Seminar on Lie groups, Lie algebras and their representations

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General

The seminar will be on Wednesday 16-18 during the winter term 2020/21. It probably will be organized as a digital seminar.

We assume the following:

- The teaching period starts on Nov 2nd, 2020
- The teaching period ends on Feb 12th, 2021

Then we have the dates 4.11., 11.11., 18.11., 25.11., 2.12., 9.12., 16.12., 23.12., 13.1., 20.1., 27.1., 3.2., 10.2.

1 Introduction

Lie groups are important to describe symmetries, both in mathematics and in applications (physics, chemistry, engineering,...). The classical Lie groups are for example the orthogonal groups $O(n)$, the unitary groups $U(n)$, but mathematicians and physicists are also fascinated by more exotic examples such as the symmetry group of the octonions which is discussed a lot in modern mathematical physics. Many of these Lie groups can be represented as subgroups of $Gl(k,\mathbb{R})$ for some sufficiently large k, but there are also Lie groups which cannot. Lie groups are manifolds G together with a multiplication $\mu : G \times G \to G$ which is a smooth map, such that (G,μ) is a group.[1]

Lie groups and their representation is a mighty theory which allows effect calculations both for problems inside mathematics and also for applications outside.

2 Talks

2.1 Basic Theory

The main reference for this part is [9]. Note that in this source a *submanifold* of a manifold M is defined as an injective immersion of a manifold N into M which is weaker than the usual definition, but more appropriate in the theory of Lie groups. One of the main aims of this part is to explain the correspondence of Lie groups and Lie algebras, a short overview of which can be found in [4, Ch. 3.8].

[1]One also assumes that $g \mapsto g^{-1}$ is smooth.

Talk No. 1: Distributions and Frobenius theorem *(N.N.)*. Introduce distributions, then state and prove the Frobenius theorem (Theorem 1.60) with its maximal version (Theorem 1.64) as in [9, 1.56-1.64]. It will be necessary to treat the one-dimensional case (Theorem 1.48) first. You should also shortly explain the notion of a submanifold of a manifold and some of its properties [9, 1.27-1.36], cf. also the comment above.

Talk No. 2: Basic defintions and first examples *(N.N.)*. Introduce Lie groups and Lie algebras and Lie subgroups as in [9, 3.1-3.10 & 3.17-3.21]. Discuss the examples given, in particular $GL(n, R)$, where the Lie bracket should be calculated. Also, give examples of classical subgroups of $GL(n, R)$ such as $SO(n)$ or $U(n)$, cf. [9, 3.37].

Talk No. 3: Differential ideals and homomorphisms of Lie groups *(N.N.)*. Following [9, 2.24-2.34], introduce the Lie derivative, differential ideals and explain, how this can be used to obtain a different formulation of the Frobenius theorem. In the second part of the talk, apply this to homomorphisms of Lie groups [9, 3.11-3.16].

Talk No. 4: Simply connected Lie groups and the exponential map *(N.N.)*. This talk should cover the material in [9, 3.22-3.34]: Give a short introduction to covering space theory. Although everything relevant is contained in [9, 3.22 & 3.23], it might be useful to consult a book on algebraic topology, such as [2], to gain some more intuition. Establish the correspondence between Lie algebras and simply connected Lie groups. Introduce the exponential map. You can leave out [9, 3.37], as this was already covered by the previous talk. Also discuss the sections on continuous homomorphisms and closed subgroups, where the exponential map is used as a major tool.

Talk No. 5: Adjoint representation and homogeneous spaces *(N.N.)*. The first part of the talk should deal with the adjoint representation and automorphisms as in [9, 3.44-3.57]. Also have a short look on [5, Ch. 1.2], which will be dealt with in more detail in talk 7. However, make sure to cover the contents of [5, Ch 1.11] (except for Prop. 1.98) or [6, Ch. 1.14]. In the remaining time, discuss the main results of the section on homogeneous spaces [9, 3.58-3.68]. Here, you may leave out some of the proofs if necessary.

2.2 Structure and representation theory for Lie groups

This part is devoted to a few results about the structure of Lie groups and Lie algebras, which is closely tied to their representations. The basic reference for this part is [5]. Note, that the second edition [6] has a slightly different numbering.

