cap F_n \neq \emptyset$.

So $x$ is adherent to $F$
$\Leftrightarrow \forall F \in F$ and $U \in B(x)$, $F \cap U \neq \emptyset$.
$\Rightarrow \exists$ a filter $G$ finer than $F$ and $B(x)$. Indeed consider for $U \in B(x)$ the filter $F_U = \{V \mid U \subseteq V\}$, and apply the proposition with $\Phi - \{F_U \mid U \in B(X)\} \cup \{F\}$.
$\Rightarrow \exists$ a filter $G$ finer than $F$, which converges to $x$.

Corollaire 2.1 Let $U$ be a ultrafilter. $U$ converges to $x$ iff $x$ is adherent to $U$.

Définition 2.3 (Prop) $X$ is quasi compact if it satisfies one of the following properties :

1. every filter has an adherent point
2. every ultrafilter is convergent
3. Every family of closed set whose intersection is empty has a finite subfamily whose intersection is empty.
4. Every open cover has a finite subcover.

D : (i) $\Rightarrow$ (ii) Let $U$ be a ultrafilter, it has an adherent point, and so converges to it. (ii) $\Rightarrow$ (i) : let $F$ be a filter, $U$ a ultrafilter which is finer, it converges to $x$ say, so $x$ is adherent to $U$ and also to $F$.
(i) $\Rightarrow$ (iii) Let $\{F_i\}$ be a family of closed subsets whose intersection is empty. Let’s suppose that every finite intersection is non empty. Then there exists a filter $F$ that contains all the $F_i$. Let $x$ be an adherent point, so $x \in F_1 - F_i$ for all $i$, which contradicts $\cap F_i = \emptyset$.
(iii) $\Rightarrow$ (i) Let $F$ be a filter, and suppose it has no adherent point. Then $\forall x \in X, \exists F_x \in F$ with $x \notin F_x$, and since $F_x \in F$, $F_x$ too. So the $F_x$ have the finite intersection property, however by construction, their intersection is empty, which contradicts (iii).
(iii) and (iv) are dual. $\blacksquare$

2.2 remark on compactness

A (open) basis of $X$ is a family $B$ (of open subsets necessarily) from the following definition) , such that the open of $X$ are the (arbitrary) union of elements of $B$. Dually, it will be called a closed basis, if the closed sets are the intersection of elements of $B$.

A (open) sub-basis of $X$ is a family $C$ (of open subsets necessarily) from the following definition), such that the open of $X$ are the (arbitrary) union of finite intersection of elements of $C$. Dually, it will be called a closed sub-basis, if the closed sets are the intersection of finite union of elements of $C$. The family $B$ of finite intersection of $C$ is then clearly a basis, called the basis generated by $C$.

Let $C$ be a subbasis, and $B$ the basis it generates. Taking the complementary, we give the same name to the (sub)-basis of closed or open sets by taking the complementary Then the following are equivalent :

$X$ is quasi compact
$\Leftrightarrow$ Every open cover has a finite subcover
$\Leftrightarrow$ Every open cover by elements of $B$ has a finite subcover.
$\Leftrightarrow$ Every family of closed set of $B$ whose intersection is empty has a finite subfamily whose intersection is empty.
Proposition 2.2 The following are equivalent:
1. Every family of closed set of \( B \) whose intersection is empty has a finite subfamily whose intersection is empty.
2. Every family of closed set of \( C \) whose intersection is empty has a finite subfamily whose intersection is empty.

D: clearly since \( C \subseteq B \) \( (i) \Rightarrow (ii) \).
Let’s suppose \( (ii) \), and let \( F = \{ F_i \} \) be a family of closed subsets of \( B \) with the finite intersection property. Let’s show that \( \cap_i F_i \neq \emptyset \).
Let \( A \) be a maximal family of closed subsets of \( B \) such that \( A \supseteq \{ F_i \} \) and has the finite intersection property. (such an \( A \) exists with Zorn’s Lemma). So \( \cap_{F \in A} F \subseteq \cap_i F_i \) so it is enough to show that \( \cap_{F \in A} F \neq \emptyset \). We now suppose \( \{ F_i \} \) maximal. It is easy to see that it implies that the family is stable by finite intersection. Every \( F_i \) can be written \( F_i = F_i^1 \cup \ldots F_i^n \) (should write \( n_i \) instead of \( n \)) Let’s show that for every \( i \) there exists a \( j \) with \( F_i^j \in F \).
Let \( j \in \{ 1, \ldots n \} \). If \( \forall G \in F \ G \cap F_j^i \neq \emptyset \) then the family \( F \cup \{ F_j^i \} \) still has the finite intersection property and we are done. Otherwise, for all \( j \) there exists a \( G_j \in F \) such that \( G_j \cap F_j^i = \emptyset \). Then \( G = \cap_j G_j \in F \), but \( \forall j \ G \cap F_j^i = \emptyset \) so \( G \cap F = \emptyset \), which is a contradiction (with the FIP).
So \( \forall i \) there exists a \( j_i \) such that \( F_i^{j_i} \in F \). Then \( \cap_i F_i^{j_i} \subseteq \cap_i F_i \) and since by construction \( F_i^{j_i} \in C \) \( (ii) \) implies that \( \cap_i F_i^{j_i} \neq \emptyset \), so \( \cap_i F_i \neq \emptyset \).

Here is another proof: let’s show that \( X \) is quasi compact, i.e. satisfies the property, every ultrafilter converges to some \( x \). Indeed let \( \mathcal{U} \) be a ultrafilter and let’s suppose it doesn’t converge to any \( x \). Then for every \( x \) we can find \( F \subseteq \mathcal{U} \) such that \( x \notin F \). Then there exists \( F_1 \ldots F_n \) some closed of \( C \) such that \( G = F_1 \cup \ldots F_n \supseteq F \), and \( x \notin F_1 \cup \ldots F_n = G \). Then \( G \in \mathcal{U} \) so there exists one \( i \) such that \( F_i = F_i = x \) since it is an ultrafilter. But the \( F_x \in C \), they have the FIP, but have empty intersection since \( \forall x, x \notin F_x \).

We can then deduce that:

Proposition 2.3 \( X \) is quasi compact
\( \iff \) Every family of closed set of \( C \) whose intersection is empty has a finite subfamily whose intersection is empty.
\( \iff \) Every open cover by elements of \( C \) has a finite subcover.

2.3 constructible sets

cf EGA§9.

Définition 2.4 \( Z \subseteq X \) is retrocompact iff \( \forall U \) qc open, \( Z \cap U \) is qc (note that it is equivalent that \( Z \cap U \) is qc in \( Z \) or in \( U \) since this only depend on the topology of \( Z \cap U \), i.e. if \( i : Z \subseteq X \rightarrow X \) is quasi-compact.

Définition 2.5 \( S \subseteq X \) is constructible if it is in the boolean algebra \( (\cap, \cup, \cdot) \) generated by the open retrocompact.

Proposition 2.4 Let \( V \subseteq X \) retrocompact and \( U \) open in \( X \), then \( V \cap U \) is retrocompact in \( U \).

D: let \( W \subseteq U \) a qc open. Then \( (V \cap U) \cap W = V \cap W \). Since \( W \) is qc in \( X \), and \( V \) retrocompact, \( V \cap W \) is qc in \( X \), so also in \( U \).

cf rq après 9.1.1:

Remarque 5 1. if \( V_1 \) and \( V_2 \) are retrocompact, \( V_1 \cup V_2 \) too.
2. If \( V_1, V_2 \) are retrocompact open, then \( V_1 \cap V_2 \) too.

Indeed; (i) if \( U \subseteq X \) is open qc, \( V_1 \cap U \), and \( V_2 \cap U \) are qc. Quasi compact sets are stable by finite union so, \( (V_1 \cup V_2) \cap U \) is qc.

(ii) If \( U \subseteq X \) is qc open, \( V_1 \cap U \) is qc open, so \( V_2 \cap (V_1 \cap U) \) too.

This is probably the reason why in EGA, the retrocompact sets are introduced, because, the retrocompact open are stable by intersection, whereas qc not necessarily (unless you make the assumption \( X \) is quasi-separated...which is tantamount).

If \( X \) is Hausdorff, \( U \) is qc open, iff \( U \) is compact open iff \( U \) is compact open-closed.

**Proposition 2.5 (EGA 0 9.1.8)** If \( U \subseteq X \) is open.

1. If \( T \) is constructible in \( X \), \( T \cap U \) is constructible in \( U \).
2. If \( U \) is in addition retrocompact, the converse is true: if \( T \subseteq U \) is constructible in \( U \), it is also constructible in \( X \).

**Définition 2.6** \( T \subseteq X \) is locally constructible, if for every \( x \in X \) there exists \( V \) an open neighborhood of \( x \) such that \( T \cap V \) is constructible in \( X \).

**Définition 2.7 (EGA 4 1.9)** \( E \subseteq X \) is pro-constructible (resp. ind-constructible) if for every \( x \in X \) there exists \( V \) a neighborhood of \( x \) such that \( V \cap E \) is an intersection of locally constructible sets (resp union).

**Remarque 6** (cf EGA 0 9.1.11)** If \( U \subseteq X \) is open, \( T \) locally constructible in \( X \), then \( U \cap T \) is locally constructible in \( U \).

(EGA 0 9.1.10)** If \( X \) is quasi compact, and has a basis of open retrocompact, then \( T \) is constructible iff it is locally constructible.

