Nerves of Hopf algebras and Chern-Simons.

Based on joint work
with Pavol Ševera
arxiv: 1906.10616

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Motivation and context: Goal for today is:

- <u>Thm</u>: There is an equivalence of categories

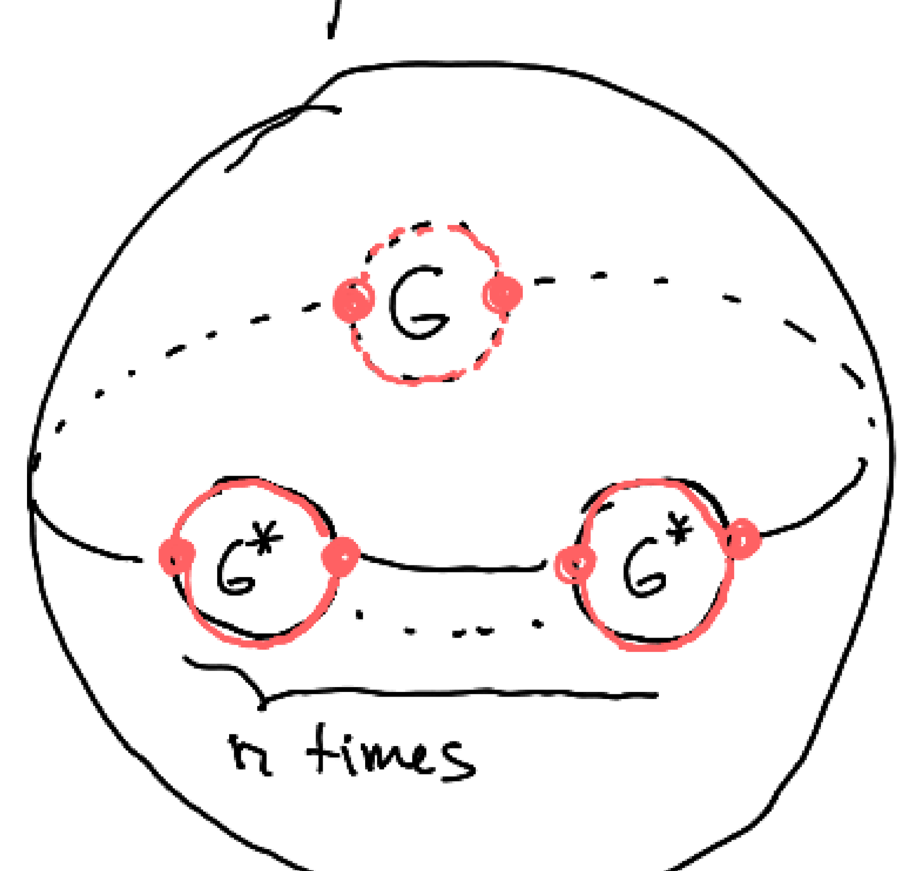
$$\underset{\substack{\uparrow \\ \mathrm{BMC}}}{\mathrm{HopfAlg}(\mathcal{C})_{S^1}} \xrightarrow{\text{Nerve } N_H} \mathrm{Br\,LaxMonFunc}\big(\underset{\substack{\cup \quad\ \cup \\ \Delta_{aug} \quad \mathrm{Braid}_n}}{\mathrm{BrCom}}, \mathcal{C}\big)_{\substack{\text{Segal}\\ \text{nerve}\\ \text{cond.}}} .$$

- Why talk about it at this conference?
  The nerve $N_H : \mathrm{BrCom} \longrightarrow \mathcal{C}$ should be seen as a part (sector) of a 3,2,1 (defect?) TFT, assigned to the Drinfeld double of H [based on Ševera's 1401.6164 Sec.6, 1106.1366, 0811136]

- Remark: This is at the moment an imprecise motivation, not a statement about field theories. Nevertheless:

