More differential geometry

Matthias Ludewig

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5.5 Riemannian coverings and the theorem of Cartan-Hadamard

Recall that a smooth map $\varphi : N \to M$ between manifolds is called a *covering* if for all $p \in M$, there exists an open neighborhood U of p such that

$$\varphi^{-1}(U) = \coprod_{\alpha \in I} U_\alpha,$$

where I is some index set, U_α is an open subset of N for each $\alpha \in I$ and $\varphi|_{U_\alpha} : U_\alpha \to U$ is a diffeomorphism. Any connected manifold M has a *universal cover*, which is a connected and simply connected cover $\pi : \tilde{M} \to M$ of M, uniquely determined up to diffeomorphism. The fundamental group $\pi_1(M)$ acts freely and transitively on the fibers $\pi^{-1}(p)$, for each $p \in M$.

Example 5.1. (a) If F is a countable set and $N = M \times F$, then the projection $N \to M$ onto the first factor is a covering map. Such coverings are called *trivial*.

(b) For $N = \mathbb{R}^n$ and $M = \mathbb{R}^n/\mathbb{Z}^n$ the torus, the quotient map is a covering map.

Definition 5.2 (Riemannian covering). Let (N, h) and (M, g) be Riemannian manifolds. A covering map $\varphi : N \to M$ is a *Riemannian covering* if it is a local isometry.

Exercise 5.3. Let (N, h) and (M, g) be Riemannian manifolds and let $\varphi : N \to M$ be a covering map. Show that if (M, g) is complete, then so is (N, h).

Proposition 5.4 (Consequence of Bonnet-Myers). *Let (M, g) be a compact Riemannian manifold with positive Ricci curvature. Then its fundamental group $\pi_1(M)$ is finite.*

Proof. Let κ be such that $\mathrm{ric} \geq \kappa(n-1)g$. Let $\tilde{M}$ be the universal cover of M and equip if with the Riemannian metric $\tilde{g}$ obtained by pulling back the metric g along the projection $\pi : \tilde{M} \to M$. Then π is a Riemannian covering. In particular, π is a local isometry, hence $\tilde{M}$ again satisfies $\mathrm{ric}_{\tilde{g}} \geq \kappa(n-1)\tilde{g}$. Moreover, by Exercise 5.3, $(\tilde{M}, \tilde{g})$ is complete.
This shows that $(\tilde{M}, \tilde{g})$ satisfies the assumptions of the Bonnet-Myers theorem, hence $\tilde{M}$ is compact. It is now a standard fact from topology that if a covering space is compact, then the cover must be finite. Hence $\pi : \tilde{M} \to M$ is a finite cover, so $\pi_1(M)$ (which bijectively corresponds to the fibers of this cover) must be finite. □

For the next result, we need the following exercise.

Exercise 5.5. Let (N, h) and (M, g) be connected and complete Riemannian manifolds. Show that a surjective local isometry $\varphi : N \to M$ is always a covering map.

Theorem 5.6 (Cartan-Hadamard). *Let (M, g) be a complete Riemannian manifold with sectional curvature $K \leq 0$. Then for all $p \in M$, $\exp_p : T_pM \to M$ is a covering map.*

Proof. Let $p \in M$. Since M is complete, the exponential map $\exp_p$ is defined on all of T_pM.
We show that the differential $d\exp_p|_X : T_X T_pM \cong T_pM \to T_{\exp_p(X)}M$ is an isomorphism for each $X \in T_pM$. To this end, we check that $d\exp_p|_X$ has trivial kernel. We have $d\exp_p|_0 = \mathrm{id}$, so let now $X \neq 0$ and let $\gamma(t) = \exp_p(tX)$ is the geodesic with $\dot\gamma(0) = X$. Then we have

$$d\exp_p|_X(Y) = J(1),$$

where J is the Jacobi field along γ with $J(0) = 0$ and $\frac{\nabla}{dt}J(0) = Y$. We claim that J has no zeros. To this end, we set $f(t) = \frac{1}{2}\|J(t)\|^2$ and calculate

$$\begin{aligned} f''(t) &= \frac{d}{dt}\left\langle \frac{\nabla}{dt}J(t), J(t)\right\rangle \\ &= \left\langle \frac{\nabla^2}{dt^2}J(t), J(t)\right\rangle + \left\|\frac{\nabla}{dt}J(t)\right\|^2 \\ &= \big\langle R(\dot\gamma(t), J(t))\dot\gamma(t), J(t)\big\rangle + \left\|\frac{\nabla}{dt}J(t)\right\|^2 \\ &= -K\big(\mathrm{span}\{\dot\gamma(t), J(t)\}\big)\cdot\Big(\|J(t)\|^2\|\dot\gamma(t)\|^2 - \langle J(t), \gamma(t)\rangle^2\Big) + \left\|\frac{\nabla}{dt}J(t)\right\|^2 \end{aligned}$$

Because $K \leq 0$, this is non-negative. Moreover, f satisfies

$$f(0) = 0, \qquad f'(0) = \left\langle \frac{\nabla}{dt}J(0), J(0)\right\rangle = 0, \qquad f''(0) \geq \|Y\|^2 > 0.$$

In total, this implies that $f(t) > 0$ for all $t > 0$. Hence $J(t)$ is never zero, so $d\exp_p|_X$ has trivial kernel and is therefore an isomorphism. We conclude that $\exp_p$ is a local diffeomorphism.
Define a Riemannian metric $\tilde g$ on T_pM by $\tilde g = \exp_p^* g$. This turns $\exp_p$ into a local isometry. We claim that $(T_pM, \tilde g)$ is complete. Since $\exp_p$ is a local isometry, the geodesics through $0 \in T_pM$ are precisely the straight lines going through zero; indeed, by definition of the exponential map, they are mapped to the geodesics $\gamma(t) = \exp_p(tX)$ under $\exp_p$, and preimages of geodesics under local isometries are again geodesics. Hence $(T_pM, \tilde g)$ is geodesically complete at 0. By the Hopf-Rinow theorem, $(T_pM, \tilde g)$ is complete, as claimed. By Exercise 5.5 and completeness of (M, g), $\exp_p$ is a covering map. □

Remark 5.7. If M is simply connected, the theorem of Cartan-Hadamard implies that M must be diffeomorphic to $\mathbb{R}^n$.

5.6 The injectivity radius

Let (M,g) be a Riemannian manifold and let $\gamma : (a,b) \to M$ be a geodesic. Recall that two points $t_1, t_2 \in (a,b)$ are conjugate if there exists a non-trivial Jacobi field J along γ with $J(t_1) = 0$ and $J(t_2) = 0$.

Lemma 5.8. *Let (M,g) be a Riemannian manifold and let $\gamma : [0,r] \to M$ be a geodesic. Suppose that $t_1 \in [0,r]$ is conjugate to zero along γ. Then γ is not length minimizing among piecewise C^1-curves between $\gamma(0)$ and $\gamma(t)$, for any $t > t_1$.*

The proof of this lemma will use the second variation formula for the energy functional at a geodesic. Recall that if $\gamma : [a,b] \to M$ is a geodesic and $(\gamma_s)_{s\in(-\varepsilon,\varepsilon)}$ is a variation with fixed endpoints and variational vector field $X(t) = \frac{\partial}{\partial s}|_{s=0}\gamma_s(t)$, then we have

$$D^2E|_\gamma(X,X) := \frac{\partial^2}{\partial s^2}\Big|_{s=0} E[\gamma_s] = \int_a^b \left(\left\| \frac{\nabla}{dt} X(t) \right\|^2 - \big\langle R\big(X(t), \dot\gamma(t)\big)\dot\gamma(t), X(t) \big\rangle \right) dt. \tag{5.1}$$

The same formula is true if the variational vector field is *piecewise* smooth, in the sense that there exists a subdivison $a = t_0 < t_1 < \cdots < t_N = b$ of $[a,b]$ such that the curves γ_s are still continuous, but smooth only on the subintervals $[t_{j-1}, t_j]$. In particular, (5.1) defines a quadratic form on the space of piecewise C^1 vector fields along γ. Polarizing this quadratic form, we obtain the bilinear form

$$D^2E|_\gamma(X,Y) = \int_a^b \left(\left\langle \frac{\nabla}{dt} X(t), \frac{\nabla}{dt} Y(t) \right\rangle^2 - \big\langle R\big(X(t), \dot\gamma(t)\big)\dot\gamma(t), Y(t) \big\rangle \right) dt,$$

defined for two piecewise C^1 vector fields X, Y along γ.

Proof. Fix $t_2 > t_1$. We will show that there exists a vector field Y along γ with $Y(0) = 0$ and $Y(t_2) = 0$ such that the second variation $D^2E|_\gamma$ of the energy is negative on Y. Then any variation with fixed end points and variational vector field Y will then produce a curve between $\gamma(0)$ and $\gamma(t_2)$ shorter than γ.
Because 0 and t_1 are conjugate along γ, there exists a non-trivial Jacobi field J such that $J(0) = 0$ and $J(t_1) = 0$. Let J^* be the Jacobi field such that $J^*(t) = J(t)$ for $t \le t_1$ and $J^*(t) = 0$ for $t > t_1$. Then J^* is a piecewise C^1 vector field. Let moreover X be a piecewise C^1 vector field such that $X(0) = 0$ and $X(t_2) = 0$ and $X(t_1) = -\frac{\nabla}{dt}J(t_1)$. Then $X(t_1) \neq 0$, because if we had $\frac{\nabla}{dt}J(t_1) = 0$, then J would be trivial. Now for $\varepsilon > 0$, set $Y = J^* + \varepsilon X$. Then

$$D^2E|_\gamma(Y,Y) = D^2E|_\gamma(J^*,J^*) + 2\varepsilon D^2E|_\gamma(J^*,X) + \varepsilon^2 D^*E|_\gamma(X,X). \tag{5.2}$$

For the first term, we have

$$\begin{aligned} D^2E|_\gamma(J^*,J^*) &= \int_0^{t_1}\left(\left\|\frac{\nabla}{dt}J(t)\right\|^2 - \big\langle R\big(J(t),\dot\gamma(t)\big)\dot\gamma(t),J(t)\big\rangle\right)dt \\ &= \int_0^{t_1}\left(-\left\langle\frac{\nabla^2}{dt^2}J(t),J(t)\right\rangle + \big\langle R\big(\dot\gamma(t),J(t)\big)\dot\gamma(t),J(t)\big\rangle\right)dt \\ &= 0. \end{aligned}$$