(EGA 4 1.9.4)** Under these hypothesis, \( T \subseteq X \) is pro-constructible iff it is an intersection of constructible : indeed then we can cover \( X \) by some finite retrocompact open (since retrocompact open \( \Rightarrow \text{qc} \), \( X_i \), say \( T = \cup_i T \cap X_i \), and \( T \cap X_i = \cup_j T_{ij} \) is constructible in \( X \), then \( T = \cup_i \cap_j E_{ij} \cup_{i,j} (1_{ij})_{i,j} E_{ij} \cup_{i,j} (1_{ij})_{i,j} E_{ij} \) is an intersection of constructible sets.

### 2.4 spectral spaces

**Définition 2.9** \( T_0 \), quasi-compact, \( X \) spectral if it is quasicompact, the qc open form a basis IS quasi-separated iff the qc open are retrocompact.

Moreover, if \( X \) is quasi-compact quasi-separated, the qc open are precisely the retrocompact open.

To give a counter-example, let \( X = \text{Spec}(k[T_{1,20}]) \), and \( U = X \setminus (T_{1}) \). Define \( Y \) as two copies of \( X \) (say \( X_1 \) and \( X_2 \), glued along \( U \). Then like \( X \), \( X_1 \) are qc, but \( X_1 \cap X_2 \) is not qc.

**Remarque 7** Let \( X \) be a topological space such that there exists a basis for the topology \( \mathcal{B} \setminus \{U\} \) which are qc, and stable by finite intersection. Then, if \( V \) is open of \( X \), \( V \cap W \) is also qc. Indeed write \( V = \cup_{i=1}^n V_i \) with \( V_i \subseteq B \). (This is possible because \( V \) is qc and \( B \) a basis. Do the same for \( W \), then \( V \cap W = \cup_{i,j} V_i \cap W_j \) is then a finite union of qc sets, so is qc. Hence \( X \) is quasi-separated.

If \( X \) is a separated scheme, it verifies these hypotheses, so the intersection of two quasi compact is quasicompact.

**Définition 2.9** [3] 0] \( X \) is spectral if it is \( T_0 \), quasi-compact, the qc open form a basis and are stable by finite intersection, and every non-empty closed irreducible subset has a unique generic point.

**Remarque 8** In [2] the definition is with a unique generic point, but without \( T_0 \). This is equivalent: suppose that \( X \) is \( T_0 \) if \( \bar{x} = \bar{y} x \neq y \), let \( U \) be an open \( s \in x \in U, y \notin U \). Then \( x \in \bar{y} \subseteq U^c \) contradiction. Conversely if the generic points are unique, let \( x \neq y \), then \( \bar{x} = \bar{y} \), say, \( \bar{x} \notin \bar{y} \), it implies \( x \notin y \) then \( x \not\in \bar{y} \) which separates \( x \) and \( y \).

From the previous remark, if \( X \) is spectral, \( X \) is quasi-separated.

It also implies, that in the definition, you can only require that there exists a basis of the topology which with qc open, which are stable by finite intersection (this is the statement [2], prop 4 (i) \( \Leftrightarrow \) (ii)).
**Proposition 2.6** Let $X$ be a spectral space. An open $U$ is retrocompact iff it is quasi compact.

$D: \Rightarrow$ If $U$ is retrocompact, since $X$ is qc, $X \cap U - U$ is qc.

$\Leftarrow$ Let $V$ be an open qc. Then $U \cap V$ is qc. $\Rightarrow$

In the following part, $X$ will always be a spectral space.

**Proposition 2.7** $T \subseteq X$ is locally constructible iff $T$ is constructible.

$D: T$ loc constructible iff $\forall x \exists V_x x-$neighborhood, such that $T \cap V_x$ is constructible in $V_x$

$\Leftarrow$ $X = X_1 \cup \ldots \cup X_n$ such that $T \cap X_i$ is constructible in $X_i$, and $X_i$ qc, using the fact that qc open form a basis, and that $X$ is qc, and that intersecting with an open preserves constructible sets

$\Rightarrow$ $T$ constructible, since the $X_i$ being quasi-compact, they are retrocompact, and then $T \cap X_i$ constructible in $X_i$ implies it is constructible in $X$, and $T = \cup_i (T \cap X_i)$. $\Rightarrow$

**Proposition 2.8** $T$ is proconstructible iff $T$ is an intersection of constructible sets in $X$.

We only have to show $\Rightarrow$.

$T$ proconstructible iff $\forall x \exists V_x$ such that $E \cap V_x$ is an intersection of locally constructible in $V_x$

$\Rightarrow$ $\forall x \exists V_x \text{qc such that } T \cap V_x$ is an intersection of locally constructible. (using the fact that qc form a basis, and the fact that locally constructible are preserved by intersecting with an open, so intersection of locally constructible are preserved when intersecting with an open)

$\Rightarrow$ $\forall x \exists V_x$ open qc such that $T \cap V_x$ is an intersection of constructible (using the fact that $V_x$ is an open retrocompact of $X$)

$\Rightarrow$ $X = X_1 \cup \ldots \cup X_n$ with $X_i$ qc and $T \cap X_i$ intersection of constructible in $X_i$. Then since $X_i$ is qc so retrocompact , we see that $T \cap X_i$ is an intersection of constructible in $X$, say $T \cap X_i = \cap C_{i,j}$, for some $C_{i,j}$ union.

Then $T = \cup_{i=1,n} \cap C_{i,j} = \cap_{j=1,n} \cup_{i=1,n} C_{i,j}$ which is an intersection of constructible of $X$.

**Remarque 9** So what [4] calls the patch topology $X_{\text{patch}}$, is what EGA 1.9.13 , calls the constructible topology $X_{\text{cons}}$. The open subsets are the ind-constructible subsets , and the closed pro-constructible.

**Proposition 2.9** $X_{\text{cons}}$ is compact.

$D: It$ is Hausdorff, because 3 open qc $U$ that separates two points $x, y$, so $U$ and $U^c$ are open that separate $x, y$.

If we use the remark on compactness, let’s note $C$ the subsbasis of closed sets of $X_{\text{cons}}$ formed by (arbitrary) closed and qc open (from $X$). Then we have to check that a family $A$ of $C$ which has FIP has non empty intersection. With Zorn, if we take $B$ a maximal family with the FIP containing $B$, its intersection will be smaller than that of $A$, so we can restrict to $B$, i.e. suppose that $A$ is maximal with the FIP.

$A = F \cup U$, the closed, and the qc open. Then, because $X$ is quasi-compact , $G = \cap_{F \in F} F$ is a closed non-empty. Then $G$ has the FIP $F, U$ (for the qc open , this is because the $F, U$ have the FIP , that $U$ is qc so their intersection, which is $G \cap U$ is non empty), so by maximality , $G \in A$.

If it wasn’t irreducible, let’s write it $G = G_1 \cup G_2$. Then if $A \cup \{G_i\} i = 1,2$ doesn’t have the FIP , we would have an $A_i \cap G_i = \emptyset$, whence $G \cap (A_1 \cap A_2) = \emptyset$ but $A_1 \cap A_2$ is $G$ is irreducible, say $G = \bar{g}$, then $g \in F$ for all closed, and if $g \notin U$ for one open , $\bar{g} \cap U = \emptyset$, absurd.

second proof: Let $F$ be a family of $B$, the closed-basis of $X_{\text{cons}}$ of pro-constructible sets formed by the $F \cup U$, $F$ closed, and $U$ qc open, with the FIP. With Zorn’s Lemma, we can assume it is maximal. Then for $F \cup U \in F$, $F$ or $U \in F$, indeed otherwise, there are $A, B \in F$ with $A \cap F \neq \emptyset$, and $B \cap U \neq \emptyset$, then $F \cap A \cap B = \emptyset$ which contradicts FIP. Let then $F_1$ be the closed sets of $F$ and let then $F_2$ be the open sets of $F$. One has $\cap A = \cap A \cup F A$. Then $F = \cap A$ is a non empty closed set (by hypothesis). It is irreducible, because if $F = F_1 \cup F_2$ with the same argument that above one shows one of the $F_i$ is $F$. So since $X$ is spectral, $F = \{x\}$ for some $x \in X$, and as above one shows $x \in \cap F_i$. $\Rightarrow$

**Corollaire 2.2** If $X$ is spectral, the constructible subsets of $X$ are exactly the closed-open subsets of $X_{\text{cons}}$. 12
D : $\Rightarrow$ If $U$ is qc open in $X$, by definition, it becomes a closed open of $X_{cons}$, and since the closed-open are stable by finite boolean combination we are done.  \\
$\Leftarrow$  Let $U$ be a closed open of $X_{cons}$. It is then compact, since closed in a compact. Since by definition the $U \cap V^c$ form a basis of $X_{cons}$ for $U, V$ qc open of $X$, we can write $U = \cup_{i=1...n} U_i \cap V_i^c$ with $U_i, V_i$ qc open, so $U$ is constructible.  \\

**Proposition 2.10 ([4] Prop 4)** Let $X$ be quasi-compact, $T_0$, has a basis formed by qc open that are closed under finite intersection. The following are equivalent :  \\
1. $X$ is spectral  \\
2. Every nonempty irreducible closed subspace has a generic point  \\
3. every family of qc open of a closed subspace with the FIP has finite intersection.  \\
4. $X_{cons}$ is compact and has a basis of closed-open sets.  \\
5. $X_{cons}$ is quasi-compact  \\
6. A family of pro-constructible sets with the FIP has non empty intersection.