Talk No. 6: Lie's theorem and Engel's theorem *(N.N.)*. Introduce solvable, nilpotent and (semi-)simple Lie algebras and prove Lie's theorem and Engel's theorem as in [5, Ch. 1.1-1.6]. You may always restrict your attention to the case of real or complex base field. Of course, you don't have to define those things again that have already seen in this seminar before, e. g. Lie (sub-)algebras, ideals or the adjoint representation. Furthermore, you can leave out the complexification and the Killing form. However, you should cover semidirect products and some of the examples.

Talk No. 7: Cartan's criterion for semisimplicity *(N.N.).* Prove Cartan's criterion for semisimplicity and its consequences as in [5, Ch. 1.7] (a further consequence is [5, Prop 1.98]). You will need to define Killing form and complexification first, see [5, Ch. 1.3], but try to be short on the complexification. Throughout, you can restrict to Lie algebras over the real and complex numbers. Also discuss some of the examples.

Talk No. 8: Nilpotent Lie groups *(N.N.).* Define semidirect products of Lie groups and prove the structural results on nilpotent and solvable Lie groups given in [5, Ch. 1.12-1.13] or [6, Ch. 1.15-1.16]. It might be necessary to state some of the results from [5, Ch. 1.10-1.11] or [6, Ch. 1.10 & 1.14], most of which should have been covered by the first part of the seminar. If time permits, present a proof of [9, Ch. 3 Ex. 18], where abelian Lie groups are "classified".

Talk No. 9: Representations of $\mathfrak{sl}(2,\mathbb{C})$ and examples of simple Lie algebras *(N.N.).* Discuss representations of $\mathfrak{sl}(2,\mathbb{C})$, cf. [5, Ch. 1.9]. Note that Schur's lemma (Lemma 1.66) holds in greater generality [5, Proposition 4.8]. Then present the examples in [5, Ch. 2.1] showing that certain classical Lie algebras are simple. If you are interested in how these two parts are connected later, you may take a look at the discussion around [5, (2.27)].

Talk No. 10: Cartan subalgebras *(N.N.).* Introduce Cartan subalgebras, prove existence, uniqueness and their main properties. The material can be found in [5, 4.2-4.3]. Also have a look at [5, (2.58)] to get an idea of the overall strategy in the remaining part of the seminar.

Talk No. 11: Roots *(N.N.).* Define root systems and prove (as time permits) the concerning statements in [5, 4.4-4.5] until (including) [5, Proposition 4.28]. Despite the technical proofs, make sure your audience gets the idea of what is going on from diagrams similar to [5, Figures 2.1 & 2.2].

Talk No. 12: Cartan matrix, Dynkin diagrams and Weyl group *(N.N.).* Introduce an ordering on abstract root systems and explain how this leads to a Cartan matrix and Dynkin diagrams. Then explain how the Weyl group can be used to show that the Dynkin diagram is essentially independent of the choices involved. You can follow [5, 4.5-4.6], starting with the paragraph following the proof of [5, Proposition 4.28].

Talk No. 13: Classification of Dynkin diagrams and complex semisimple Lie algebras *(N.N.).* The plan for this talk is to classify Dynkin diagrams following [5, 4.7]. Mention, how this leads to a classification of simply connected complex semisimple Lie groups, although not all the steps of the "existence part" have been carried through. As this is the last talk, you may also vary the contents a bit, explaining more on the surjectivity of the Dynkin diagram construction [5, 4.8-4.11], while being shorter on the classification.

References

[1] A. Arvanitoyeorgos, *An Introduction to Lie groups and the geometry of homogeneous spaces*, 2003.

[2] A. Hatcher, *Algebraic Topology*, 2001.

[3] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, 1972.

[4] A. Kirillov, *An introduction to Lie algebras and Lie groups*, 2008.

[5] A. Knapp, *Lie groups beyond an introduction*, first edition, 1996.

[6] A. Knapp, *Lie groups beyond an introduction*, second edition, 2002.

[7] W. Miller, *Symmetry groups and their applications*, 1972.

[8] B. O'Neill, *Semi-Riemannian Geometry*, 1983.

[9] F. W. Warner, *Foundations of differentiable manifolds and Lie groups*, 1983.