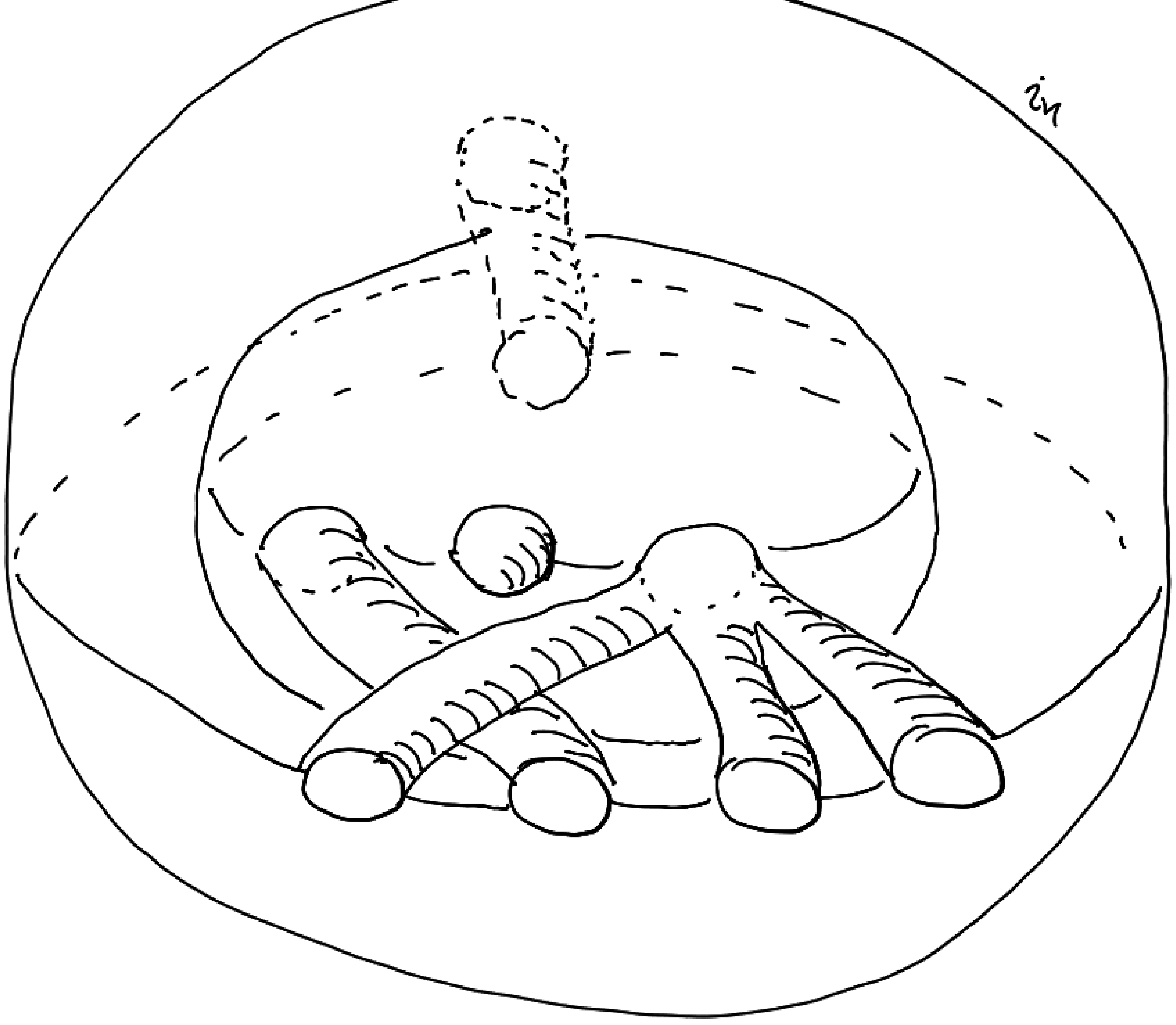
- Let $\mathfrak{g}$ be a Lie bialgebra, i.e. $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$ is a quadratic Lie algebra with $\mathfrak{g}, \mathfrak{g}^*$ Lie subalgebras. $(H = \mathcal{O}_q(G))$