D : (i) $\iff$ (ii) this is a consequence of [7]  \\
(i) $\Rightarrow$ (v) is 2.9  \\
(v) $\Rightarrow$ (vi) is just the alternative definition of quasi-compactness, and the fact that the pro-constructible are the closed sets of $X_{cons}$.  \\
(vi) $\Rightarrow$ (iv) $X_{cons}$ is then quasi-compact. Since theqc open form a basis of $X$, $X_{cons}$ is Hausdorff (so compact), and in fact by definition, the sets of the form $F \cap U$ with $U$ qc open, and $F$ the complementary of a qc open form by definition a basis of $X_{cons}$. Their complementary is $F^c \cup U^c$ are also open in $X_{cons}$, so $F \cap U$ is closed-open.  \\
(iv) $\Rightarrow$ (iii) : Let $F$ be a closed set and $\{U_i\}$ a family of quasi-compact open of $F$ with the FIP. Each $U_i$ is qc , $U_i \subset F \cap V_i$ where $V_i$ is open in $X$. Hence since the qc open form a basis of $X$, we can write $V_i = \cup_j W_j$, and since each $U_i$ is quasi-compact, there exists a finite subset (say $\{1...n\}$) such that $U_i = \cup_{1...n} W_j \cap F$. Hence, each $U_i$ is pro-constructible in $X$, and since $X_{cons}$ is compact they have non empty intersection.  \\
(iii) $\Rightarrow$ (ii) Let $F$ be an irreducible closed subset. Put $G = \cap \text{non-empty qc open of } F \cap U$. It is non empty (a space $Z$ is irreducible $\iff$ finite intersections of non empty open are non empty), so the set of non-empty qc open of $F$ has FIP and we use the hypothesis. Suppose $x \neq y \in G$. Then, $\exists U$ a qc open of $X$ such that $x \in U$ and $y \notin U$ (because $X$ is $T_0$ and there is a basis of qc open. Then $U \cap F$ is qc open and non empty, so $y \notin G$ which is absurd. So $G = \{x\}$. Suppose $\{x\} \neq F$. Then $V = F \setminus \{x\}$ is a non empty open of $F$ so contains a non empty qc open of $V$ of $F$, but $x \notin V$ contradiction.  \\

**Proposition 2.11 ([4] Prop 7 , cf also [5] (rem 2.1 (vi) )** Let $(X,S)$ a compact space , $B = \{U\}$ a family of closed-open sets (hence compact) of $X$. Let $T$ the topology of $X$ which has $B$ as a sub-basis.  \\
Then $(X,T)$ is $T_0$ $\iff$ $(X,T)$ is spectral, and in that case the constructible subsets of $(X,T)$ are precisely the closed-open subsets of $(X,S)$

D : $\Rightarrow$ is clear .  \\
$\Leftarrow$ Taking finite intersection of $B$ doesn’t change the fact it is formed by closed-open sets of $(X,S)$ , so we can assume $B$ is a basis stable under intersection of $T$.  \\
By definition, $T \subseteq S$, hence it remains quasi-compact, and has a basis $(B)$ stable under intersection of quasi-compact open and is $T_0$ by hypothesis. So according to 2.10 we just have to prove that $(X,T)_{cons}$ is compact.  \\
Now, let $V$ be a quasi-compact open of $(X,T)$ , so by a quasi-compactness $V = \cup_{i=1...n} U_i$ with $U_i \in B$. And $V^c = \cap_{i=1...n} U_i^c$. We can deduce from that :  \\
$\iota : (X,S) \rightarrow (X,T)_{cons}$ is continuous. Since it is bijective, and $(X,S)$ is compact and $(X,T)_{cons}$ is Hausdorff, it is a homeomorphism (the direct image of a closed is the direct image of a compact so compact). So in fact $(X,S) = (X,T)_{cons}$ which is then compact.  \\
The fact that constructible of $(X,T)$ are the closed-open of $(X,S)$ is then 2.2.  \\

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**Remark 10** Let $X$ a spectral space; $T$ proconstructible. Then $[\mathcal{D}, 2.1(i)]$ $T$ is qc in the topology of $X$ and $X_{cons}$. In particular, since $X$ is constructible, it is quasi-compact in $X_{cons}$. Note that $X_{cons}$ is Hausdorff: indeed, $X$ is $T_0$, so if $x \neq y$, let say $U$ a neighborhood of $x$ not containing $y$. Since $X$ is spectral, we can assume $U$ isqc, so $U$ and $U^c$ are open in $X_{cons}$ and separate $x$ and $y$. So $X_{cons}$ is compact.

**Définition 2.10** ([cf.\emph{H0} or \emph{D.2.1}]) A map $f: X \to Y$ between spectral spaces is said spectral if it is continuous and $f^{-1}$ preserves the qc open (which actually implies continuity).

**Proposition 2.12** $f$ is spectral if and only if $f: X_{cons} \to Y_{cons}$ is continuous.

$\Rightarrow$ let $V$ be an open of $Y_{cons}$, i.e. a union $\cup_i V_i$ with each $V_i$ constructible, i.e. boolean combination of qc open. Since $f^{-1}$ commutes with boolean combination and preserves qc open $f^{-1}(V_i)$ is constructible.

$\Leftarrow$: if $V$ is a qc open, it is constructible and $f^{-1}(V)$ is constructible open, so ([\mathcal{D} 2.1 (i)]) open qc.

$\square$

**Proposition 2.13** (DICKMANN p.90) Let $f: X \to Z$ and $g: Y \to Z$ spectral maps. Then $\pi: X \times_Z Y \to Y$ (taken in Top) is spectral.

$D: \pi$ factizes as $X \times_Z Y \xrightarrow{\pi} X \times Y \xrightarrow{\pi} Y$. $X \times_Z Y$ is closed, so proconstructible in $X \times Y$, so $[D 3.3.1]$ $a$ is spectral, with $[D p88]$, $b$ is spectral too. $\square$

**Définition 2.11** A topological space $X$ is locally spectral if there exists a covering $X_i$ such that each $X_i$ is spectral.

**Remark 11** In [H] Theorem 9, it is proved that locally spectral spaces are precisely the underlying topological spaces of schemes (what he called prescheme).

**Proposition 2.14** (cf.\emph{H} p44) A locally spectral space $X$ is spectral if and only if it is quasi-separated and quasi-compact.

$D: \Leftrightarrow$ is obvious. Then suppose $X$ locally spectral, quasi-compact and quasi-separated. Then cover it with $X_i, i = 1, \ldots n$ that are spectral. For each $i$ consider the inclusion $f_i: X_i \to X$. If $U$ is a qc open of $X$, then $f_i^{-1}(U)$ is quasi-open (since $X$ is quasi-separated). We deduce that $f_i: X_{cons} \to X_{cons}$ is continuous. Then since one easily sees that $(X_1 \times \ldots \times X_n)_{cons} = X_{1cons} \times \ldots \times X_{ncons}$, one sees from the second one that it is compact, and $f: (X_1 \times \ldots \times X_n)_{cons} \to X$ is continuous and surjective, so $X_{cons}$ is compact. Now, $X$ is $T_0$, quasi compact and quasi-separated, (this is local property (contrary to being $T_2$)), so [H] Prop 4 (v) $X$ is spectral. $\square$

**Proposition 2.15** If $f: X \to Y$ spectral, and $T \subseteq X$ proconstructible, then $f(T)$ is proconstructible.

$D: T$ proconstructible mean $T$ closed in $X_{cons}$, since $f$ spectral $\Leftrightarrow f: X_{cons} \to Y_{cons}$ continuous and that $X_{cons}$ and $Y_{cons}$ are compact, $f(T)$ is compact, so closed in $Y_{cons}$, so proconstructible. $\square$

**Proposition 2.16** if $f: X \to Y$ spectral and surjective, $S \subseteq Y$ is constructible (resp., proconstructible) iff $f^{-1}(S)$ is.

$\Rightarrow$ is OK.

Conversely, by surjectivity of $f$, we have $S = f(f^{-1}(S))$ and $S^c = f(f^{-1}(S)^c)$. If $f^{-1}(S)$ is constructible i.e. closed open in $X_{cons}$ then $f^{-1}(S)^c$ is closed, so $f(S)$ is constructible and $S^c = f(S)^c$. $\square$
Proposition 2.17 (Dickmann, schwartz Tressel, Spectral Spaces, theorem 3.3.1). Let $X$ be a spectral space, and $T \subseteq X$. Then $T$ is proconstructible iff $T$, with the induced topology, is spectral, and $i : T \hookrightarrow X$ is a spectral map. Moreover constructible (resp qc open, resp complementary of qc open) in $T$, are the traces of constructible(resp. qc open, resp complementary of qc open) in $X$.

From that we deduce some consequences of [3, 2.1] when $T$ is a proconstructible of $X$

(i) $T$ is quasi-compact for $X$ and $X_{cons}$. In particular an open subset is constructible iff it is qc, and a closed subset is constructible iff its complementary is quasi compact (as a closed subset of a quasi-compact set, it is anyway quasi-compact). Indeed $T = i^{-1}(X)$ with $X$ qc and $i$ spectral so $T$ is qc in $X$. And $T$ is closed in $X_{cons}$ so compact in $X_{cons}$.

(ii) $T$ is constructible iff $T^c$ is proconstructible. Indeed $\Rightarrow$ is clear, and if $T^c$ is proconstructible, $T$ is closed-open in $X_{cons}$ so constructible 

(iii) $T = \cup_{T \subseteq X}$.