Here we used that J^* is non-zero only on $[0,t_1]$, then integrated by parts and then used that J vanishes at $t=0$ and $t=t_1$ and satisfies the Jacobi equation. For the next term, we calculate

$$\begin{aligned} D^2E|_\gamma(J^*,X) &= \int_0^{t_1}\left(\left\langle\frac{\nabla}{dt}J(t),\frac{\nabla}{dt}Y(t)\right\rangle - \big\langle R\big(J(t),\dot\gamma(t)\big)\dot\gamma(t),X(t)\big\rangle\right)dt \\ &= \left\langle\frac{\nabla}{dt}J(t),X(t)\right\rangle\Big|_{t=t_1} \\ &\qquad + \underbrace{\int_0^{t_1}\left(-\left\langle\frac{\nabla^2}{dt^2}J(t),X(t)\right\rangle + \big\langle R\big(\dot\gamma(t),J(t)\big)\dot\gamma(t),X(t)\big\rangle\right)dt}_{=0,\quad \text{as before}} \\ &= -\left\|\frac{\nabla}{dt}J(t_1)\right\|^2 < 0, \end{aligned}$$

by the definition of X. We conclude that the first term (5.2) is zero and the second is negative. Since the second term is linear in ε and the third is quadratic in ε, we can choose ε small enough to obtain $D^2E|_\gamma(Y,Y)<0$, as desired. □

Theorem 5.9. *Let (M,g) be a Riemannian manifold and let $r>0$ be such that $\exp_p$ is defined on $B_r(0)\subseteq T_pM$. Then the restriction of $\exp_p$ to $B_r(0)$ is injective if and only if it is a diffeomorphism onto its image.*

Proof. Clearly, if $\exp_p|_{B_r(0)}$ is a diffeomorphism onto its image, then it is injective.
Conversely, suppose that $\exp_p|_{B_r(0)}$ is not a diffeomorphism onto its image. We need to show that it is not injective. Since $\exp_p|_{B_r(0)}$ is not a diffeomorphism onto its image, it is either not injective (in which case there is nothing to show) or there is a point $X\in B_r(0)$ such that $d\exp_p|_X$ is not injective. So let us assume the latter. Since $d\exp_p|_0=\mathrm{id}_{T_pM}$, we have $X\neq 0$. Let $\gamma(t)=\exp_p(tX)$ be the geodesic with $\dot\gamma(0)=X$.
Since $d\exp_p|_X$ is not injective, there exists a non-zero $Y\in T_XT_pM\cong T_pM$ such that $d\exp_p|_X(Y)=0$. On the other hand, we have

$$d\exp_p|_X(Y)=J(1),$$

where J is the Jacobi field along γ with $J(0)=0$ and $\frac{\nabla}{dt}J(0)=Y$. Hence $t=0$ and $t=1$ are conjugate along γ.

By Lemma 5.8, γ is not minimizing between $\gamma(0)$ and $\gamma(s)$, for any $s > 1$. On the other hand, since $B_r(p) \subseteq M$ is open, there exists some $s > 1$ such that $\gamma(s) \in B_r(p)$. Let γ' be a minimizing geodesic between p and $\gamma(s)$. Since γ' is shorter than γ, we have $X' := \dot{\gamma}'(0) \in B_r(0)$. We obtain that

$$\exp_p(sX) = \gamma(s) = \gamma'(1) = \exp_p(X').$$

Since γ' is shorter than γ as a geodesic between p and $\gamma(s)$, they cannot be reparametrizations of each other, hence $sX \neq X'$. We conclude that $\exp_p$ is not injective. □

Recall that the *injectivity radius* $\mathrm{injrad}(p)$ at a point p in a Riemanian manifold (M, g) is the supremum over all $r > 0$ such that $\exp_p$ is a diffeomorphism onto its image when restricted to $B_r(0) \subset T_pM$. By the above theorem, an alternative definition is

$$\mathrm{injrad}(p) = \sup\{r > 0 \mid \exp_p|_{B_r(0)} \text{ is injective}\}. \tag{5.3}$$

We also have the following characterization.

Lemma 5.10. *If $d(p, \exp_p(X)) = \|X\|$ for all $X \in B_r(0) \subset T_pM$, then $\exp_p$ is injective on $B_r(0)$.*

Proof. Suppose that there exist vectors $X, X' \in B_r(0)$ such that $\exp_p(X) = \exp_p(X')$. We have to show $X = X'$. By the assumption of the lemma,

$$\|X\| = \exp_p(p, X) = \exp_p(X') = \|X'\|.$$

Let $\gamma(t) = \exp_p(tX)$ and $\gamma'(t) = \exp_p(tX')$. Let $s > 1$ such that $\gamma(s)$ is still contained $B_r(p)$. Then since γ and γ' are both length minimizing between p and $\gamma(1) = \gamma'(1)$, the curve

$$\eta(t) = \begin{cases} \gamma'(t) & t \in [0, 1] \\ \gamma(t) & t \in [1, s] \end{cases}$$

is minimizes the length between p and $\gamma(s)$. But since shortest curves are geodesics, η must be in fact smooth, hence $\dot{\gamma}(1) = \dot{\gamma}'(1)$, which implies $\gamma = \gamma'$ and hence $X = X'$. □

Theorem 5.11. *Let (M, g) be a complete Riemannian manifold. Then* $\mathrm{injrad}(p)$ *is a continuous function of $p \in M$.*

Proof. We show that $\mathrm{injrad}(p)$ is both upper and lower semicontinuous in p, which implies that injrad is continuous.
(a) We first show that $\mathrm{injrad}(p)$ is upper semicontinuous in p, in other words,

$$\limsup_{k \to \infty} \mathrm{injrad}(p_k) \leq \mathrm{injrad}(p)$$

whenever $(p_k)_{k \in \mathbb{N}}$ is a sequence in M converging to p. We verify the assumption of Lemma 5.10. Write $r_k = \mathrm{injrad}(p_k)$ and $r^* = \limsup_{k \to \infty} r_k$. For $X \in B_{r^*}(0) \subset T_pM$, choose a sequence of vectors $X_k \in T_{p_k}M$ converging to X. Then for $k \in \mathbb{N}$ large enough, $X_k \in B_{r_k}(0) \subset T_{p_k}M$. Therefore, setting $q_k = \exp_{p_k}(X_k)$, the geodesic $\gamma_k(t) = \exp_{p_k}(tX_k)$ between p_k and q_k is minimizing, hence $d(p_k, q_k) = \|X_k\|$. Since $X_k \to X$, we have $q_k \to \exp_p(X)$, hence by continuity of the distance function,

$$d(p, \exp_p(X)) = \lim_{k\to\infty} d(p_k, q_k) = \lim_{k\to\infty} \|X_k\| = \|X\|.$$

As this holds for any $X \in B_{r^*}(0)$, Lemma 5.10 implies that $\exp_p$ is injective on $B_{r^*}(0) \subset T_pM$. By (5.3), we therefore obtain $\operatorname{injrad}(p) \geq r^*$.
(b) We show that $\operatorname{injrad}(p)$ is lower semicontinuous in p, in other words,

$$\liminf_{k\to\infty} \operatorname{injrad}(p_k) \geq \operatorname{injrad}(p)$$

whenever $(p_k)_{k\in\mathbb{N}}$ is a sequence in M converging to p. Suppose this is not true. Then, setting $r_k = \operatorname{injrad}(p_k)$ and $r = \operatorname{injrad}(p)$, there exists $\varepsilon > 0$ such that

$$\liminf_{k\to\infty} r_k < r - \varepsilon.$$

Therefore, after possibly passing to a subsequence, we have $r_k < r - \varepsilon$. By Thm. 5.9, $\exp_{p_k}$ is not injective when restricted to $B_{r-\varepsilon}(0) \subset T_{p_k}M$, so there exist two distinct vectors $X_k, Y_k \in T_{p_k}M$ such that $\exp_{p_k}(X_k) = \exp_{p_k}(Y_k)$ and such that $\|X_k\|, \|Y_k\| \leq r - \varepsilon$. By compactness, after passing to subsequences, X_k and Y_k converge to vectors $X, Y \in T_pM$ with $\exp_p(X) = \exp_p(Y)$ and $\|X\|, \|Y\| \leq r - \varepsilon$. If $X \neq Y$, we can conclude that exp is not injective on $B_{r-\varepsilon}(0) \subset T_pM$, a contradiction to $\operatorname{injrad}(p) = r$.
It therefore remains to show that $X \neq Y \in T_pM$. To this end, we consider the smooth map

$$F := \pi \times \exp : TM \to M \times M,$$

where $\pi : TM \to M$ is the bundle projection. We claim that the differential $dF|_X : T_XTM \to T_pM \times T_{\exp(X)}M$ has full rank. To see this, we choose a subspace $H \subset T_XTM$ (a space of "horizontal vectors") such that $d\pi|_p : H \to T_pM$ is a linear isomorphism. Then T_XTM splits as a direct sum

$$T_XTM = H \oplus T_XT_pM,$$

where both summands are isomorphic to T_pM, the first via $d\pi|_p$ the second via the canonical isomorphism. With respect to this direct sum decomposition, dF_X has the form

$$dF|_X = \begin{pmatrix} d\pi|_p & 0 \\ * & d\exp_p|_X \end{pmatrix}. \tag{5.4}$$

As by construction, the length of X is less than the injectivity radius of p, $d\exp_p|_X$ has full rank. We obtain that $dF|_X$ has full rank as well. By the inverse function theorem, F must be a diffeomorphism onto its image when restricted to a neighborhood $U \subset TM$ of X; in particular F is injective on this neighborhood. If we now assume that $X = Y$, then both $(X_k)_{k\in\mathbb{N}}$ and $(Y_k)_{k\in\mathbb{N}}$ converge to X. For $k \in \mathbb{N}$ large enough, X_k and Y_k are both contained in U, but we have

$$F(X_k) = (p_k, \exp_{p_k}(X_k)) = (p_k, \exp_{p_k}(Y_k)) = F(Y_k),$$

a contradiction to the fact that F is injective on U. □

Definition 5.12 (Injectivity radius). We define the (global) *injectivity radius* of (M, g) by

$$\mathrm{injrad}(M, g) = \inf_{p\in M} \mathrm{injrad}(p).$$

We generally have $\mathrm{injrad}(M, g) \geq 0$, but the injectivity radius may be zero. However, we have the following lemma.

Corollary 5.13. *If the Riemannian manifold (M, g) is compact, then $\mathrm{injrad}(M, g) > 0$.*

Proof. We have $\mathrm{injrad}(p) > 0$ for each $p \in M$. By Thm. 5.11, the injectivity radius is continuous. Hence because M is compect, there exists $p \in M$ such that

$$0 < \mathrm{injrad}(p) = \inf_{p\in M} \mathrm{injrad}(p) = \mathrm{injrad}(M, g).$$ □

5.7 Closed geodesics

Definition 5.14. Let (M, g) be a Riemannian manifold. A *closed geodesic* in M is a smooth map $\gamma : S^1 \to M$ such that γ satisfies the geodesic equation at every point of S^1.

Theorem 5.15. *Let (M, g) be a compact Riemannian manifold and let $c_0 : S^1 \to M$ be a piecewise C^1 curve. Then there exists a closed geodesic homotopic to c that minimizes the energy among all piecewise C^1 curves homotopic to c_0.*

For the proof of the theorem, we need the following lemma.