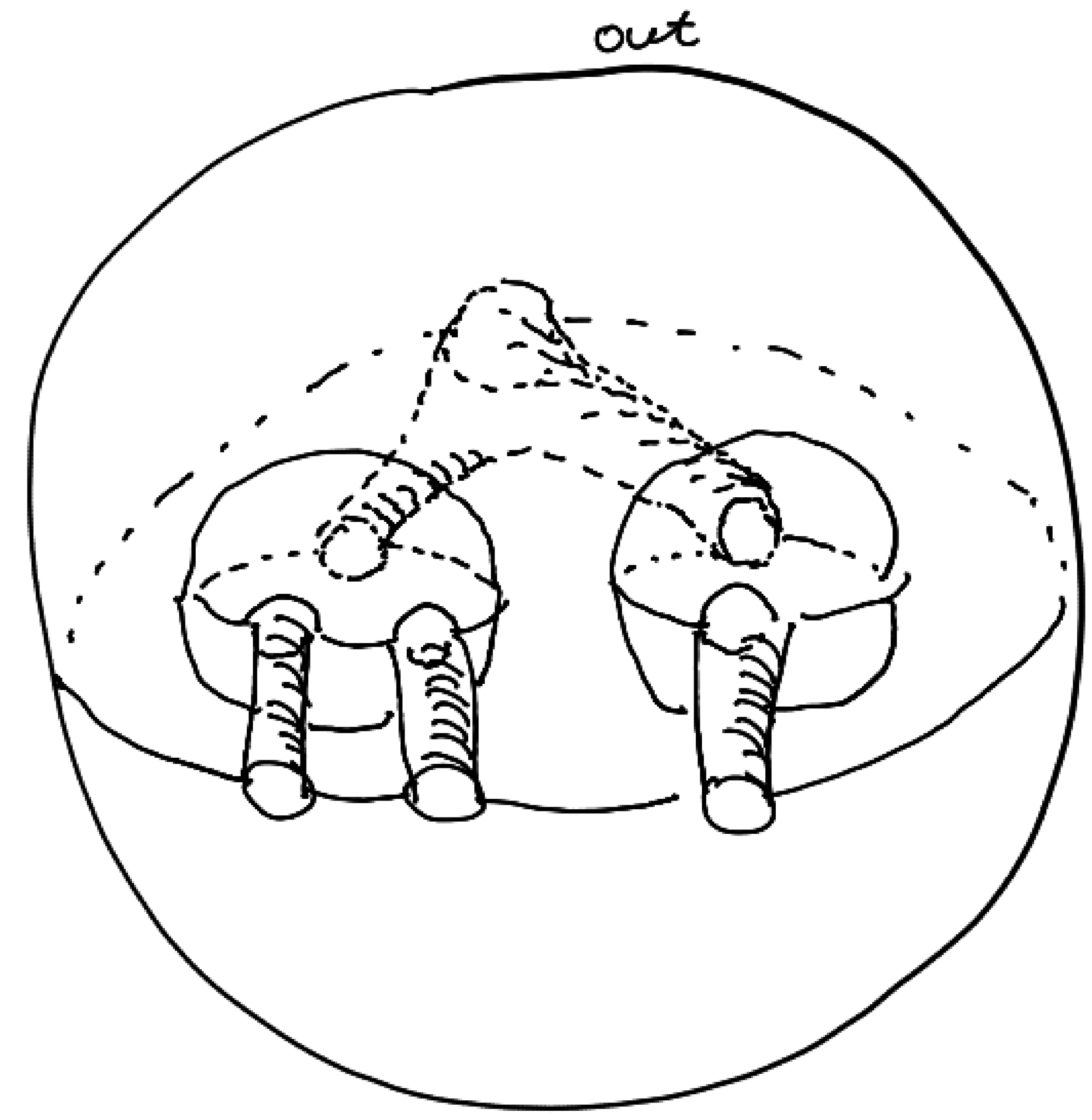
- Consider, for each $n \in \mathbb{N}$, a sphere with $n+1$ punctures and boundary conditions (vague) for the CS theory for associated with D:



(i.e. classically: fields are flat connections on this $\Sigma \times I$ w/ holonomy / gauge tr. given by G or $G^* \subset D$)

and, moreover, consider bordisms:



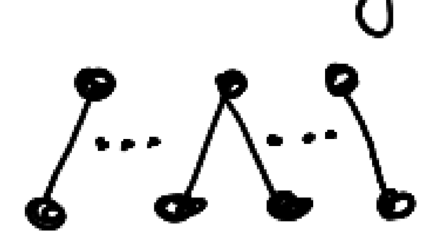
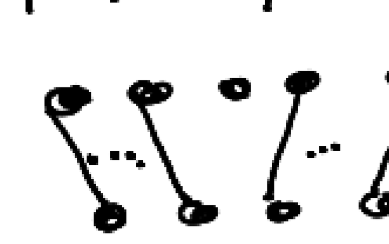
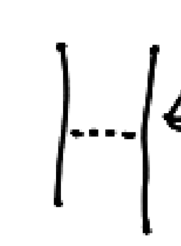
,


Quantum Chern-Simons should assign:

- $N(\underbrace{\bullet\cdots\bullet}_{n})$ the (Hilbert) space associated to the n+1-punctured sphere
- $N($  $) : N(\bullet^4) \longrightarrow N(\bullet^3)$
- $\underbrace{N(\bullet^n)\otimes N(\bullet^m)}_{\text{the two inner spheres}} \longrightarrow N(\bullet^{n+m})$

to the bordisms above

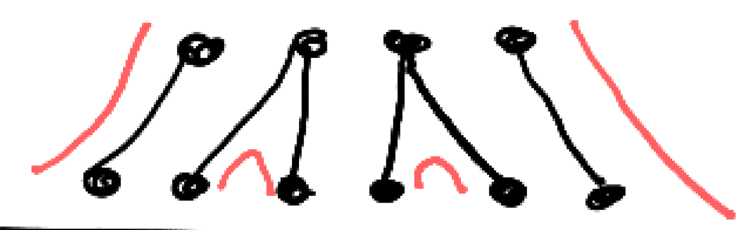
Def: BrCom is the braided monoidal category generated by a single object $\bullet$ and morphisms  , $\emptyset$ modulo: $\bullet$ is a braided commutative algebra ( $)_\Delta$.