Let $x \in T$ and $U = \{u \mid U \in X$ is a quasi-compact open neighborhood of $x\}$. Hence $x \in \cap_{U \subseteq X} U$. Since $x \in T$, $\forall U \in U$, $U \cap T \neq \emptyset$. More generally, if $U_1 \ldots U_n \in U$, $U_1 \cap \ldots \cap U_n \cap T = (U_1 \cap Y) \cap \ldots \cap (U_n \cap Y) \neq \emptyset$. So the $\{U \cap T | U \subseteq X\}$ have the FIP, and are proconstructible in $X$ since $T$ is and the $U$ are, so their intersection is not empty (since $X_{cons}$ is compact). Then let $t \in \cap_{U \subseteq X} (U \cap T)$, and $T^c = \{T \subseteq X\}$ which is then open. If $x \in T^c$, there exists $U \in U$ such that $x \in U \subseteq V$. But by hypothesis $t \in \cap \subseteq V$ which is absurd. So $x \in T$.

2.5 Valuations

Proposition 2.18 ([2], Prop 9 §3) A valuation ring $A$ is noetherian iff a discrete valuation ring (i.e. $\Gamma = \mathbb{Z}$) iff principal (because a finitely generated ideal of a valuation ring is principal)

For instance take $v : k(x_1, x_2) - A \to \mathbb{Z}_{\leq 0}$.

$\sum_{a \in A} x^a \mapsto -\min(\nu | a \neq 0)$. Then $R$ the associated valuation ring is not noetherian. Indeed, its ideal correspond to the interval of $\mathbb{Z}_{\geq 0}$. Among them $\cup_{n \geq 0} [0, (-1, n)]$ is not principal, i.e. of the form $[0, \infty, a]$. This ideal is $R(\frac{a}{x_1})_{n \geq 0}$.

Remark 12 Let $v : A \to \Gamma_0$ a valuation, $R$ its valuation ring of the residual field of $v$. What is the link between $A$ and $R$ being noetherian?

If $R \subseteq K$ is a valuation ring and $R \subseteq A \subseteq K$ an intermediate valuation ring, what link between $A$ and $R$ being noetherian?

No link because $R$ will be noetherian iff $\Gamma = \mathbb{Z}$. So for the first question, take $v : A - C[T] \to (\mathbb{Z}, +, \leq)$, $\sum_{a \in A} x^a \mapsto \max(\{a\} | A$ is noetherian but $\Gamma \neq \mathbb{Z}$, i.e. $A$ noetherian but not $R$.

On the contrary, $v : A - k[x]_{\mathbb{Z}} \to \mathbb{Z}, \sum_{a \in A} x^a \mapsto -\min(\{a | \Gamma \neq \mathbb{Z} \}$ where the $A_i \in k[x, \ldots]$. Now if $R \subseteq A \subseteq K$ then if $R$ is noetherian it is a discrete valuation ring, and the only possibilities for $A$ are $R$ and $K$.

But if $R$ is not noetherian, there will exist an intermediate $A$ noetherian (i.e. discrete valuation ring) iff there exists a quotient $\Gamma/H = \mathbb{Z}$ with $H$ necessarily a (the) maximal proper comes subgroup. Sometimes it is the case, for instance $v : k[x, y] \to \mathbb{Z}, \sum_{a_{(n,m)}} x^a y^m \mapsto -\min((n, m) | \sum_{a_{(n,m)}} \neq 0)$, sometimes not, for instance in the case of a non nonsingular valuation.

Proposition 2.19 Let $v : A \to \Gamma_0$ a valuation with $\Gamma = \Gamma_v$, and $\alpha : \Gamma \to G$ such that $v \circ \alpha$. Then $v$ is injective.

D : Otherwise let $h \in \ker(\alpha) \setminus \{1\}$. Then there exists $x \in K$ the residual field of $v$ with $v(x) = h \neq 1$, but the same calculus in $K_v$ leads $\alpha(x) = 1$. n

Remark 13 A valuation on $A : v : A \to \Gamma_0$ is equivalent to give $p = v^{-1}(0)$ a prime ideal of $A$ and a valuation $\nu$ on $A/p$ which is also equivalent to give an equivalent class of morphism $A \to k$ with $k$ a valued field, where $(\nu, k) \sim (k', \nu')$ if there exists a morphism of valued field $\nu : k \to k'$ such that $\nu' = \nu \circ \phi$. 
2.6 Embedding in $\mathcal{P}(A \times A)$

Proposition 2.20 (2.2) \textit{Let $A$ be a ring, $|$ a binary relation on $A$ such that:

1. $a \mid b$ or $b \mid a \forall a, b$.
2. If $a \mid b$ and $b \mid c$ then $a \mid c$.
3. $a \mid b$ and $a \mid c$ implies $a \mid b + c$.
4. $a \mid b$ implies that $a \mid bc$
5. $a \mid bc$ if and only if $0 \not\mid c$ then $a \mid b$.
6. $0 \not\mid 1$.

Then there exists a unique equivalence class of valuation $v$ s.t. $-1_v$ where $a \mid b$ iff $v(a) \geq v(b)$.
}

\textbf{D:} Let $\sim$ be the binary relation defined by $a \sim b$ iff $a \mid b$ and $b \mid a$. This is an equivalence relation. Indeed reflexivity is a consequence of (i), transitivity from (ii), and symmetry is obvious.

Let's not $p - \{a \in A \mid a \sim 0\} - \{a \mid 0 \mid a\}$. Indeed, $a \sim 0$ then $0 \mid a$, and conversely, $0 \mid a$, since anyway $0 \mid 0$ (because of (i) and (vi)) with (vi) taking $c - a$ we get $0 \mid a$ then $a \sim 0$. $p$ is a prime ideal: first, if $a, b \in p$, $0 \mid a$ and $0 \mid b$ so with (ii), $0 \mid a + b$, hence $p$ stable by $+$, and with (iv) taking $c - 1$ we have $0 \mid a$ so $p$ is a subgroup. In fact (iii) gives that $p$ is an ideal. Now if $a \not\in p$ and $ab \in \text{mathfrak}{p}$ then $0 \not\mid a$ (cf previous remark), $0 \mid ab$, i.e. $0 \mid b, a$ and then with (v), $0 \mid b$, i.e. $b \not\in p$. So $p$ is prime.

Put $B - A/p$. Then $\mid$ factorises through $B$. Indeed let $a, b \in A$ and $\not\mid c$. If $a \mid b$, then since anyway $0 \mid 0$ and $0 \mid c$ by hypothesis, $a \mid c$ so (iii) $a \mid b + c$ hence if $a \sim d$ mod $p$ we have $0 \mid b$ iff $a \mid b$. In particular $a \sim a'$ and we conclude using the transitivity of $\mid$ that $\mid$ factorises through $A/\sim$, and hence also through $B$, and that this relation satisfies also (i) - (vi). Actually (v) becomes even $ab \mid c$ and $c \not\mid 0$ implies $a \mid c$.

Let $K = qf(B)$. Let $x \in K$, with $x = \frac{a}{b} - \frac{c}{d}$. Then $v(u)$ $u'/v'$.

Indeed if $u - 0$ then $u' - 0$ and the two assertions are true. Otherwise if $v' \mid u'$ then $v' \mid uw'$ but $uw' - u'v$ so $v' \mid uw'$ and since $u' \not\mid 0, v \mid u$.

It then makes sense to define $R = \{x \in K, x = \frac{u}{v} \mid v \mid u\}$.

This is a valuation ring.

1. $R$.

$\frac{u}{v}$ and $\frac{u'}{v'} \in R$ then $v(u)$ hence $v' \mid uw', v' \mid d'$ so $uw' \mid u'$ hence $v' \mid u'$.

Also $v' \mid uw'$ and $v' \mid v' \mid u' \mid + v'$, hence $\frac{u}{v} \mid u' \mid + \frac{v}{v' \mid u'}$. Finally, if $x = \frac{u}{v} \in K$, then by (i), $v \mid u$ or $v \mid u$ so $x$ or $x^{-1} \in R$.

$R$ is then a valuation ring say with $K \rightarrow R$, $\Gamma$ defining its valuation, and if $f$ is the natural morphism $f : A \rightarrow B$, then $v - w \circ f$ is a valuation, and by definition of $R$, if $b \not\mid p$, $a \mid b$ then $\frac{b}{b'} \not\mid 0$.

\textbf{Remarque 14} \textit{We could consider $\Gamma_0 - (A/\sim, \times)$, check it is an ordered monoid with \textit{a} $\geq b$ iff $a \mid b$. Then $\Gamma = \Gamma_0[\{0\}$ would be an ordered submonoid. Then $v : A \rightarrow (\Gamma_0, \leq)$ is "a valuation in an ordered monoid". So if we could find $(\Gamma, \leq) \hookrightarrow (\Gamma, \leq)$ an injection of ordered monoid with $G$ a group, we could affirm that $v$ comes from a "real" valuation (with value in a group).