Lemma 5.16. *Let (M, g) be a complete Riemannian manifold. Whenever two piecewise C^1 loops $c_1, c_2 : S^1 \to M$ satisfy $d(c_1(t), c_2(t)) < \mathrm{injrad}(c_1(t))$ for all $t \in S^1$, then c_1 and c_2 are homotopic.*

Proof. Let $r_t = \mathrm{injrad}(c_1(t))$. Then the exponential map $\exp_{c_1(t)}$ is injective on $B_{r_t}(0) \subset T_{c_1(t)}M$ and since $d(c_1(t), c_2(t)) < r_t$, there exists a unique $X(t) \in B_{r_t}(0) \subset T_{c_1(t)}M$ such that $c_2(t) = \exp_{c_1(t)}(X(t))$.
We show that X is a piecewise C^1 vector field along c_1. To this end, consider the map $F = \pi \times \exp$ as in the proof of Thm. 5.11. by construction, X satisfies $F(X(t)) = (c_1(t), c_2(t))$. By the assumption $d(c_1(t), c_2(t)) < r_t$, the differential $d\exp_{c_1(t)}|_{X(t)}$ is non-singular for each $t \in S^1$, so from (5.4), we get that also the differential $dF|_{X(t)}$ is non-singular for each

$t \in S^1$. Hence F is a local diffeomorphism. Now if for some $t \in S^1$, $U \subset TM$ is an open neighborhood of $X(t)$ such that F is a diffeomorphism when restricted to U, we have

$$X(t) = (F|_U)^{-1}(c_1(t), c_2(t)).$$

Since c_1 and c_2 are piecewise C^1 and F is smooth, we obtain that X is also piecewise C^1. We can now define a homotopy by

$$H(s,t) = F(sX(t)) = \exp_{c_1(t)}(sX(t)),$$

which satisfes $H(0,t) = c_1(t)$, $H(1,t) = c_2(t)$. □

Proof (of Thm. 5.15). Set

$$E_0 = \inf\{E[c] \mid c \text{ homotopic to } c_0\}$$

and let $c_1, c_2, \ldots$ be a sequence of piecewise C^1 loops homotopic to c_0 such that $E[c_k] \to E_0$ as $k \to \infty$. As reparametrization does not change the homotopy class, we may assume that c_i are parametrized proportionally to arc length. By the estimate $L[c_k]^2 \leq 2E[c_k]$, we obtain that the lengths $L[c_k]$ are bounded uniformly over k. We think of the loops c_k as maps $[0,1] \to M$ such that $c_k(0) = c_k(1)$.
Pick $r \in \mathbb{R}$ satisfying $0 < r < \operatorname{injrad}(M,g)$, which is possible by (5.13). Then since the c_i are parametrized by arc length and their lengths are uniformly bounded, there exists some $N \in \mathbb{N}$ such that, setting $t_j = j/N$, the points $p_j^k := c_k(t_j)$ $(j = 0, \ldots, N)$, satisfy

$$d(p_{j-1}^k, p_j^k) \leq \frac{r}{3}$$

for any $j = 1, \ldots, N$ and any $k \in \mathbb{N}$ (observe that $p_0^k = p_N^k$ since the c_i are loops). We obtain N sequences $(p_1^k)_{i\in\mathbb{N}}, \ldots, (p_N^k)_{k\in\mathbb{N}}$ in M. Since with M also the N-fold product M^N is compact, after passing to a subsequence, we may achieve that for each $j = 0, \ldots, N$, the sequence $(p_j^k)_{i\in\mathbb{N}}$ converges to some point $p_j \in M$, where $p_0 = p_N$.
By continuity, we also have

$$d(p_{j-1}, p_j) \leq \frac{r}{3} < \operatorname{injrad}(M,g),$$

for each j, so there exists a unique minimizing geodesic between p_{j-1} and p_j. Let $c : S^1 \to M$ be the piecewise geodesic loop such that $c(t_j) = p_j$ and such that on the subintervals $[t_{j-1}, t_j]$ of $[0,1]$ $(j = 1, \ldots, N)$, c is a the unique minimizing geodesic between p_{j-1} and p_j. Let $k_0 \in \mathbb{N}$ be so large that $d(p_j^k, p_j) \leq r/3$ for all $k \geq k_0$. Then for any $k \geq k_0$ and any $t \in S^1$, we have

$$d(c(t), c_k(t)) \leq d(c(t), p_j) + d(p_j, p_j^k) + d(p_j^k, c_k(t)) \leq \frac{r}{3} + \frac{r}{3} + \frac{r}{3} = r < \operatorname{injrad}(M,g).$$

where j is such that $t \in [t_{j-1}, t_j]$. From Lemma 5.16, we obtain that c and c_k are homotopic, hence c is also homotopic to c_0. Moreover, for each $j = 1, \dots, N$ and each $k \in \mathbb{N}$, we get

$$\begin{aligned} L[c|_{[t_{j-1},t_j]}] &= d(p_{j-1}, p_j) \leq d(p_{j-1}, p_{j-1}^k) + d(p_{j-1}^k, p_j^k) + d(p_j^k, p_j) \\ &\leq d(p_{j-1}, p_{j-1}^k) + L[c_k|_{[t_{j-1},t_j]}] + d(p_j^k, p_j). \end{aligned}$$

Taking the sum over j, we get

$$\begin{aligned} L[c] &\leq L[c_k] + 2\sum_{j=1}^n d(p_j^k, p_j) \\ &\leq \sqrt{2E[c_k]} + 2\sum_{j=1}^n d(p_j^k, p_j) \qquad \overset{k\to\infty}{\longrightarrow} \sqrt{2E_0}. \end{aligned}$$

Let γ be the reparametrization of c proportionally to arc length, which is again homotopic to c_0. Then since γ is parametrized proportionally to arc length, we have

$$E[\gamma] = \frac{1}{2}L[\gamma]^2 = \frac{1}{2}L[c]^2 \leq E_0.$$

On the other hand, since E_0 is is infimum of all energies of loops homotopic to c_0, we must have $E[\gamma] = E_0$. Hence γ realizes the infinimum of the energy among all piecewise C^1 curves homotopic to c_0. It follows that all variations $(\gamma_s)_{s\in(-\varepsilon,\varepsilon)}$ of γ must satisfy $\frac{d}{ds}|_{s=0}E[\gamma_s] = 0$, so it follows from Thm. 5.3.1 that γ is a geodesic (and in particular smooth, not only piecewise smooth). □

5.8 Synge's theorem

Let M be a manifold. For a point $p \in M$, denote by $\mathrm{Fr}_p(M)$ the set of vector space bases of T_pM. If n the dimension of the manifold, there is a right action of $\mathrm{GL}(n, \mathbb{R})$ on $\mathrm{Fr}_p(M)$, given by

$$(E_1, \dots, E_n) \cdot \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} = \left(\sum_{i=1}^n a_{i1}E_i, \dots, \sum_{i=1}^n a_{in}E_i \right)$$

which is free and transitive. Hence the choice of a basis in T_pM gives a bijection to $\mathrm{GL}(n, \mathbb{R})$. $\mathrm{Fr}_p(M)$ is then given the unique manifold structure making this bijection to the open set $\mathrm{GL}(n, \mathbb{R}) \subset \mathbb{R}^{n\times n}$ is a diffeomorphism (this is independent of the choice of basis). The manifolds $\mathrm{Fr}_p(M)$ fit together to the *frame bundle*

$$\mathrm{Fr}(M) = \coprod_{p\in M} \mathrm{Fr}_p(M).$$

There is then a unique manifold structure on $\mathrm{Fr}(M)$ such that the footpoint projection $\pi : \mathrm{Fr}(M) \to M$ is a surjective submersion (this is constructed similarly to the smooth

structure on TM). In fact, this shows that $\mathrm{Fr}(M)$ is even a fiber bundle with typical fiber $\mathrm{GL}(n, \mathbb{R})$ (in fact, it is even a *principal bundle*).
$\mathrm{GL}(n, \mathbb{R})$ has two different connected components: The matrices with positive and negative determinant. Since $\mathrm{Fr}_p(M)$ is diffeomorphic to $\mathrm{GL}(n, \mathbb{R})$ for every $p \in M$, this shows that also each $\mathrm{Fr}_p(M)$ has two connected components. We then have the following lemma.

Lemma 5.17. *If M is connected, then $\mathrm{Fr}(M)$ has at most two connected components.*

Proof. This follows at once from the long exact sequence for homotopy groups, applied to the fibration

$$\mathrm{GL}(n, \mathbb{R}) \longrightarrow \mathrm{Fr}(M) \longrightarrow M.$$

Here we have that

$$\cdots \longrightarrow \pi_1(M) \longrightarrow \underbrace{\pi_0\big(\mathrm{GL}(n, \mathbb{R})\big)}_{\mathbb{Z}_2} \longrightarrow \pi_0\big(\mathrm{Fr}(M)\big) \longrightarrow \underbrace{\pi_0(M)}_{=0}$$

is a short exact sequence of pointed sets. Hence since M is connected, $\pi_0(\mathrm{GL}(n, \mathbb{R})) \cong \mathbb{Z}_2$ surjects onto $\pi_0(\mathrm{Fr}(M))$.
An explicit argument is the following. Fix a base point $E_0 \in \mathrm{Fr}(M)$. Denote by p_0 the foot point of E_0. Because M is connected (hence path-connected, as M is a manifold), given any other frame $E \in \mathrm{Fr}(M)$, say with foot point p, there exists a continuous path $c : [0, 1] \to M$ with $c(0) = p_0$ and $c(1) = p$. Now, because $\mathrm{Fr}(M)$ is a fiber bundle, it has the *path-lifting property*, which implies that we can lift c to a path $\bar{c} : [0, 1] \to \mathrm{Fr}(M)$, satisfying $\widehat{c}(0) = E_0$ and $\pi(\bar{c}(t)) = c(t)$ for all t. In particular, $c(1)$ and E are two points in the same fiber $\mathrm{Fr}_{p_0}(M)$. If these two points lie in the same component of $\mathrm{Fr}_{p_0}(M)$, then we connect them by a path in $\mathrm{Fr}_p(M)$ and, by concatenation, obtain a path connecting E_0 and E.
Now, if E and E' are two points in $\mathrm{Fr}(M)$, we can connect the foot points of both with p_0 through paths c and c', and lift these to $\mathrm{Fr}(M)$, obtaining two elements $c(1), c'(1) \in \mathrm{Fr}_{p_0}(M)$. Now since $\mathrm{Fr}_{p_0}(M)$ has two connected components, at least two of the three elements $E_0, c(1), c'(1)$ must lie in the same connected component. This shows that given two frames E and E', either one of them lies in the same path component of E_0 or they both lie in the same path component. Hence $\pi_0(\mathrm{Fr}(M))$ has at most two elements. $\square$

Definition 5.18 (Orientability). A connected manifold M is *orientable* if the frame bundle $\mathrm{Fr}(M)$ has two different connected components. An *orientation* of an orientable connected manifold M is a choice of one of these connected components. It is denoted by $\mathrm{Fr}^+(M)$ and its elements are called *positively oriented* bases.
A (possibly non-connected) manifold is called orientable if each connected component is orientable, and an orientation of such a manifold is the choice of an orientation for each connected component.

For an oriented manifold M, the $\mathrm{GL}(n, \mathbb{R})$-action on $\mathrm{Fr}(M)$ restricts to an action of $\mathrm{GL}_n^+(\mathbb{R})$ (the group of matrices with positive determinant) on $\mathrm{Fr}^+(M)$.