- $\Delta_{aug} \subset BrCom$ by $\{0,..,n\} \mapsto \bullet^{n+1}$, $\sigma \mapsto$ , $\delta \mapsto$ 
- "classical" version: $Com = FinSet$, "semiclassical" version: $iCom : Com +$  chord

⇝ which part of the TFT we remember this way?
(lucky coincidence)

Thm [–Ševera '19] $\mathcal{C}$ be a braided monoidal category. Then the category of Hopf algebras with an invertible antipode in $\mathcal{C}$ is equivalent to the category of braided, lax monoidal functors $N: BrCom \longrightarrow \mathcal{C}$ s.t. the

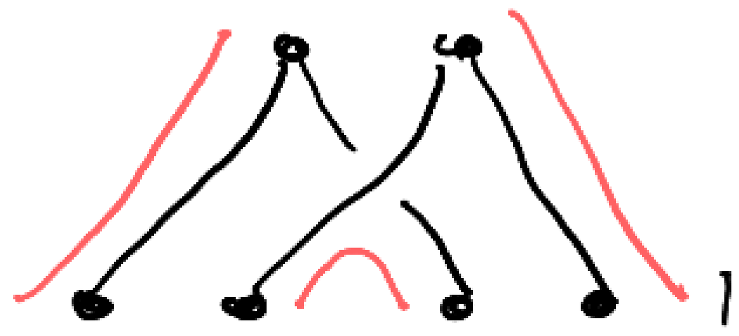
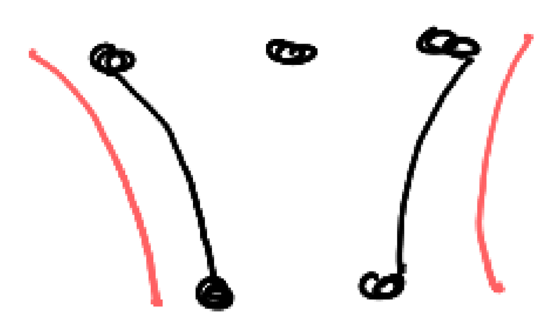
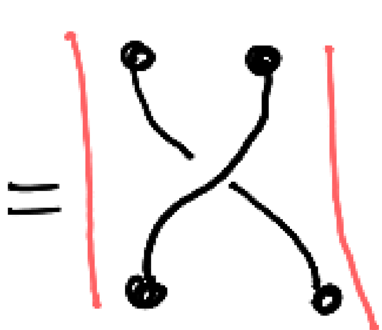
Segal/nerve condition holds:

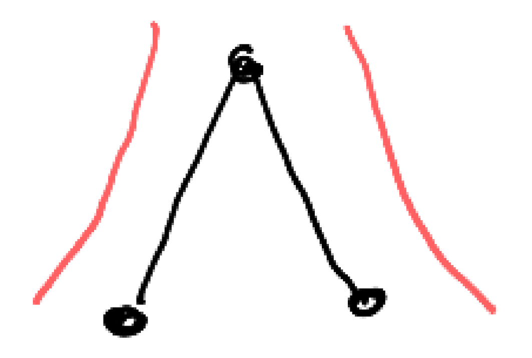


$$N(\bullet\bullet)^{\otimes n} \xrightarrow{\cong} N(\bullet^{n+1})$$

$$N(\bullet) \cong N(\emptyset) \cong I_{\mathcal{C}}$$

Given such N, the Hopf algebra is defined by $H := N(\bullet\bullet)$ and:

product = , coproduct =  $\cong N(\bullet\bullet)^{\otimes 2}$, antipode = .

unit = , counit = ,

Remarks

- the restriction of N_H to $\Delta \subset BrCom$: the cosimplicial nerve of the augmented coalgebra H.
- main application: $Com \xrightarrow[\text{nerve}]{\text{s. lax}} \mathcal{C}$ $\longleftrightarrow$ commutative Hopf algebras

  $iCom \xrightarrow[\text{nerve}]{\text{i. br. lax}} \mathcal{C}$ $\longleftrightarrow$ —— " —— with a compat. Poisson bracket

  a Drinfeld associator Φ directly lifts this to a functor $BrCom \to \mathcal{C}_{\hbar}$, get a quantization of the Poisson-Hopf algebra
- Higher groups: relaxing the nerve condition: functors

  $Com \xrightarrow{\text{s. lax}} Vect$ are (functions on) higher groups & groupoids

  $BrCom \xrightarrow{\text{br lax}} Vect$: higher Hopf algebras?

Thank you for your attention.