This could lead to consider the forgetful functor:\n
for $: Ab \rightarrow \text{commutative monoids}$, check that it has a right adjoint $i$ defined by $i(M) = (M \cup \{M + 1\} \times \{a, b\}) - (a \mid b)$, $i(-a) = (b \mid a)$, $i(-a) = (a \mid b)$. Then wonder if $\ast$\underline{the natural morphism} $M \rightarrow i(M)$ is injective?\n\n$\ast$\underline{Can we extend the ordering of $M$ to $i(M)$?}\n\nIt won't be automatic: indeed if $\Gamma = \{-n, -(n - 1), \ldots, -1, 0\}$ monoid for $a \mid b = \max(-n, a + b)$. Then\n
$v : k[X] \rightarrow \Gamma_0$ $n \not\mid 0 \rightarrow -\min(n + 1, \text{val}_X(P))$ $0 \rightarrow -n + 1$\n
and identifying $-n + 1$ with a null element is a "monoidal" valuation. But $(\Gamma, \leq)$ doesn't in an ordered group (it has torsion, and ordered groups don't), $v$ doesn't comes from a valuation : $|v$ verifies (i) - (vi) and (vii) but not (v) :\n
$X \mid x^n = X^{n+1}, 0 | X$, but however $0 \not\mid X^n$.\n
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We then consider \( \phi : S(A) \to \mathcal{P}(A \times A) \) defined by 
\( \phi(v) = \{ a \} \) with \( \phi(v) = \{ a \} \) if \( v(a) \geq v(b) \). Then the 6 conditions in the previous proposition show that \( \text{im}(\phi) \) is a closed set of \( \mathcal{P}(A \times A) \), that we endow with the product topology. 
Moreover \( \phi \) is injective: indeed if \( \{ a \} = \{ b \} \), then we easily see that \( \text{supp}(v) = \text{supp}(w) = \{ a \} \in A \) such that \( 0\{a\} = 0 \text{ hence, } K \) the residual field of \( v \) and \( w \) are the same, and the valuation ring on them induced by \( v \) and \( w \) are the same (because \( v = w \)) so they induce the same valuation on \( K \). Hence through \( \phi \) we identify \( S(A) \) with a closed subset of \( \mathcal{P}(A \times A) \). It then induces a topology \( (S(A), T_1) \) \( \cdot \mathcal{P}(A \times A) \) being compact, and \( S(A) \) closed, \( (S(A), T_1) \) is compact. In it the subsets of the form 
\( \{ v \} \{ v(a) \leq v(b) \} \in \{ v \} \{ v(a) \leq v(b) \} \in \{ v \} \{ v(a) \leq v(b) \} \in \mathcal{P}(A \times A) \). The topology \( T \) they generate is \( T_0 \), if \( v \neq w \in S(A) \) then there exists \( a, b \in A \) such that \( v(a) < v(b) \) and \( w(a) > w(b) \). If \( v(b) \neq 0 \) then \( v \in \{ x \} \{ x(a) < x(b) \} \) and not \( w \). Otherwise \( v(a) = v(b) = 0 \) so \( w \in \{ x \} \{ x(b) < x(a) \} \) and not \( v \).

**Lemma 2.1** Endow \( \mathcal{P}(X) \) with the product topology. Then the closed open subsets are the finite boolean combination of subsets \( P_\alpha = \{ U \subset X \mid x \in U \} \).

**Proposition 2.21** \( S(A) \) be endowed with the topology whose subbasis is the \( \{ v \} \{ v(a) \leq v(b) \} \). \( S(A) \) is spectral and its constructible subspaces are the boolean combination of \( \{ v \} \{ v(a) \leq v(b) \} \).

### 2.7 Specializations

**Proposition 2.22** (cf [5] 2.2) Let \( v : A \to \Gamma_0 \) and \( H \) a convex subgroup, \( v \to H : A \to (\Gamma/H)_0 \) is called a secondary specialization, \( v \in \{ w \} \) in \( \text{Spec}(A) \).

D : Let \( U = \{ x \mid x(f) \leq x(g) \neq 0 \} \) be a basic neighborhood of \( v \), i.e \( v(f) \leq v(g) \neq 0 \). Then \( w(f) \leq w(g) \) and \( v(g) \neq 0 \), i.e. \( v(g) \in \Gamma \), so \( w(g) \in \Gamma/H \) and is \( 0 \), so \( w \in U \).

\[ \begin{align*}
\text{example : } v : A &\to k[x,y] \to \mathbb{Z}_2^2, \\
\sum a_{n,m} x^n y^m &\to -\min((n,m) | a_{n,m} \neq 0) \\
\text{There are 3 convex subgroups : } &\{1\} - \Gamma_0 \\
\{0,1\} &- \Gamma_1 \\
\{0,2\} &- \Gamma_2 \\
\phi_v &- [1], \text{ and then : } \\
v/\Gamma_0 &- v, v/\Gamma_1 - v_x \text{ (valuation of } x \text{).} \\
v/\Gamma_2 &- v_{\text{discr}}. \\
v/\Gamma_0 : A &\to \{0,1\} \text{ with } v/\Gamma_0(f) - 1 \text{ if } v(f) - 1, \text{ i.e. if } f(0,0) \neq 0, \text{ i.e. it factorises through } A \to k, f \mapsto f(0,0), \text{ and then with the discrete valuation on } k. \\
v/\Gamma_1 : A &\to \mathbb{Z}_0, \\
f \mapsto v(f) \text{ if } v(f) \in (0,\mathbb{Z}), \text{ 0 otherwise. factorises through } A \to k[y], f \mapsto f(0,y) \text{ and then the } y-\text{adic valuation.} \\
v/\Gamma_2 &- v. \\
\end{align*} \]

**example of the unit ball** Let \( A - k[T], r - [\text{lambda}] < 1, \lambda \in k \) with \( k \) a non-archimedean field. Define: 
\[ \eta_\lambda : \sum a_r T^r \mapsto \max(|a_r| r^i) \subseteq \mathbb{R} \]
\[ \eta_{r+} : \sum a_r T^r \mapsto \max(|a_r| r^i, -i) \subseteq \mathbb{R} \times \mathbb{Z} \]
\[ \eta_{r-} : \sum a_r T^r \mapsto \max(|a_r| r^i, i) \subseteq \mathbb{R} \times \mathbb{Z} \]

From what we have seen above since \( \eta_{r+} \) and \( \eta_{r-} \) are secondary specializations of \( \eta_\lambda \) (with \( H - 0 \times \mathbb{Z} \)), they both belong to \( \{ \eta_\lambda \} \). The contrary is false (this is a consequence of \( \text{Spec}(A) \) being \( T_0 \)), concretely, if \( U = \{ x \| x(T) \leq x(\lambda) \neq 0 \} \) then \( \eta_\lambda \) and \( \eta_{r+} \in U \), but \( \eta_{r-} \) doesn’t. In the same way : \( V = \{ x \| x(\lambda) \leq x(T) \neq 0 \} \) then \( \eta_\lambda \) and \( \eta_{r+} \in V \), but \( \eta_{r-} \) doesn’t. So the specialization described above is the only one existing between these three points.

This shows the difference between the topology of Berkovich and Huber. Indeed if \( U = \{ v \| v(X) < \)
Let $v(\lambda)$ then $\eta_{c\tau} \in U$ and if $U$ was open in the Huber topology, it should contain $\eta_r$ but this is not the case.

2.8 $c\Gamma_v(I)$

$I = (t_1, \ldots, t_n)$ an ideal.

Lemma 2.2 (2.4) If $v(I) \cap c\Gamma - \emptyset$ there exists a greatest convex subgroup $H$ such that $v(i)$ is cofinal in $H$, $\forall i \in I$. Furthermore $v(I) \neq \{0\}$ and $v(I) \cap H \neq \emptyset$.

$D$: The existence of $H$ is a consequence of [1]. But in this particular case, we have $v(i) < 1 \forall i \in I$. Otherwise $v(i) \geq 1$ and then $c\Gamma$ by definition of it. Let $h = \max(v(t_I))_{j=1}^n - v(t_I)$ say. If $h = 0$ then $v(I) = 0$ and $H = \Gamma_v$. Otherwise, if $i \in I$, $i = \sum a_k t_k$ and $v(i) \leq \max(v(a_k))h$ say $v(i) \leq v(a)h$. So $v(i^2) \leq v(a^2)h^2 - v(a^2t_I) < v(t_I)$ and $v(a^2t_I) \leq 1$.

So $v(i)$ is cofinal in $H < v(t_I) >_{\text{conv}}$. Conversely, if $v(i) < H$ then $v(t_I)$ is cofinal in $H$ and $H \subseteq H >_{\text{conv}}$. So the great convex subgroup in which $v(I)$ is cofinal is $H = v(t_I) >_{\text{conv}}$ which then contains $v(t_I) \in v(I) = c\Gamma_v(I)$ is then the union of $c\Gamma_v$ and this subgroup $H$ if $v(I) \cap c\Gamma_v = \emptyset$.

Lemma 2.3 (2.5) If $\Gamma_v \neq c\Gamma_v$ (otherwise $c\Gamma_v(I) = \Gamma_v$). Then the following are equivalent

1. $c\Gamma_v(I) = \Gamma_v$
2. $v(i)$ is cofinal in $\Gamma_v$, for all $i \in I$
3. $v(i)$ is cofinal in $\Gamma_v$ for a set of generators of $I$.

$D: 1 \Rightarrow 2$: since $c\Gamma_v \neq \Gamma_v$, we can’t have $v(I) \cap c\Gamma_v = \emptyset$, and by definition $v(I)$ is cofinal in $\Gamma_v$.

$2 \Rightarrow 1$: then $v(I) \cap c\Gamma_v = \emptyset$. Otherwise if $v(i) \in c\Gamma_v$, $g \in c\Gamma_v$ then exists a $n$ such that $v(i)^n < g \in c\Gamma_v$, which is absurd. Hence $v(I) \cap c\Gamma_v = \emptyset$ and $c\Gamma_v(I) = \Gamma_v$.