Lemma 5.19. *Let (M, g) be a Riemannian manifold. For a piecewise smooth loop $c : S^1 \to M$, denote by P_c the parallel transport around c.*

(a) *If M is oriented, then every piecewise smooth loop $c : S^1 \to M$ has* $\det(P_c) = 1$.

(b) *If M is not orientable, then there exists a piecewise smooth loop $c : S^1 \to M$ such that* $\det(P_c) = -1$.

Proof. We think of c as a map $c : [0, 1] \to M$ such that $c(0) = c(1)$. We know that $P_c : T_{c(0)}M \to T_{c(1)}M$ is an orthogonal transformation. Hence $\det(P_c) = \pm 1$.
(a) Suppose that M is orientable. Let $E = (E_1, \dots, E_n) \in \mathrm{Fr}^+_{c(0)}(M)$ be a positively oriented basis of of $T_{c(0)}M$ and let $E(t) = (E_1(t), \dots, E_n(t)) \in \mathrm{Fr}_{c(t)}(M)$ be the corresponding parallel transported basis along c. Then $E(t)$ is a continuous path in $\mathrm{Fr}^+(M)$, hence $E(t)$ must be positively oriented for every $t \in [0, 1]$. In particular, for $t = 1$, we have $E(1) = (P_cE_1, \dots, P_cE_n) \in \mathrm{Fr}^+_{c(1)}$, so

$$(P_cE_1, \dots, P_cE_n) = (E_1, \dots, E_n) \cdot A$$

for some $A \in \mathrm{GL}^+(\mathbb{R})$. It follows from the definition of the action that A is just a matrix representation of P_c with respect to the basis E. Hence

$$\det(P_c) = \det(A) > 0.$$

(b) Suppose that M is not orientable. Then $\mathrm{Fr}(M)$ is connected (hence path-connected). So given $p \in M$ and any two bases $E, E' \in \mathrm{Fr}_p(M)$, there exists a smooth path $\bar{c}$ in $\mathrm{Fr}(M)$ with $\bar{c}(0) = E$ and $\bar{c}(1) = E'$. Since E and E' lie in the same fiber, its foot point curve c is a closed curve. Let $E(t)$ be the frame obtained by parallel transport of E along c. This gives another smooth path in $\mathrm{Fr}(M)$, which lies in the same fiber as $\bar{c}(t)$ for every $t \in [0, 1]$. Hence for each $t \in [0, 1]$, there exists a unique $A(t) \in \mathrm{GL}(n, \mathbb{R})$ such that $E(t) = \bar{c}(t) \cdot A(t)$. Since both $E(t)$ and $\bar{c}(t)$ are smooth in t, the matrix $A(t)$ must depend smoothly on t also. Since $E(0) = E = \bar{c}(0)$, we have $A(0) = \mathrm{id}$, so $\det(A(t)) > 0$ for each $t \in [0, 1]$. We obtain that

$$(P_cE_1, \dots, P_c(E_n)) = E(1) = \bar{c}(1) \cdot A(1) = E' \cdot A(1).$$

Now if E' does not lie in the same connected component as E, we have $E' = E \cdot B$ for a matrix B with $\det(B) < 0$. We therefore have

$$\det(P_c) = \det(BA(1)) = \det(B)\det(A(1)) < 0.$$

□

Lemma 5.20. *Let (M, g) be a Riemannian manifold and let $c, c' : S^1 \to M$ be two piecewise smooth loops that are homotopic. Then the parallel transports P_c and $P_{c'}$ around c, respectively c', satisfy* $\det(P_c) = \det(P_{c'})$.

Remark 5.21. The point is here that the homotopy is only required to be continuous.

Proof. Let $c : S^1 \to M$ be a loop (which we think of as path $c : [0,1] \to M$ with $c(0) = c(1)$) and let $E \in \mathrm{Fr}_{c(0)}(M)$ be a basis of $T_{c(0)}M$. Let moreover $E(t)$ be the parallel transport of E around c. As seen in the proof of Lemma 5.19, we have $\det(P_c) = 1$ if and only if $E(1)$ lies in the same connected component as E. Let now c' be another loop and let H be a homotopy between c and c'. Since $\mathrm{Fr}(M)$ is a fiber bundle, we can lift H to a map $\overline{H} : [0,1] \times [0,1] \to \mathrm{Fr}(M)$ such that $H(0,t) = E(t)$ for all $t \in [0,1]$. Write $E' = H(1,0)$ and let $E'(t)$ be the parallel transport of E' along c'. Then as in the proof of Lemma 5.19, $E'(t) = H(1,t) \cdot A(t)$ for a smooth map $A : [0,1] \to \mathrm{GL}(n, \mathbb{R})$. Now, if $E(1)$ lies in the same connected component as E if and only if $H(s,1)$ and $H(s,0)$ lie in the same component of $\mathrm{Fr}_{H(s,0)}(M)$ for every $s \in [0,1]$, if and only if $E'(1)$ lies in the same connected component as E'. □

Theorem 5.22 (Synge). *Let (M, g) be a compact Riemannian manifold with positive sectional curvature.*

(a) *If M is even-dimensional and oriented, then it is simply connected.*

(b) *If M is odd-dimensional, then it is orientable.*

The proof is based on the following.

Linear Algebra Lemma 5.23. *Let V be an oriented Euclidean vector space and let P be an orthogonal transformation of V. If V is odd-dimensional, assume that* $\det(P) = 1$. *If V is even-dimensional, assume that* $\det(P) = -1$. *Then P fixes a one-dimensional subspace.*

Proof. As an orthogonal transformation, the eigenvalues of P all lie on the complex unit circle. Since P is real, the spectrum is symmetric with respect to the real axis, hence if λ is an eigenvalue, then so is $\overline{\lambda}$. Let $E \subset V$ be the direct sum of eigenspaces to eigenvalues with non-zero imaginary part. Then E is an even-dimensional invariant subspace, since all eigenvalues of $P|_E$ come in complex conjugate pairs. As $\lambda\overline{\lambda} = 1$, we have $\det(P|_E) = 1$. On the orthogonal complement V' of E, P has the eigenvalues $+1$ and -1. We have to show that $e = \dim\{X \mid PX = X\}$ is not zero. We have

$$\det(P) = \det(P|_{V'})\det(P|_E) = \det(P|_{V'}) = (-1)^{\dim(V')-e} = (-1)^{\dim(V)-e}.$$

since V' has the same parity as V. Now observe that the assumptions imply that e must always be odd, in particular not zero. □

Proof (of Thm. 5.22). (a) Assume that M is even-dimensional and suppose that $c : S^1 \to M$ defines an non-trivial element of $\pi_1(M)$. By Thm. 5.15, there exists a closed geodesic γ homotopic to c that minimizes the energy in its homotopy class. We may parametrize γ by arc length. The parallel transport P_γ around γ satisfies

$$P_\gamma \dot{\gamma}(0) = \dot{\gamma}(0),$$

hence it preserves the orthogonal complement $V = \dot\gamma(0)^\perp \subset T_{\gamma(0)}M$. Since M is even-dimensional, V is odd-dimensional. Hence $P_\gamma|_V$ is orthogonal transformation of an odd-dimensional vector space. By Lemma 5.19(a), we have $\det(P_\gamma) = 1$. By Lemma 5.23, $P_\gamma|_V$ fixes a line, hence there exists $X \perp \dot\gamma(0)$ with $\|X\| = 1$ and $P_\gamma X = X$. Let $X(t)$ be the parallel vector field around γ corresponding to this eigenvector (the point is here that X closes up continuously after going around γ). Let $(\gamma_s)_{s\in(-\varepsilon,\varepsilon)}$ be a variation with variational vector field X. Then since X is parallel, we get

$$\begin{aligned}\frac{\partial^2}{\partial s^2}E[\gamma_s] &= \int_0^1 \left(\left\|\frac{\nabla}{dt}X(t)\right\|^2 - \big\langle R\big(X(t),\dot\gamma(t)\big)\dot\gamma(t), X(t)\big\rangle\right)dt\\ &= -\int_0^1 R\big(X(t),\dot\gamma(t)\big)\dot\gamma(t), X(t)\rangle dt\\ &= -\int_0^1 K\big(\operatorname{span}\{X(t),\dot\gamma(t)\}\big)dt < 0.\end{aligned}$$

Here we use that as $X \perp \dot\gamma(0)$, we also have $X(t) \perp \dot\gamma(t)$ for all other t, hence $X(t)$ and $\dot\gamma(t)$ span a plane in $T_{\gamma(t)}M$ and we can identify $R(X(t),\dot\gamma(t)\big)\dot\gamma(t), X(t)$ with the sectional curvature of this plane. We obtain that for $|s|$ small, γ_s is a curve homotopic to γ such that $E[\gamma_s] < E[\gamma]$, a contradiction to the assumption that γ is energy minimizing in its homotopy class. Hence every element of $\pi_1(M)$ must be trivial.

(b) Suppose M is not orientable. Then by Lemma 5.19(b), there exists a piecewise smooth loop $c : S^1 \to M$ such that $\det(P_c) = -1$. c must define a non-trivial element of $\pi_1(M)$, because if c is homotopic to a constant loop, then by Lemma 5.20, we have $\det(P_c) = \det(\mathrm{id}) = 1$ (using that parallel transport around constant loops is the identity). Let now γ be an energy-minimizing geodesic homotopic to c (Thm. 5.15). By Lemma 5.20, we also have $\det(P_\gamma) = -1$. Since M is odd-dimensional, the orthogonal complement $V = \dot\gamma(0)^\perp$ is an even-dimensional invariant subspace for P_γ, and we still have $\det(P_\gamma|_V) = -1$. Hence by Lemma 5.23, $P_\gamma|_V$ fixes a line. As before, we derive a contradiction to the fact that γ minimizes the energy in its homotopy class. Hence there cannot be a piecewise smooth loop $c : S^1 \to M$ with $\det(P_c) = -1$, so M must be orientable. □

Remark 5.24. The necessity of the assumptions of Synge's theorem can be seen via the following examples.

(a) The odd-dimensional spheres can be realized as $S^{2n-1} \subset \mathbb{C}^n$. For any $p \in \mathbb{N}$ and $q_1, \dots, q_n$ coprime integers to p, there is an action of the cyclic group $\mathbb{Z}/p\mathbb{Z}$ on S^{2n-1}, where the generator of $\mathbb{Z}/p\mathbb{Z}$ acts via

$$(z_1, \dots, z_n) \longmapsto (e^{2\pi q_1/p}z_1, \dots, e^{2\pi q_n/p}z_n).$$

This is a free and orientation preserving action by isometries, hence taking the quotient by this action, we obtain a new Riemannian manifold, the so-called *lens space* $L(p; q_1, \dots, q_n)$. The corresponding Riemannian manifold still has constant positive sectional curvature and non-trivial fundamental group. This shows that the manifold in Thm. 5.22 must indeed be even-dimensional for the conclusion of (a) to hold.

(b) The real projective spaces $\mathbb{R}P^n$ are non-orientable if n is even, but have positive sectional curvature. This shows that the conclusion (b) of Synge's theorem needs the assumption that the dimensional of M is odd.