$3 \Rightarrow 2$: The set $J = \{a \in A \mid v(a)$ is cofinal in $\Gamma_v\}$ is an ideal. Indeed first $v(J) < 1$, and if $g \in \Gamma_v, a, b \in J$, $\exists n$ such that $v(a^n) < g$ and $v(b^n) < g$ then $v((a+b)^n) < g$. If $x \notin A$, then by definition, $v(a) \leq v(x)$ if $v(x) = 1$, then $v(x) \in c\Gamma_v$, $v(ax) \leq v(x)$. Now if $1 \leq v(ax)$ we have $v(ax) \in \in c\Gamma_v$ and $v(a)$ too, which is impossible since $c\Gamma_v \neq \Gamma_v$ and $v(x)$ is cofinal. Hence $v(x)|1 < 1$ for all $x \in A$. Let then $g \in \Gamma_v$. There exists $n$ such that $v(a^n) < g$, then $v(ax)^n + v(a^n) < v(a^n) < g$. So $J$ is an ideal, and $3 \Rightarrow 2$.

Remark 15 If $I - A$ then $v(I) \cap c\Gamma_v \neq \emptyset$ so $c\Gamma_v(I) - c\Gamma_v$, and then $Spv(A, A) = \{v \mid c\Gamma_v = \Gamma_v\}$.

3 Continuous valuation of $\mathfrak{f}$adic rings

Remark 16 Let $A$ be a Tate ring, and $v: A \rightarrow \Gamma - \Gamma_v$ be a continuous valuation. Then $c\Gamma - \Gamma$. Indeed take $x$ a nilpotent unit. Then $x$ is cofinal in $\Gamma$, and $< x >_{\text{conv}}, \Gamma$. It is even true that the subgroup generated by $\{v(a) \geq 1\}$ is $\Gamma$. So in that case there are only secondary specializations.

Theorem 1 (3.1) $\text{Cont}(A) = \{v \in Spv(A, A') \mid v(A^+) < 1\}$.

$D$: If $v$ is continuous. Then clearly $v(A^+) < 1$. Then, if $c\Gamma_v = \Gamma_v$ OK. Otherwise let $A \in A^+$, $v(a^n) \rightarrow 0$ so $v(a^n)$ is cofinal in $\Gamma_v$ and according to [28] $v \in Spv(A, A')$, i.e. $c\Gamma_v(A,A') - \Gamma_v$.

Conversely let $v \in Spv(A, A')$ such that $v(A^+) < 1$. First let’s show that $\forall a \in A^+$, $v(a)$ is cofinal in $\Gamma_v$. If $\Gamma_v \neq c\Gamma_v$, then this is true by definition of $Spv(A, A')$ and $c\Gamma_v(A,A')$. Otherwise $\Gamma_v = c\Gamma_v$, hence if $g \in \Gamma_v$ $\exists a \in A$ such that $v(g) \gg v(g)^{-1}$ i.e. $v(g) \ll v(g)^{-1}$. Hence if $A \in A^+$, $\exists n$ such that $v(a^n) < 1 \Rightarrow v(a^n) < g$.

So let $\Lambda_0, I = (b_1, \ldots, b_n)$ be an adic ring of definition for $A$. Since the $b_i \in A^-$, the $v(b_i)$ are cofinal from what we’ve just seen, in particular $v(b_i) < 1$, and we easily see that for $v = (k_1, \ldots, k_n)$ with $|v| \geq N$ for a big enough $N$, $v(b^n) < g$. Hence since $v(I) < 1$ we have $v(I^{N+1}) < g$ which shows $v$ is continuous.

Theorem 2 If $A$ is a ring which is Tate, $A^- - A$ and hence $\text{Cont}(A) = \{v \in Spv(A) \mid c\Gamma_v = \Gamma_v$ and $v(A^+) < 1$.
3.1 counter-example continuous valuations

1. $v(A^+) \leq 1$, $v(A^-) < 1$; inspired by [2] §10 lemma 1 which says that if $v : k \to \Gamma_0$ is a valuation of the field $k$ and $g \in \Gamma$, $w : k[X] \to \Gamma$, $\sum a_iX^i \mapsto \max(v(a_ig))$ is a valuation. Let $v : A = k[X] \to \mathbb{Z} \times \mathbb{R}$, $\sum a_iX^i \mapsto \max(-n, |a_i|)$. This is a valuation, $(v(A^-) < 1$ and $v(A^+) \leq 1$ but it is not continuous: $\mathbf{Cl}_v = 0 \times \mathbb{Z}$. Or if $\pi \in k$, $v(\pi n)$ isn’t arbitrary small although $\pi n \to 0$. Here $v \in L(A)^\text{cont}(A)$.

2. A Tate ring and a $v$ such that $v(A^-) < 1, v(A^+) \leq 1$ but not continuous. Take $A - \mathbb{Z}_p[X, X^{-1}] \not\supset A_0 - \mathbb{Z}_p[X] \not\supset I - A_0.X$. The $v$-adic topology on $A_0$ extended to $A$ makes it a $v$-adic ring. Mainly because if $P \in A \cdot f_n \to 0$ then $Pf_n \to 0$ and if $g_n \to 0, f_ng_n \to 0$. Then $A^+ = I; A^+ = A_0$. Let $v : A \to \mathbb{Z}^2, v(\sum a_iX^i) = \max(|a_i|, -i)$. Then $v(I) < 1, v(A^+) \leq 1$ but $X_n \to 0$ however $v(X^n)$ doesn’t converges to $0$, i.e. $v(X)$ isn’t cofinal in $\Gamma_v$. (Also because $\mathbf{Cl}_v = 1 \times \mathbb{Z} \neq \mathbb{Z}^2$).

3. A $v$-adic ring, $v : A \to \Gamma_0$ such that $v(A^-) < 1$ but not continuous.

Proposition 3.1 (cf [5] 3) The integral closure of the subring $\mathbb{Z} + A^- = \{n + a \mid n \in \mathbb{Z}, a \in A^\text{ad} \}$, $B$ is the smallest ring of integral elements of $A$.

D : First note that $A^-$ is open. Indeed if $(A', I)$, an adic ring of definition of $A$, $I \subset A^-$ is open. So $\mathbb{Z} + A^-$ is a subring of $A^-$ (note that $\mathbb{Z} + A^-$ is well a subring, because $A^-$ is stable by $+$ and $\times$).

Now let’s show that $A^-$ is integrally closed in $A$.

First let’s prove:

Lemme 3.1 If $B$ is a bounded subring in $A$ $v$-adic, and $A \subset A^-$ then $B[a]$ is bounded.

D : Let $(A_0, I)$ be a ring of definition. $I^n$ a neighborhood, $\exists m$ such that $a^k I^m \subset I^n$, and $p$ such that $BP \subset I^n$. Then $b^n I^m \subset a^n I^m \subset I^n$. Hence since $\mathbb{Z}$ is also bounded, we see that if $a_0, \ldots, a_n \in A$, $\mathbb{Z}[a_0, \ldots, a_n]$ is also bounded.

So let $x \in A$ be integer on $A$, i.e. $x^n = a_0 + a_1x + \ldots + a_{n-1}x^{n-1}$. Call $B = \mathbb{Z}[a_0, \ldots, a_{n-1}]$. By induction we get that $x^n \in B + Bx + \ldots + Bx^{n-1}$. So if $I^n$ is a neighborhood, we can find $k$ such that $Bx^k I^k \subset I^n$ for all $l \geq 0, n-1$. Then $x^n I^k \subset I^n$, $\forall p$. Hence $x \in A^-$.

So $A^-$ is integrally closed in $A$, so $(\mathbb{Z} + A^-)^{\text{closure}} A \subset A^-$.

Conversely, if $B \subset A^-$ is open and integrally closed in $A$, let $x \in A^-$, so that $x^n \to 0$, hence there exists a $n$ such that $x^n \in B$ since it is open, hence $x \in B$ since it is integrally closed. $\square$

[36] supposes that $\{0\}$ is an ideal : let $x \in 0$, $a \in A$ a 0-neighbourhood. We can assume $V = -V$, then $0 \in ax + V \Leftrightarrow ax \in V$, but $A \to A, u \mapsto au$ is continuous so $\exists$ a neighborhood of 0, $W$ such that $aW \subset V$, and $x \in W$ because $x - W$ is a neighborhood of $x \not\in x - W$, i.e. $x \in W$. $\square$

3.2 Affinoid rings

Définition 3.1 A subring of $A$ is called a ring of integral elements if it is open, integrally closed, and contained in $A^-$. An affinoid ring is a pair $(A, A^+)$ with $A$ a $v$-adic ring and $A^+$ a subring of integral elements. By a ring homomorphism of affinoid ring it is meant $f$ such that $f(A^+) \subset B^+$.

Lemme 3.2 Let $J$ be an ideal of $A$. $J$ is open $\iff$ $A^\times \subset \sqrt{J}$

D : $\Rightarrow$ Let $a \in A^\times$, so that $\exists n$ such that $a^n \in J$.

$\Leftarrow$ Let $(a_0, I)$ be a ring of definition, $I = \langle i_1, \ldots, i_n \rangle$. Since $I \subset A^- \subset \sqrt{J}$, for each $j = 1 \ldots n$ there exists $k_j$ such that $i_{kj} \in J$. Then $k := \sum k_j$ and $I^k \subset J$ which is then open. $\square$

This explains that if $T.A$ is open, $A^\times \subset \sqrt{J}$ so $U = \{v \in Spv(A, A^\times) \mid v(a_i) \leq v(t) \neq 0\}$ a rational subset.
Remarque 17 On rational domains. Let’s restrict to the case $A$ a ring Tate. Then for $[5]$ a rational subset of $\text{Spa}(A)$ is $R(\frac{1}{2}) - R - \{v \mid v(t_i) \leq v(s) \neq 0\}$ with $(t_i) - A$. Hence if $v \in \{v \mid v(t_i) \leq v(s)\} R$ since $\sum a_i t_i - 1$ if $v(s) - 0$ we have $v(t_i) - 0$ so $v(1) - 0$ which is impossible, so $R - \{v \mid v(t_i) \leq v(s)\}$.