6 Lie groups and homogeneous spaces

6.1 Lie groups and their Lie algebras

Definition 6.1. A *Lie group* is a group G together with a manifold structure on G such that the group multiplication $m : G \times G \to G$ and the inversion map $i : G \to G$ are smooth. A *homomorphism* between Lie groups G and H is a smooth group homomorphism $\varphi : G \to H$.

Example 6.2. Any countable group is a Lie group when endowed with the discrete topology. (It cannot be uncountable because then as a topological space with the discrete topology, it would not be second-countable.)

Example 6.3. $\mathbb{R}^n$ with its usual smooth structure becomes a Lie group with respect to addition of vectors. Its quotient $T^n = \mathbb{R}^n/\mathbb{Z}^n$, the n-dimensional torus, is also a Lie group. In fact, every connected abelian Lie group is of the form $T^n \times \mathbb{R}^m$, for numbers $n, m \in \mathbb{N}_0$.

Example 6.4. It is a general (non-trivial) fact that the isometry group $\mathrm{Isom}(M, g)$ of a semi-Riemannian manifold (M, g) is a Lie group.

Example 6.5. If V is a real or complex or quaternionic vector space, then the *general linear group* $\mathrm{GL}(V)$ is a Lie group with the manifold structure coming from viewing it as an open subset of $\mathrm{End}(V)$. For any $n \in \mathbb{N}$, we thus obtain Lie groups $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{GL}(n, \mathbb{H})$. There are canonical inclusions $\mathrm{GL}(n, \mathbb{C}) \subset \mathrm{GL}(2n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{H}) \subset \mathrm{GL}(2n, \mathbb{C}) \subset \mathrm{GL}(4n, \mathbb{R})$, which for $n = 1$ are given by

$$a + bi \longmapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \qquad z + wj \longmapsto \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix}.$$

The most important theorem for generating examples of Lie groups is the following. A proof can be found in any textbook on Lie groups.

Theorem 6.6. *If G is a Lie group and $H \subset G$ is a closed subgroup, then it is in fact a submanifold and a Lie group with the induced smooth structure.*

Example 6.7. It follows that any closed subgroup $G \subset \mathrm{GL}(V)$ is a Lie group. We have the following concrete examples.

(a) The *special linear group* $\mathrm{SL}(V) = \det^{-1}(1) \subset \mathrm{GL}(V)$. We also use the notations $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{SL}(n, \mathbb{C})$ and $\mathrm{SL}(n, \mathbb{H})$. In the quaternionic case, one has to take the real determinant, coming from the inclusion $\mathrm{GL}(n, \mathbb{H}) \subset \mathrm{GL}(4n, \mathbb{R})$.

(b) The *orthogonal group* $\mathrm{O}(n) = \{A \in \mathrm{GL}(n,\mathbb{R}) \mid AA^t = \mathrm{id}\}$.

(c) The *special orthogonal group* $\mathrm{SO}(n) = \mathrm{O}(n) \cap \mathrm{SL}(n,\mathbb{R})$.

(d) The *unitary group* $\mathrm{U}(n) = \{A \in \mathrm{GL}(n,\mathbb{C}) \mid AA^* = \mathrm{id}\}$. Here $A^* = \overline{A}^t$ denotes the transpose, followed by complex conjugation.

(e) The *special unitary group* $\mathrm{SU}(n) = \mathrm{U}(n) \cap \mathrm{SL}(n,\mathbb{C})$.

(f) The unitary groups are not to be confused with the *complex orthogonal groups* $\mathrm{O}(n,\mathbb{C}) = \{A \in \mathrm{GL}(n,\mathbb{C}) \mid AA^t = \mathrm{id}\}$ and $\mathrm{SO}(n,\mathbb{C}) = \mathrm{O}(n,\mathbb{C}) \cap \mathrm{SL}(n,\mathbb{C})$. $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are compact, while these are not.

(g) The (compact) *symplectic group* $\mathrm{Sp}(n) = \{A \in \mathrm{GL}(n,\mathbb{H}) \mid AA^\dagger = \mathrm{id}\}$. Here $A^\dagger$ denotes transpose followed by quaternionic conjugation.

Given a Lie group G, we write

$$\lambda_g(h) = gh, \qquad \rho_g(h) = hg,$$

for the actions of G on itself by left and right multiplication. It follows from the axioms of a Lie group, that λ_g and ρ_g are diffeomorphisms for every $g \in G$.

Definition 6.8 (Left-invariant vector fields). A vector field $\tilde{X}$ on a Lie group G is called *left invariant* if for all $g \in G$, we have

$$d\lambda_g(\tilde{X}) = \tilde{X} \circ \lambda_g.$$

Lemma 6.9. *For two left-invariant vector fields $\tilde{X}$, $\tilde{Y}$ on a Lie group G, the Lie bracket $[\tilde{X}, \tilde{Y}]$ is again left invariant.*

Proof. Observe here that left-invariance of a vector field $\tilde{X}$ on G can be reformulated as $(\lambda_g)_* \tilde{X} = \tilde{X}$. Therefore the lemma follows from the fact that the Lie bracket of vector fields is *natural*, in the sense that for any diffeomorphism $f : M \to N$ between manifolds and any two vector fields $\tilde{X}$, $\tilde{Y}$ on M, we have $f_*[\tilde{X}, \tilde{Y}] = [f_*\tilde{X}, f_*\tilde{Y}]$, where $f_*\tilde{X} = df(\tilde{X}) \circ f^{-1}$ is the pushforward of X by f. □

For any manifold M, the Lie bracket on vector fields satisfies the so-called *Jacobi identity*

$$[[\tilde{X}, \tilde{Y}], \tilde{Z}] + [[\tilde{Y}, \tilde{Z}], \tilde{X}] + [[\tilde{Z}, \tilde{X}], \tilde{Y}] = 0. \tag{6.1}$$

Abstractly, a vectorspace V together with a skew-symmetric bracket $[\cdot,\cdot] : V \times V \to V$ satisfying (6.1) is called a *Lie algebra*. By Lemma 6.9, the space of left invariant vector fields $\mathcal{X}_\lambda(G)$ is a Lie subalgebra of the Lie algebra $\mathcal{X}(G)$ of all vector fields on G.

There is an alternative description of the Lie algebra $\mathcal{X}_\lambda(G)$ of left invariant vector fields for a Lie group G that we discuss now.

Lemma 6.10. *Evaluation at $e \in G$ gives a vector space isomorphism between the space of left invariant vector fields on G and T_eG.*

Proof. For any tangent vector $X \in T_eG$, there is a left invariant vector field $\tilde{X}$ with $\tilde{X}|_e = X$, which is given by

$$\tilde{X}|_g = d\lambda_g(X).$$

Conversely, every left invariant vector field $\tilde{X}$ is determined by its value at $e \in G$, as $\tilde{X}|_g = (\tilde{X} \circ \lambda_g)|_e = d\lambda_g(\tilde{X}|_e)$. □

From now on, we set

$$\mathfrak{g} := T_eG,$$

which by the above Lemma is isomorphic to the Lie algebra of left invariant vector fields on G. To obtain a description of the Lie bracket of $\mathcal{X}_\lambda(G)$ in terms of $\mathfrak{g}$, consider the conjugation action

$$\alpha : G \times G \longrightarrow G, \qquad (g, h) \longmapsto ghg^{-1}$$

of G on itself. Differentiating α with respect to the second variable at the unit element $e \in G$, we obtain an action

$$\mathrm{Ad} : G \times \mathfrak{g} \longrightarrow \mathfrak{g}.$$

of G on $\mathfrak{g}$. This action is called the *adjoint action* of G on $\mathfrak{g}$.

Example 6.11. If $G \subset \mathrm{GL}(V)$ is a closed subgroup, the adjoint action is just given by $\mathrm{Ad}_g(X) = gXg^{-1}$. The tangent space at an arbitrary $g \in G$ is related to $\mathfrak{g}$ by

$$T_gG = \{Xg \mid X \in T_eG\} = \{gX \mid X \in T_eG\}.$$

The equality of the two descriptions above follows from the invariant of $\mathfrak{g}$ under the adjoint action.

Further differentiating Ad with respect to the first variable, we obtain a map

$$\mathrm{ad} : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}.$$

Lemma 6.12. *Let $X, Y \in \mathfrak{g}$ and let $\tilde{X}$ and $\tilde{Y}$ be the corresponding left invariant vector fields. Then*

$$\mathrm{ad}_X(Y) = [\tilde{X}, \tilde{Y}]|_e.$$

Proof. We show that $\mathrm{ad}_X(Y)$ and $[\tilde{X}, \tilde{Y}]|_e$ coincide as derivations on $\mathfrak{g} = T_eG$. To this end, let $\xi, \eta : (-\varepsilon, \varepsilon) \to G$ be smooth curves with $\xi(0) = \eta(0) = e$ and $\dot{\xi}(0) = X$, $\dot{\eta}(0) = Y$. Then

$$\mathrm{ad}_X(Y) = \frac{\partial}{\partial t}\Big|_{t=0} \mathrm{Ad}_{\xi(t)}(Y).$$

On the other hand, $\sigma(s) = \xi(t)\eta(s)\xi(t)^{-1}$ is a curve such that $\dot\sigma(0) = \mathrm{Ad}_{\xi(t)}(Y)$. Hence for $f \in C^\infty(G)$, we have

$$\partial_{\mathrm{Ad}_{\xi(t)}(Y)} f(e) = \frac{\partial}{\partial s}\Big|_{s=0} f\big(\alpha_{\xi(t)}(\eta(s))\big).$$

We therefore get

$$\partial_{\mathrm{ad}_X(Y)} f(e) = \frac{\partial}{\partial t}\Big|_{t=0} \partial_{\mathrm{Ad}_{\xi(t)}(Y)} f(e) = \frac{\partial^2}{\partial t \partial s}\Big|_{t=s=0} f\big(\alpha_{\xi(t)}(\eta(s))\big)$$

Exchanging the differentiation variables and using the chain rule, we get

$$\begin{aligned}\frac{\partial^2}{\partial t \partial s}\Big|_{t=s=0} f\big(\alpha_{\xi(t)}(\eta(s))\big) &= \frac{\partial^2}{\partial s \partial t}\Big|_{s=t=0} f\big(\xi(t)\eta(s)\xi(t)^{-1}\big) \\ &= \underbrace{\frac{\partial^2}{\partial s \partial t}\Big|_{s=t=0} f\big(\xi(t)\eta(s)\big)}_{①} + \underbrace{\frac{\partial^2}{\partial s \partial t}\Big|_{s=t=0} f\big(\eta(s)\xi(t)^{-1}\big)}_{②}.\end{aligned}$$

We have

$$\frac{\partial}{\partial s}\Big|_{s=0} \xi(t)\eta(s) = \frac{\partial}{\partial s}\Big|_{s=0} \lambda_{\xi(t)}\big(\eta(s)\big) = d\lambda_{\xi(t)}|_e(Y|_e) = \tilde{Y}|_{\xi(t)},$$

hence swapping derivatives again, we get

$$① = \frac{\partial^2}{\partial t \partial s}\Big|_{t=s=0} f\big(\xi(t)\eta(s)\big) = \frac{\partial}{\partial t}\Big|_{t=0} \partial_{\tilde{Y}} f(\xi(t)) = \partial_{\tilde{X}} \partial_{\tilde{Y}} f(e)$$