Now let’s consider $S - \{v \mid v(t_i) \leq v(s), i = 1, \ldots, n\}$ where $(t_i, s) - A$. If we had $t_n + 1 - s$ we still have $S - \{v \mid v(t_i) \leq v(s), i = 1, \ldots, n\}$ which is rational in Huber’s sense. So the two possible definition of $R(\frac{1}{2})$, with $(T) - A$ or $(T, s) - A$ give the same class of subsets.

3.3 analytic points

There exists a finite set $T \subseteq A^-$ such that $T.A$ is open. Indeed let $(B, I)$ a ring of definition , with $I - (b_1, \ldots, b_n)$. Then $T - \{b_1, \ldots, b_n\}$ works.

Proposition 3.2 $\{\text{Spa}(A)\}_a = \{x \mid \text{supp}(x) - v^{-1}(0) \text{ is not open} \}$

$- \{x \mid x(t) \neq 0 \text{ for one } t \in T\}$

$- \bigcup_{x \in \text{Spa}(A)} R(\frac{x}{I})$

D : if $x \in \{\text{Spa}(A)\}_a$ then $\text{supp}(x)$ is an ideal, not open, so $T \subseteq \text{supp}(x)$ so $\exists x \mid x(t) \neq 0$. Conversely, if $x(t) - 0$, $t \in A^-$, $x( I^n ) - 0$ but converges to 0 so $\text{supp}(x)$ is not open. $\Box$

Proposition 3.3 If $v \in \text{Spa}(A)$ is analytic, then $v$ is microbial.

D : We have $A \to A/\text{supp}(v) - B - qf(B) - K$ and let $R$ be the valuation ring of $K$ associated to $v$.

Let also $A_0, I$ be a ring of definition of $A$. First note that saying that $v$ is analytic means that $\text{supp}(v)$ is not open (it is closed, but we don’t care), so $I \not\subseteq \text{supp}(v)$ and in fact neither $I^n$ for any $n$. Then the topology on $R$ is $I$-adic (more precisely we should define $J - R, [i \mid i \in I]$ and say $R$ is $J$-adic. Indeed, first $J$ is an ideal of $R$ which is not $\{0\}$ (because $I \not\subseteq \text{supp}(v)$), so it is open. And if $g \in \Gamma_v$, we have a $n$ such that $v(I^n) - g$. Then $v(J^n) - g$ which shows that the topology of $R$ is the $J$-adic topology. Since $J^n - [0]$ for all $n$, the topology is not discrete. So according to the criterion 4, $v$ is microbial. $\Box$

3.4 constructible sets in the respective spaces

In $\text{Spa}(A)$, the constructible subsets are finite boolean combination of subsets of the form $\{v \mid v(a) - v(b)\}$ (prop 2.2 in [5]). This includes for instance the subsets $\{v \mid v(a) - v(b)\}$ and $\{v \mid v(a) = 0\}$.

For instance $U - \{v \in \text{Spa}(A) \mid v(a) \neq 0\}$ is quasi compact open (i.e. constructible and open, i.e. proconstructible and open), because indeed it is open, $\{v \mid v(a) - v(b) \neq 0\}$ and constructible.

In $\text{Spa}(A, I)$ the constructible subsets are the boolean combinations (finite) of rational domains : $U - R(\frac{1}{2}) - \{v \in \text{Spa}(A, I) \mid v(t) \leq v(s) \forall t \in T\}$ where $T$ is finite and $I \subseteq \sqrt{T.A}$.

$\text{Cont}(A) - \{v \in \text{Spa}(A, A^-) \mid v(A^-) - 1\}$ is a proconstructible subset of $\text{Spa}(A, A^-)$.

Let’s consider $\text{Cont}(A)$ Then $a$ and $b$ are not spectral in general.

$\xrightarrow{a} \text{Spa}(A, A^-)$ $\xrightarrow{b} \text{Spa}(A)$

Indeed otherwise $b^{-1}\{v(a) - v(b)\}$ would be constructible, but it is hard to imagine how it could be a boolean combination of rational subset , particularly when $b - 0$ in which case it is a Zariski closed subset. More precisely if $A$ is an afffine algebra , $f - 0$, $\pi \in k^\times\{0\}$ , then $b^{-1}\{v(f) = 0\}$ $- \bigcup_{n\geq0}\{v(f) - v(\pi^n)\}$ and you can’t extract a finite cover from the right hand side, so it is not quasi compact. So $b$ isn’t spectral , and since $b - c \circ a$, and $a$ is spectral, $c$ isn’t spectral.

according to [5, 2.5, 2.6] :

$r : \text{Spa}(A) \to \text{Spa}(A, I)$ $v \mapsto v|\Gamma_v(I)$ is spectral. Then take $A - k[I], I = A$

It is false to say that $r^{-1}\{v \in \text{Spa}(A, A) \mid v(a) - v(b)\} - \{v \in \text{Spa}(A) \mid v(a) - v(b)\}$.

For instance $U - r^{-1}\{v \mid v(T) - 1\}$. Let $v_T$ be the $T$-adic valuation $\Gamma_v \rightarrow \{1\}$ so $\Gamma_v - \{0\}$
and \( v_T([0]) = w \) is the trivial valuation. Hence since \( w(T) \geq 1 \), \( v_T \in U = r^{-1} [v \mid v(T) \geq 1] \) but \( v_T \notin \{ v \in \text{Spv}(A) \mid v(T) \geq 1 \} \) so \( r^{-1} [v \mid v(T) \geq 1] \neq \{ v \in \text{Spv}(A) \mid v(T) \geq 1 \} \).

This helps to understand the fact that:

\[ r : \text{Spv}(A) \to \text{Spv}(A,I) \text{ is spectral } [5] \text{ 2.6(ii)}, \text{ and surjective, so } T \subseteq \text{Spv}(A,I) \text{ is constructible iff } r^{-1}(T) \text{ is}. \]

So if it was true that \( r^{-1} \{ v \in \text{Spv}(A,I) \mid v(a) \leq v(b) \} = \{ v \in \text{Spv}(A) \mid v(a) \leq v(b) \} \), the subsets \( \{ v \in \text{Spv}(A,I) \mid v(a) \leq v(b) \} \) would be constructible.

## 4 Tate rings of topologically finite type over fields

**Proposition 4.1** Let \( A \) be a \( k \)-affinoid algebra, and \( v \in \text{Spv}(A,A^\circ) \). Then \( v|_k \) is the initial valuation of \( k \).

Indeed, \( v \) is a valuation with \( v(x) \geq 1 \) whenever \( x \in A^\circ \cap k \). Conversely if \( x \notin A^\circ \cap k \), then \( x^{-1} \in A^\circ \) so \( v(x^{-1}) < 1 \) (cf 3.1 [5]), i.e. \( v(x) > 1 \). So \( v(x) \leq v(y) \) iff \( \|x\| \leq \|y\| \) so they are the same.

More conceptually \((k,k') \to (A,A^\circ)\) is a continuous morphism of affinoid rings, so induces \( f : \text{Spv}(A,A^\circ) \to \text{Spv}(k,k') \). What we’ve shown is somehow the fact that \( \text{Spv}(k,k') \cap \{ [1] \} \) where \( [\cdot] \) is the valuation on \( k \), because in general (cf [5] 1.1.6) if \( A \to (A^\circ,A^\circ) \) is an affinoid field, \( \text{Spv}(A) \) is the set of valuation ring \( B \), such that \( A^\circ \subseteq B \subseteq (A^\circ)^\circ \). In our case it gives \( k' \subseteq B \subseteq k' \), so the only possibility is \( k' = k \).

If \( A \) is a Tate algebra, \( L_A = \{ v \in \text{Spv}(A) \mid v(A^\circ) \leq 1 \} \). We note (abuse of notation) \( \text{Spv}(A) := \text{Spv}(A,A^\circ) \). Then

\[
\text{Max}(A) \subseteq \text{Spv}(A) \subsetneq \text{Spv}(A) \subseteq L_A \subseteq \text{Spv}(A)
\]

**Proposition 4.2** ([5] the. 10.2) Let \( K \) be a field, \( A \subseteq K \) a subring \( p \) a prime ideal of \( A \). Then there exists a valuation ring \( R \) of \( K \) such that \( A \subseteq R \) and \( R \cap A \subseteq p \).

**Corollary 4.1** Let \( k \) be a valued field and \( K \) an extension of field, then there exists a valuation on \( K \) that extends the one of \( k \).

\( D : \text{let } A \) be the valuation ring of \((k,v), p - m_A \). Then there exists \( R \) a valuation ring of \( K \) with \( A \subseteq R \) and \( m_R \cap A \subseteq m_A \). So let \( C = k \cap R \). Then \( C = A \) (for instance because \( A \) is a valuation ring that extends \( A \), with the same maximal ideal, or because \( x \in C \cap A, v(x) > 1 \), \( x^{-1} \in m_A \) but since \( m_R \supseteq m_A \), \( x^{-1} \in m_R \) which contradicts \( x \in R \).

**Proposition 4.3** Let \( A \) be a ring, \( I = (a_1, ..., a_n) \) an ideal. Then \( \pi : A \to A/I \) induces \( \text{Spv}(\pi) : \text{Spv}(A/I) \to \text{Spv}(A) \). Its image is \( \{ v \mid v(I) = 0 \} \) and it is a homeomorphism on its image. In particular, if \( I \) is of finite type, this image is a constructible subset.

**Proposition 4.4** (cf [5] 4.1 or [7] Prop. 2.1.1) Let \( f : A \to B \) a morphism of finite presentation and \( U \subseteq \text{Spv}(B) \) a constructible subset. Then \( \text{Spv}(f)(U) \) is constructible.

\( D : f \) decomposes as \( A \xrightarrow{f_1} A[X_1, ..., X_n] \xrightarrow{f_2} B = A[X_1, ..., X_n]/I \) where \( I = (a_1, ..., a_n) \) is finitely generated. Let \( U \) be a boolean combination of \( \{ v(a) \mid v(b) \} \) with \( a, b \in A[X_1, ..., X_n] \), then \( \text{Spv}(f_2) \) is the same boolean combination of \( \{ v(a) \mid v(b) \} \cap \{ v(a_i) = 0 \} \). So we can restrict to the case \( B = A[X_1, ..., X_n] \) and \( U \) a boolean combination of \( \{ v(P) < v(Q) \} \).

Since \( P \in A[X_1, ..., X_n] \), \( Q \in A[X_1, ..., X_n] \) are polynomials, \( p \in Z[1, ..., Y_m, X_1, ..., X_n] \) and \( q \in Z[1, ..., Y_m, X_1, ..., X_n] \), and \( Q = q(q_1, ..., q_M, X_1, ..., X_n) \).

An element \( w \in \text{Spv}(A[X]) \) represented by \( B \xrightarrow{\psi} k \) is in \( U \) iff the combination of formula \( p(t_1, ..., t_j) < q(q_{l_1}, ..., q_{l_j}) \) is true where \( t_j = \psi(X_j) \).

Hence \( A \xrightarrow{f} k \) corresponds to a valuation \( v \) of \( A \), it is in \( \text{Spv}(f)(U) \) iff there exists a diagram

\[
\begin{array}{ccc}
B & \xrightarrow{\psi} & L \\
\downarrow & & \downarrow \\
A & \xrightarrow{f} & k
\end{array}
\]
But when \( \phi \) and \( \iota \) are fixed, a \( \psi \) giving rise to a commutative diagram as this one is equivalent to the data of \( l_1, \ldots, l_n \in L \).

Hence \( v \in \text{Spv}(f)(U) \) iff

\[ \exists \text{ an extension } L \text{ of } k \text{ such that the formula } P(l_1, \ldots, l_n) < Q(l_1, \ldots, l_n) \text{ is true iff} \]

\[ \exists : k \rightarrow L \text{ an extension with } L \text{ algebraically closed valued field (using } \ref{4.1} \text{), and such that the following formula holds} \]

\[ \exists l_1 \ldots l_n \text{ boolean combination}(\| p(p_i, l_i) \| < \| q(q_k, l_i) \| ) \]

For such a \( L \), if it is trivially valued, we can embed it in \( L(X) \) with the \( X \)-adic valuation so that it isn’t trivially valued anymore so that

\[ \Leftrightarrow \exists : k \rightarrow L \text{ an extension with } L \text{ algebraically closed non-trivially valued field (using } \ref{4.1} \text{), and such that the following formula holds} \]

\[ \exists l_1 \ldots l_n \text{ boolean combination}(\| p(p_i, l_i) \| < \| q(q_k, l_i) \| ) \]

But using elimination of quantifiers for the non-trivially valued fields (warning you can’t eliminate \( \exists x \neq y \neq 0[x] \neq [y] \), which precisely defines the non-trivially valued fields), this formula is equivalent to a universal (meaning independant of \( L \)) formula \( \varphi(p_i, q_k) \), which defines a constructible subset of \( \text{Spv}(A) \).

\textbf{Theorem 3} \( \ref{4.1} \) \textit{\( L_A \) is the closure of \( \text{Max}(A) \) in the constructible topology of \( \text{Spv}(A) \)}

\( \text{D : \text{First } } L(A) \text{ is well closed in this topology (cf Prop 2.2) which says that a basis for the constructible topology is the sets } \{ v \mid v(a) \circ v(b) \} \text{, } a \in \{ <, \leq \}. \)

\textbf{Proposition 4.5} Let \( f : X \rightarrow Y \) a continuous map between topological spaces, \( A \subseteq Y \).

\begin{enumerate}
  \item \( f^{-1}(A) \subseteq f^{-1}(\bar{A}) \)
  \item if \( f \) is open \( f^{-1}(A) = f^{-1}(\bar{A}) \)
\end{enumerate}

\( \text{D : } 1f^{-1}(\bar{A}) \text{ is closed and contains } f^{-1}(\bar{A}). \)

2 Let \( x \in f^{-1}(\bar{A}) \) and \( U \) a neighborhood of \( x \). We have to show that \( U \cap f^{-1}(\bar{A}) \neq \emptyset \). But \( f(U) \) is open, so neighborhood of \( y = f(x) \in A \), so \( f(U) \cap A \neq \emptyset \). So if \( z \in f(U) \cap A, z = f(u) , u \in f^{-1}(\bar{A}) \cap f^{-1}(f(u)) \subseteq f^{-1}(A) \cap U, \Rightarrow f^{-1}(A) \cap f(u) \neq \emptyset \).

\textbf{4.1 Prime filters}

\( \text{Max}(A) \) denotes the set of prime filters of \( \text{Max}(A) \), (precisely the prime filters of the lattice of finite union of rational subsets (cf Dickmann).

\textbf{Cor 4.5 :} Let \( \mathcal{F} \) be a prime filter, define \( \mathcal{F}^* = [\text{Max}(A) \setminus R | R \notin \mathcal{F}] \) and define \( \mathcal{W} = \mathcal{F} \cup \mathcal{F}^* \).

\[ \| \text{Let } W_1, \ldots, W_n \in \mathcal{W}, \text{ then } \cap_{i=1}^n W_i \neq \emptyset \]

\( \text{D : in this intersection there is in fact one rational domain } R \text{ (because they are stable par } \cap \text{), and some } R_i \text{ with } R_i \notin \mathcal{F} \text{. Then if we had } R \subseteq \cup_i R_i , R \neq \cup(R_i \cap R) \text{ is an element of } \mathcal{F} \text{ so one of the } R \cap R_i \text{ must also be in, so } R_i \text{ also which is absurd. So } R \subseteq \cup_i R_i, \text{ i.e. } R \cap R_i \neq \emptyset \).

\[ \| D := \cap_{W \in \mathcal{W}} W \neq \emptyset \). Let \( x \in D \), then \( s(x) = \mathcal{F} \)

\( \text{D : } s(x) \supseteq \mathcal{F} ; \text{ if } F \in \mathcal{F} , \ x \in D \text{ so } x \in \bar{F} \).

Conversely let \( \bar{U} \cap \text{Max}(A) = U \in s(x) \), i.e. \( x \in \bar{U} \). If we had \( U \notin \mathcal{F} \), then \( V = \text{Max}(A) \setminus F \in \mathcal{W} \), and then \( x \in \bar{V} \), so \( x \notin \bar{U} \), absurd.

\[ 4.7.2 : \mathcal{F} \in \text{Max}(A), \text{ then :} \]

\begin{enumerate}
  \item \( \{ a \in A \mid \forall F \in \mathcal{F} \exists x \in F \mid a(x) - 0 \} \)
  \item \( \{ a \mid \forall e \in k^* \exists F \in \mathcal{F} \mid |a|_F \leq |e| \} \)
  \item \( \{ a \mid \forall r > 0 \exists F \in \mathcal{F} \mid |a|_F \leq r \} - p \mathcal{F} \)
\end{enumerate}

2 =3 is clear.
$1 \subseteq 3$: if $a \in 1$. Let $F_1 = \{ x \in Max(A) \mid |a(x)| \leq r \}$ and $F_2 = \{ x \in Max(A) \mid |a(x)| \geq r \}$. $F_1 \cup F_2 = Max(A)$ so one of it is in $F$. $F_2 \notin F$ because $a \in 1$, and $\forall x \in F_2 a(x) \neq 0$. So $F_1 \in F$, and $a \in 3$.

If $a \in 3$. Let $F \in F$ such that $a(x) \neq 0 \forall x \in F$. Then $\exists r > 0$ such that $|a(x)| \geq 2r \forall x \in F$. But since $a \in 3 \exists g \in F$ such that $|a|_{G} \leq r$. Then $F \cap G \in F$, but is empty. Contradiction, so $\exists x \in F$ such that $a(x) = 0$.

Remark: with 3, we see that $p_F$ is prime ideal. Indeed if $a, b \in 3$ and $r > 0$, $\exists F_a, F_b$ such that $|a|_{F_a} \leq r$. Then $|a + b|_{F_a \cap F_b} \leq r$. If $c \in A$, $|ac|_{F_a} \leq ||c||_{F_a} \leq ||c||_{r}$. And if $a, b \in A$ and $ab \in 3$. Let $r > 0$. $F$ such that $|ab|_{F} \leq r^2$, $F_a = \{ x \mid |a(x)| \leq r \}$, $F_b = \{ x \mid |b(x)| \leq r \}$, Then $F_a \cup F_b \supseteq F$, so one of them is in $F$.

Rk: we proved that $p_F = \{ a \mid \exists F \in F, r > 0 \mid \forall x \in F|a(x)| \geq r \}$.

$s(\eta_1) = \{ R \mid R \text{ contains all but finitely many open balls of radius } 1 \}$. $s(\eta_{<1}) = \{ R \mid R \cong (B(0,1) \text{ minus some balls of radius } < 1) \}$.

Références