To deal with the term ②, we observe that differentiating the constant curve $t \mapsto \xi(t)\xi(t)^{-1}$ using the chain rule, we obtain that $t \mapsto \xi(0)^{-1}$ represents the tangent vector $-\dot\xi(0) = -X$. Hence

$$\frac{\partial}{\partial t}\Big|_{t=0} \eta(s)\xi(t)^{-1} = \frac{\partial}{\partial t}\Big|_{t=0} \lambda_{\eta(s)}\big(\xi(t)^{-1}\big) = -d\lambda_{\eta(s)}|_e(X|_e) = -\tilde{X}|_{\eta(s)},$$

and

$$② = -\frac{\partial}{\partial s}\Big|_{s=0} \partial_{\tilde{X}} f\big(\eta(s)\big) = -\partial_{\tilde{Y}} \partial_{\tilde{X}} f(e).$$

Putting everything together, we obtain

$$\partial_{\mathrm{ad}_X(Y)} f(e) = \partial_{\tilde{X}} \partial_{\tilde{Y}} f(e) - \partial_{\tilde{Y}} \partial_{\tilde{X}} f(e) = \partial_{[\tilde{X},\tilde{Y}]} f(e),$$

as claimed. □

Definition 6.13 (Lie algebra). Let G be a Lie group. The *Lie algebra of* G is its tangent space at the identity element $\mathfrak{g} = T_e G$, together with the Lie bracket $[X, Y] = \mathrm{ad}_X(Y)$.

Lie groups are typically denoted by capital roman letters such as G, H, K, while the corresponding Lie algebras are denoted by the corresponding fraktur letters $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{k}$. If $\varphi : G \to H$ is a Lie group homomorphism, its differential induces a Lie algebra homomorphism $\varphi_* = d\varphi|_e : \mathfrak{g} \to \mathfrak{h}$.

Example 6.14. The Lie bracket of the Lie algebra of an abelian Lie group is zero.

Example 6.15. The Lie algebra of $\mathrm{GL}(V)$ is the space $\mathrm{gl}(V) = \mathrm{End}(V)$ of vector space endomorphisms of V, with the commutator Lie bracket

$$[X, Y] = XY - YX.$$

Matrix Lie groups from Example 6.7 are the following, all with the bracket induced by the one of $\mathrm{gl}(V)$:

(a) $\mathfrak{sl}(V) = \{X \in \mathrm{gl}(V) \mid \mathrm{tr}(X) = 0\}$.

(b) $\mathfrak{o}(n) = \mathfrak{so}(n) = \{X \in \mathrm{gl}(n, \mathbb{R}) \mid X^t = -X\}$.

(c) $\mathfrak{u}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid X^* = -X\}$.

(d) $\mathfrak{su}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid X^* = -X, \mathrm{tr}(X) = 0\}$.

(e) $\mathfrak{o}(n, \mathbb{C}) = \mathfrak{so}(n, \mathbb{C}) = \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid X^t = -X\}$.

(f) $\mathfrak{sp}(n) = \{X \in \mathfrak{gl}(n, \mathbb{H}) \mid X^\dagger = -X\}$.

It is an exercise to check that the commutator bracket indeed restricts to each of these subspaces.

6.2 Differential geometry of Lie groups

Definition 6.16 (Invariant metrics). Let G be a Lie group. A semi-Riemannian metric β on G is called *left invariant*, if $\lambda_g^* \beta = \beta$ for all $h \in G$, and *bi-invariant* if $\lambda_g^* \beta = \rho_g^* \beta = \beta$ for all $h \in G$.

Lemma 6.17. *Let G be a Lie group with Lie algebra $\mathfrak{g}$.*

(a) *Restriction at the unit element e provides an isomorphism between the space of left invariant semi-Riemannian metrics on G and the space of non-degenerate inner products on $\mathfrak{g}$ (of the same signature).*

(b) *This isomorphism further refines to one between the space of bi-invariant semi-Riemannian metrics on G and the space of* Ad*-invariant, non-degenerate inner products on $\mathfrak{g}$.*

Proof. (a) Any left-invariant semi-Riemannian metric β is determined by $\beta_0 := \beta|_e$ because of the formula

$$\beta|_g(X|_g, Y|_g) = \beta_0\big(d\lambda_{g^{-1}}|_g(X|_g), d\lambda_{g^{-1}}|_g(Y|_g)\big), \qquad X, Y \in T_g G. \tag{6.2}$$

This shows that the restriction map is injective. On the other hand, given a non-degenerate inner product b on $\mathfrak{g} = T_e G$, it defines a semi-Riemannian metric on G by replacing $\beta|_e$ by b in the above formula. Therefore, the restriction map is surjective.

(b) Let β_0 be an Ad-invariant, non-degenerate inner product on $\mathfrak{g}$ and let β be the corresponding left-invariant metric given by (6.2). Then for vector fields $X, Y \in T_gG$, we have

$$\begin{aligned}
\beta|_g(X,Y) &= \beta_0\big(d\lambda_{g^{-1}}|_g(X), d\lambda_{g^{-1}}|_g(Y)\big) \\
&= \beta_0\big(\mathrm{Ad}_{h^{-1}}\, d\lambda_{g^{-1}}(X), \mathrm{Ad}_{h^{-1}}\, d\lambda_{g^{-1}}(Y)\big) \\
&= \beta_0\big(d\alpha_{h^{-1}}|_e d\lambda_{g^{-1}}|_g(X), d\alpha_{h^{-1}}|_e d\lambda_{g^{-1}}|_g(Y)\big) \\
&= \beta_0\big(d\lambda_{h^{-1}}|_h d\rho_h|_e d\lambda_{g^{-1}}(X), d\lambda_{h^{-1}}|_h d\rho_h|_e d\lambda_{g^{-1}}(Y)\big) \\
&= \beta_0\big(d\lambda_{(gh)^{-1}}|_{gh} d\rho_h|_g(X), d\lambda_{(gh)^{-1}}|_{gh} d\rho_h|_g(Y)\big) \\
&= \beta_{gh}\big(d\rho_h|_g(X), d\rho_h|_g(Y)\big),
\end{aligned}$$

where we used that the left and right actions commute, together with the product rule. So β is also right invariant, hence bi-invariant. Conversely, if β is bi-invariant, then we have for $\beta_0 := \beta|_e$ that

$$\begin{aligned}
\beta_0\big(\mathrm{Ad}_g(X), \mathrm{Ad}_g(Y)\big) &= g|_e\big(d\alpha_g|_e(X), d\alpha_g|_e(Y)\big) \\
&= \beta|_e\big(d\lambda_g|_{g^{-1}} d\rho_g|_e(X), d\lambda_g|_{g^{-1}} d\rho_g|_e(Y)\big) \\
&= \beta|_{g^{-1}}\big(d\rho_g|_e(X), d\rho_g|_e(Y)\big) \\
&= \beta|_e\big(X, Y\big) \\
&= \beta_0(X, Y)
\end{aligned}$$

for all $X, Y \in \mathfrak{g}$. $\square$

Theorem 6.18. *Every compact Lie group admits a bi-invariant Riemannian metric.*

Proof. We use that any compact Lie group has a finite right invariant measure, i.e., a measure μ with $\mu(G) < \infty$ and such that

$$\int_G \rho_g^* f(h) d\mu(h) = \int_G f(h) d\mu(h)$$

for all $f \in C^\infty(G)$ and all $g \in G$ (this can be obtained, for example, by extending a density at $e \in G$ by right invariance to all of G). Now given any positive definite inner product β_0 on G, we define

$$\beta(X,Y) = \int_G \beta_0\big(\mathrm{Ad}_h(X), \mathrm{Ad}_h(Y)\big) d\mu(h), \qquad X, Y \in \mathfrak{g}$$

Then

$$\beta\big(\mathrm{Ad}_g(X), \mathrm{Ad}_g(Y)\big) = \int_G \beta_0\big(\mathrm{Ad}_{hg}(X), \mathrm{Ad}_{hg}(Y)\big) d\mu(h),$$

which coincides with $\beta(X, Y)$ by right invariance of μ. $\square$

Example 6.19. Ad-invariant, non-degenerate inner products on $\mathfrak{gl}(n, \mathbb{R})$ and $\mathfrak{gl}(n, \mathbb{C})$ are given by

$$\beta(X, Y) = -\mathrm{tr}(XY), \qquad \text{respectively} \qquad \beta(X, Y) = -\mathrm{Re}\,\mathrm{tr}(XY),$$

inducing a semi-Riemannian metric on $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$. These inner products are positive definite on $\mathfrak{so}(n) \subset \mathfrak{gl}(n, \mathbb{R})$ and $\mathfrak{u}(n) \subset \mathfrak{gl}(n, \mathbb{C})$, hence induce bi-invariant Riemannian metrics $\mathrm{O}(n)$, $\mathrm{SO}(n)$ and $\mathrm{U}(n)$.

Lemma 6.20. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then any* Ad*-invariant, non-degenerate inner product β on $\mathfrak{g}$ satisfies*

$$\beta([X, Y], Z) = \beta(X, [Y, Z]), \qquad X, Y, Z \in \mathfrak{g}.$$

Proof. Let $\xi : (-\varepsilon, \varepsilon) \to G$ be a path with $\dot{\xi}(0) = X$. Then for $Y, Z \in \mathfrak{g}$, the number $\mathrm{Ad}^*_{\xi(t)}\,\beta(Y, Z)$ is independent of t. Hence, using the chain rule and Lemma 6.12, we get

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0} \beta\big(\mathrm{Ad}_{\xi(t)}(Y), \mathrm{Ad}_{\xi(t)}(Z)\big) \\ &= \beta\left(\frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{\xi(t)}(Y), Z\right) + \beta\left(Y, \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{\xi(t)}(Z)\right) \\ &= \beta\big(\mathrm{ad}_X(Y), Z\big) + \beta\big(Y, \mathrm{ad}_X(Z)\big) \\ &= \beta\big([X, Y], Z\big) + \beta\big(Y, [X, Z]\big), \end{aligned}$$

which implies the claim. □

Lemma 6.21. *Let G be a Lie group and let β be a bi-invariant semi-Riemannian metric on G, with Levi-Civita connection ∇. Then for left invariant vector fields $\tilde{X}$, $\tilde{Y}$, we have*

$$\nabla_{\tilde{X}} \tilde{Y} = \frac{1}{2}[\tilde{X}, \tilde{Y}].$$

Proof. By the Koszul formula, we have

$$\begin{aligned} 2\beta(\nabla_X Y, Z) = \partial_X \beta(Y, Z) + \partial_Y \beta(X, Z) - \partial_Z \beta(X, Y) \\ + \beta([X, Y], Z) - \beta([X, Z], Y) - \beta([Y, Z], X) \end{aligned} \tag{6.3}$$

For all vector fields X, Y, Z on G. If $\tilde{X}, \tilde{Y} \in \mathcal{X}_\lambda(G)$ are left invariant vector fields corresponding to $X, Y \in \mathfrak{g}$, then

$$\beta|_g(\tilde{X}|_g, \tilde{Y}|_g) = \beta|_g(d\lambda_g|_e X, d\lambda_g|_e Y) = \beta|_e(X, Y),$$

so $\beta|_g(\tilde{X}|_g, \tilde{Y}|_g$ is a constant function of g. We obtain that if $\tilde{X}, \tilde{Y}, \tilde{Z}$ are left invariant vector fields, then the first three terms of (6.3) vanish. We therefore get

$$\begin{aligned} 2\beta(\nabla_{\tilde{X}}\tilde{Y}, \tilde{Z}) &= \beta([\tilde{X},\tilde{Y}],\tilde{Z}) - \beta([\tilde{X},\tilde{Z}],\tilde{Y}) - \beta([\tilde{Y},\tilde{Z}],\tilde{X}) \\ &= \beta([\tilde{X},\tilde{Y}],\tilde{Z}) + \beta([\tilde{Z},\tilde{X}],\tilde{Y}) + \beta([\tilde{Z},\tilde{Y}],\tilde{X}) \\ &= \beta([\tilde{X},\tilde{Y}],\tilde{Z}) + \beta(\tilde{Z},[\tilde{X},\tilde{Y}]) + \beta(\tilde{Z},[\tilde{Y},\tilde{X}]) \\ &= \beta([\tilde{X},\tilde{Y}],\tilde{Z}). \end{aligned}$$

where we used Lemma 6.20. Since β is non-degenerate and each tangent space is spanned by the values of left invariant vector fields, this implies the result. □

Theorem 6.22. *Let G be a Lie group with a bi-invariant semi-Riemannian metric. Then for left invariant vector fields $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$, we have*

$$R(\tilde{X},\tilde{Y})\tilde{Z} = -\frac{1}{4}[[\tilde{X},\tilde{Y}],\tilde{Z}].$$

Moreover, we have $\nabla R = 0$, hence G is a locally symmetric space.

Proof. We calculate

$$\begin{aligned} R(\tilde{X},\tilde{Y})\tilde{Z} &= \nabla_{\tilde{X}}\nabla_{\tilde{Y}}\tilde{Z} - \nabla_{\tilde{Y}}\nabla_{\tilde{X}}\tilde{Z} - \nabla_{[\tilde{X},\tilde{Y}]}\tilde{Z} \\ &= \frac{1}{4}[\tilde{X},[\tilde{Y},\tilde{Z}]] - \frac{1}{4}[\tilde{Y},[\tilde{X},\tilde{Z}]] - \frac{1}{2}[[\tilde{X},\tilde{Y}],\tilde{Z}] \\ &= \frac{1}{4}[\tilde{X},[\tilde{Y},\tilde{Z}]] + \frac{1}{4}[\tilde{Y},[\tilde{Z},\tilde{X}]] + \frac{1}{4}[\tilde{Z},[\tilde{X},\tilde{Y}]] - \frac{1}{4}[[\tilde{X},\tilde{Y}],\tilde{Z}]. \end{aligned}$$

The first three terms vanish by the Jacobi identity (6.1).
To see that $\nabla R = 0$, calculate

$$\begin{aligned} (\nabla_{\tilde{X}}R)(\tilde{Y},\tilde{Z})\tilde{W} &= \nabla_{\tilde{X}}\big\{R(\tilde{Y},\tilde{Z})\tilde{W}\big\} - R(\nabla_{\tilde{X}}\tilde{Y},\tilde{Z})\tilde{W} - R(\tilde{Y},\nabla_{\tilde{X}}\tilde{Z})\tilde{W} - R(\tilde{Y},\tilde{Z})\nabla_{\tilde{X}}\tilde{W} \\ &= \frac{1}{8}\Big(-[\tilde{X},[[\tilde{Y},\tilde{Z}],\tilde{W}]] + \underbrace{[[[\tilde{X},\tilde{Y}],\tilde{Z}],\tilde{W}] + [[\tilde{Y},[\tilde{X},\tilde{Z}]],\tilde{W}]}_{=-[[[\tilde{Y},\tilde{Z}],\tilde{X}],\tilde{W}]} + [[\tilde{Y},\tilde{Z}],[\tilde{X},\tilde{W}]]\Big) \\ &= -\frac{1}{8}\Big([[\tilde{W},[\tilde{Y},\tilde{Z}]],\tilde{X}] + [[[\tilde{Y},\tilde{Z}],\tilde{X}],\tilde{W}] + [[\tilde{X},\tilde{W}],[\tilde{Y},\tilde{Z}]]\Big) = 0, \end{aligned}$$

by the Jacobi identity for the three vector fields $\tilde{X}$, $\tilde{W}$ and $[\tilde{Y},\tilde{Z}]$. □

Corollary 6.23. *Let β be a bi-invariant semi-Riemannian metric on a Lie group G. Then the sectional curvature of a non-degenerate 2-plane $E \subset \mathfrak{g}$ spanned by $X, Y \in \mathfrak{g}$ is given by*

$$K(E) = \frac{1}{4}\frac{\beta([X,Y],[X,Y])}{\beta(X,X)\beta(Y,Y) - \beta(X,Y)^2}.$$

In particular, a Riemannian (i.e., positive definite) bi-invariant metric on a Lie group has non-negative sectional curvature.

Proof. By Lemma 6.20, we have

$$\begin{aligned}K(E)\big(\beta(X,X)\beta(Y,Y)-\beta(X,Y)^2\big) &= \beta\big(R(X,Y)Y,X\big)\\ &= -\frac{1}{4}\beta([[X,Y],Y],X)\\ &= -\frac{1}{4}\beta([X,Y],[Y,X])\\ &= \frac{1}{4}\beta([X,Y],[X,Y]).\end{aligned}$$

□

6.3 The exponential map

Theorem 6.24. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. For each $X \in \mathfrak{g}$, there exists a unique Lie group homomorphism $\gamma : \mathbb{R} \to G$ such that $\dot{\gamma}(0) = X$.*

Proof. Let $\tilde{X}$ be the left-invariant vector field corresponding to X.
We start with showing uniqueness. Let $\gamma : \mathbb{R} \to G$ be a group homomorphism with $\dot{\gamma}(0) = X$. Then differentiating the equation $\gamma(t+s) = \gamma(t)\gamma(s)$ at $s = 0$ shows that γ satisfies the ordinary differential equation

$$\dot{\gamma}(t) = d\lambda_{\gamma(t)}|_e \dot{\gamma}(0) = \tilde{X}|_{\gamma(t)}. \tag{6.4}$$

By uniqueness of solutions to ordinary differential equations, there is at most one solution to this equation with $\gamma(0) = e$.
To show existence, let $\gamma : (-a, \varepsilon) \to G$ be the solution to the ordinary differential equation (6.4) with initial condition $\gamma(0) = e$, where $a > 0$ is chosen maximal. Suppose that $a < \infty$. Then for $0 < s < \varepsilon$, set $\tilde{\gamma}(t) = \gamma(s)^{-1}\gamma(t+s) = \lambda_{\gamma(s)^{-1}}(\gamma(t+s))$. Then $\tilde{\gamma}(0) = e$ and since $\tilde{X}$ is left invariant,

$$\dot{\tilde{\gamma}}(t) = d\lambda_{\gamma(s)^{-1}}|_{\gamma(t+s)}(\tilde{X}|_{\gamma(t+s)}) = \tilde{X}|_{\gamma(s)^{-1}\gamma(t+s)} = \tilde{X}|_{\tilde{\gamma}(t)}.$$

So $\tilde{\gamma}$ satisfies the same ordinary differential equation (6.4) as γ, with the same initial condition. However, $\tilde{\gamma}$ is defined on the interval $(-a-s, \varepsilon - s)$, a contradiction to the assumption that a is maximal. We obtain that γ is defined for all negative times. Similarly, one shows that γ is defined for all positive times.
We now show that γ is a group homomorphism. Observe that for each $s \in \mathbb{R}$, the path $\tilde{\gamma}(t) := \gamma(t+s)$ satisfies (6.4) with the initial condition $\tilde{\gamma}(0) = \gamma(s)$. But the path $\tilde{\tilde{\gamma}}(t) := \gamma(t)\gamma(s)$ also satisfies (6.4) with the same initial condition. Hence $\tilde{\tilde{\gamma}} = \tilde{\gamma}$, so $\gamma(t+s) = \gamma(t)\gamma(s)$ for all $s, t \in \mathbb{R}$. □

By the above lemma, we can make the following definition.

Definition 6.25 (Exponential map). Let G be a Lie group with Lie algebra $\mathfrak{g}$. The *exponential map* of G is the map

$$\exp : \mathfrak{g} \longrightarrow G$$

with the property that $\exp(X) = \gamma(1)$, where $\gamma : \mathbb{R} \to G$ is the unique Lie group homomorphism with $\dot{\gamma}(0) = X$.

Example 6.26. If $G \subset \mathrm{GL}(V)$ is a closed subgroup, then the exponential map of G is given by the usual matrix exponential,

$$\exp(g) = \sum_{n=0}^{\infty} \frac{g^n}{n!}$$

Theorem 6.27. *Let G be a Lie group and let β be a bi-invariant semi-Riemannian metric on G. Then each Lie group homomorphism $\gamma : \mathbb{R} \to G$ such that $\dot{\gamma}(0) = X$ is a geodesic for β. In particular, the Lie group exponential map coincides with the semi-Riemannian exponential map at the unit element.*

Proof. Let $\gamma(t) = \exp(tX)$ and let $\tilde{X}$ be the left invariant vector field corresponding to X. By definition, γ is an integral curve to $\tilde{X}$, so the result follows from the fact that

$$\frac{\nabla}{dt}\dot{\gamma}(t) = \nabla_{\tilde{X}}\tilde{X}|_{\gamma(t)} = \frac{1}{2}[\tilde{X}, \tilde{X}]|_{\gamma(t)} = 0.$$

Here we used Lemma 6.21 and the fact that the Lie bracket is skew-symmetric. □

6.4 Symmetric spaces

Definition 6.28 (Symmetric space). Let (M, g) be a connected semi-Riemannian manifold. A *(global) geodesic reflection* at $p \in M$ is an isometry $\sigma_p : M \to M$ such that $\sigma_p(p) = p$ and such that $d\sigma_p|_p = -\mathrm{id}_{T_pM}$. (M, g) is called *symmetric space* if for every $p \in M$, there exists a global geodesic reflection at p.

Symmetric spaces are closely related to *homogeneous spaces*. Recall that an *action* of a Lie group G on a manifold M is a smooth map $G \times M \to M$, $(g, p) \mapsto g \cdot p$, satisfying

$$g \cdot (h \cdot p) = gh \cdot p \qquad \text{and} \qquad e \cdot p = p, \qquad g, h \in G, \quad p \in M.$$

Such an action is called *transitive* if for any two points $p, q \in M$, there exists $g \in G$ such that $g \cdot p = q$. It is called *free* if the map $G \times M \to M \times M$ is injective, and it is called *proper* if the same map is proper (i.e., inverse images of compact sets are compact). The *quotient manifold theorem* states that if a Lie group G acts freely and properly on a manifold M, then the orbit space M/G is a topological manifold of dimension $\dim(M) - \dim(G)$ and has a unique smooth structure turning the projection map $M \to M/G$ into a surjective submersion.

Definition 6.29 (Homogeneous space). A *homogeneous space* is a manifold M together with a transitive action of a Lie group G.

If G is a Lie group and K is a closed subgroup of G, then the action of K on G by multiplication is always free and proper, so the quotient space G/K is a smooth manifold with a free and transitive action of G.

Conversely, given a homogenous space M, we can pick a point $p \in M$, and denote the corresponding *stabilizer subgroup* by

$$K := K_p := \{g \in G \mid g \cdot p = p\} \subseteq G.$$

Then the smooth map $G \to M$, $g \mapsto g \cdot p$ descends to a diffeomorphism $G/K \to M$.

Theorem 6.30. *Every symmetric space is homogenous.*

Proof. Let (M, g) be a semi-Riemannian symmetric space. Then

$$G := \mathrm{Isom}(M, g).$$

is a Lie group, see Example 6.4. We claim that G acts transitively on M. To see this, we use that, given $p, q \in M$, there exists a *broken geodesic* connecting p and q, i.e., a continuous path $c : [0, r] \to M$ together with a time partition $0 = t_0 < t_1 < \cdots < t_N = r$ such that c is a geodesic on each of the subintervals $[t_{j-1}, t_j]$ (in the case that M is Riemannian and complete, p and q can even be connected by a geodesic, but we do not want to make this assumption). For $j = 1, \ldots, N$, let $\sigma_j \in G$ be the geodesic reflection at $c(\frac{t_{j-1}+t_j}{2})$. Then σ_j exchanges $c(t_{j-1})$ and $c(t_j)$, hence the composition $\sigma_N \circ \cdots \circ \sigma_1 \in G$ sends p to q. $\square$

Fundamental Example 6.31. Let G be a Lie group and let $\sigma : G \to G$ be an involutive automorphism of G, i.e., a smooth map such that $\sigma(gh) = \sigma(g)\sigma(h)$ and $\sigma(\sigma(g)) = g$. Let

$$K \subseteq \mathrm{Fix}(\sigma)$$

be an open subgroup of the fixed point set of σ (i.e., a disjoint union of connected components). Moreover, define $\tau : G \to G$ by $\tau(g) = \sigma(g^{-1})$ and let

$$M = \mathrm{Fix}(\tau)_0$$

be the identity component of the fixed point set of τ.
A G-action on M is defined by the formula

$$g \bullet p = gp\sigma(g)^{-1}.$$

There is a smooth map $\pi : G \to M$ defined by $\pi(g) = g\sigma(g)^{-1}$. One can check that the image of this map is open and closed in M, hence all of M since M is connected. As σ fixes K, we have $\pi(g) = \pi(gk)$ for $k \in K$, hence π descends to a surjective map $G/K \to M$. It is injective if and only if $K = \mathrm{Fix}(\sigma)$. Indeed, if

$$g\sigma(g)^{-1} = h\sigma(h)^{-1} \qquad \Longleftrightarrow \qquad gh^{-1} = \sigma(gh^{-1}),$$

hence $gh^{-1} \in \mathrm{Fix}(\sigma)$. Conversely, if $g \in \mathrm{Fix}(\sigma) \setminus K$, then $\pi(g) = \pi(e)$, so the quotient map is not injective. We obtain that if $K = \mathrm{Fix}(\sigma)$, then π descends to a diffeomorphism

between M and the homogeneous space G/K, while in general, it descends to a covering map with fiber $\mathrm{Fix}(\sigma)/K$.
Assume that G carries a bi-invariant semi-Riemannian metric β for which σ (and hence also τ) are isometries, and such that β is non-degenerate when restricted to M. Then M is a totally geodesic semi-Riemannian submanifold of (G, β). We claim that for $p \in M$, a geodesic symmetry at p is given by $\sigma_p := \lambda_p \circ i \circ \lambda_{p^{-1}}$, where i is the inversion map of G. On elements of M, this boils down to

$$\sigma_p(q) = pq^{-1}p, \qquad q \in M.$$

Indeed, σ_p clearly fixes p, is an isometry (since left translations and τ are), preserves M by the calculation

$$\tau(pq^{-1}p) = \sigma(p^{-1}qp^{-1}) = \sigma(p^{-1})\sigma(q)\sigma(p^{-1}) = pq^{-1}p$$

and satisfies

$$d\sigma_p|_p = d\lambda_p|_e \underbrace{di|_e}_{-1} d\lambda_{p^{-1}}|_p = -d\lambda_p|_e d\lambda_{p^{-1}}|_p = -d(\lambda_p \circ \lambda_{p^{-1}})|_p = -\mathrm{id}_{T_pM}.$$

Since $d\tau|_e = -d\sigma|_e$, we obtain a β-orthogonal direct sum decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$$

of the Lie algebra $\mathfrak{g}$ of G into the Lie algebra $\mathfrak{k} = \{X \in \mathfrak{g} \mid d\sigma|_e(X) = X\}$ of K and the tangent space $\mathfrak{m} = T_eM$ to M at e. Since M is totally geodesic, the its curvature tensor is just the restriction of the curvature tensor of G to M, in other words,

$$R^M(X,Y)Z = -\frac{1}{4}[[X,Y],Z], \qquad X,Y,Z \in \mathfrak{m}.$$

Special Case 6.32 (Complex projective space). Set $G = \mathrm{U}(n+1)$ and σ be the automorphism of G given by conjugation with the matrix

$$\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}. \tag{6.5}$$

Then $K := \mathrm{Fix}(\sigma) = \mathrm{U}(1) \times \mathrm{U}(n) \subset \mathrm{U}(n+1)$ and $M \cong \mathrm{U}(n+1)/(\mathrm{U}(1) \times \mathrm{U}(n))$. M is diffeomorphic to $\mathbb{C}P^n$, via the map that sends a matrix $U \in \mathrm{U}(n+1)$ onto the first row vector. We have

$$\mathfrak{m} = \{\widehat{x} \mid x \in \mathbb{C}^n\},$$

where for $z \in \mathbb{C}^n$, we set

$$\widehat{x} := \begin{pmatrix} 0 & -\overline{x}^t \\ x & 0 \end{pmatrix} \in \mathfrak{u}(n+1) \subset \mathrm{Mat}(n+1, \mathbb{C}).$$

Consider the bi-invariant semi-Riemannian metric corresponding to the bilinear form

$$\beta(X,Y) = -\frac{1}{8}\cdot \operatorname{Re}\operatorname{tr}(XY)$$

on $\mathfrak{u}(n+1)$ (the conventional factor of $\frac{1}{8}$ is there to make the results look nice later on). Then the restriction $g = \beta|_{\mathfrak{m}}$ of this metric is in terms of these elements given by

$$g(\widehat{x},\widehat{y}) = -\frac{1}{8}\cdot \operatorname{Re}\operatorname{tr}\begin{pmatrix} -\overline{x}^t y & 0 \\ 0 & -x\overline{y}^t \end{pmatrix} = \frac{1}{4}\cdot \operatorname{Re}\langle x,y\rangle.$$

To determine the curvature, we calculate

$$[\widehat{x},\widehat{y}] = \begin{pmatrix} -2i\operatorname{Im}\langle x,y\rangle & 0 \\ 0 & -(x\overline{y}^t - y\overline{x}^t) \end{pmatrix},$$

where the brackets denote the standard Hermitean form on $\mathbb{C}^n$, which we take to be anti-linear in the *first* component. Hence

$$\begin{aligned} 4R(\widehat{x},\widehat{y})\widehat{z} &= -[[\widehat{x},\widehat{y}],\widehat{z}] \\ &= -\left(\begin{matrix} 0 & 2i\operatorname{Im}\langle x,y\rangle\cdot \overline{z}^t \\ -(x\overline{y}^t - y\overline{x}^t)z & 0 \end{matrix}\right) + \left(\begin{matrix} 0 & \overline{z}^t(x\overline{y}^t - y\overline{x}^t) \\ -2i\operatorname{Im}\langle x,y\rangle\cdot z & 0 \end{matrix}\right) \\ &= \big[\langle y,z\rangle\cdot x - \langle x,z\rangle\cdot y - 2i\operatorname{Im}\langle x,y\rangle\cdot z\big]^{\widehat{}} \end{aligned}$$

This can be written in a more invariant form (not using the Hermitean scalar product but the bilinear form β) as follows. Denote

$$j = \begin{pmatrix} i & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \in \mathrm{U}(n+1)$$

and write $J = \operatorname{Ad}_j$. We have $J(\widehat{x}) = \widehat{ix}$, hence J is an endomorphism of $\mathfrak{m}$ that satisfies $J^2 = -\mathrm{id}$ and leaves g invariant. In other words, J is a *complex structure* for the vector space $\mathfrak{m}$. Using J, we can express $\langle\,,\rangle$ through g. The formula is

$$\langle x,y\rangle = 4\,g(\widehat{x},\widehat{y}) + 4i\,g(J\widehat{x},\widehat{y}),$$

so

$$\begin{aligned} \big[\langle y,z\rangle\cdot x\big]^{\widehat{}} &= 4\,g(\widehat{y},\widehat{z})\widehat{x} + 4\,g(J\widehat{y},\widehat{z})J\widehat{x} \\ \big[\langle x,z\rangle\cdot y\big]^{\widehat{}} &= 4\,g(\widehat{x},\widehat{z})\widehat{y} + 4\,g(J\widehat{x},\widehat{z})J\widehat{y} \\ \big[2i\operatorname{Im}\langle x,y\rangle\cdot z\big]^{\widehat{}} &= 8\,g(J\widehat{x},\widehat{y})J\widehat{z}. \end{aligned}$$

Putting together, we obtain

$$R^M(X,Y)Z = g(Y,Z)X + g(JY,Z)JX - g(X,Z)Y - g(JX,Z)JY - 2g(JX,Y)JZ.$$

If X and Y are orthonormal, spanning a plane $E \subset \mathfrak{m}$, we get for the sectional curvature

$$K^M(E) = 1 + 3g(JX, Y)^2.$$

Hence the sectional curvature lies between 1 and 4. The maximal value of 4 is taken on 2-planes E that are invariant under J (hence spanned by X and $Y = JX$).

Special Case 6.33 (Sphere). Let $G = \mathrm{SO}(n+1)$ and let σ be conjugation by the special matrix (6.5). For $K = \{1\} \times \mathrm{SO}(n) \subset \mathrm{Fix}(\sigma)$, the quotient G/K is identified with S^n, while for $K = \mathrm{Fix}(\sigma) = (\mathrm{O}(1) \times O(n)) \cap \mathrm{SO}(n+1)$, the quotient is identified with $\mathbb{R}P^n$. The manifold M from Example 6.31 is $\mathbb{R}P^n$ in both cases.