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Exercises C^*-algebras and K-theory
Sheet 9

Exercise 1. Let A be a C^*-algebra. Show that if f and g are projections in $(SA)^+$, then the projection $f \oplus g \in M_2((SA)^+)$ is equivalent to the projection $(f * g) \oplus \mathbf{1} \in M_2((SA)^+)$, the *concatenation* of f and g, defined by

$$(f * g)(t) = \begin{cases} f(2t) & t \in [0, \frac{1}{2}] \\ g(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}.$$

Exercise 2. Let A be a C^*-algebra and let $\rho : SA \to SA$ be the $*$-homomorphism defined by $\rho(f)(t) = f(1-t)$. Show that $K_0(\rho) : K_0(SA) \to K_0(SA)$ acts by multplication with -1 (in other words, by sending an element of $K_0(SA)$ to its inverse).

Exercise 3. Identifying $S^n A \otimes S^m B \cong S^{n+m}(A \otimes B)$, we obtain an exterior product map

$$\times : K_0(S^n A) \times K_0(S^m B) \longrightarrow K_0(S^{n+m}(A \otimes B)).$$

Prove that this product map is *graded commutative*, meaning that for all $x \in K_0(S^n A)$, $y \in K_0(S^m B)$, we have

$$y \times x = (-1)^{nm} K_0(S^{n+m}\sigma)(x \times y),$$

where $\sigma : A \otimes B \to B \otimes A$, $a \otimes b \mapsto b \otimes a$, is the flip $*$-homomorphism.

Exercise 4. Show that for any C^*-algebra A and closed ideal J, the sequence

$$\begin{array}{ccccccccc} \cdots \longrightarrow & K_0(A) & \longrightarrow & K_0(A/J) & \dashrightarrow & K_0(SJ) & \longrightarrow & K_0(SA) & \longrightarrow \dots \\ & & & \downarrow & \nearrow & & & & \\ & & & K_0(S^2(A/J)) & & & & & \end{array}$$

is exact, where the dashed arrow is the composition of the boundary map and the periodicity map.

Exercise 5. Show that for the short exact sequence $0 \to \mathbb{K} \to \mathcal{T}_0 \to S\mathbb{C} \to 0$, the map $S\delta \circ \beta_{S\mathbb{C}} : K_0(S\mathbb{C}) \to K_0(S\mathbb{K})$ is just $K_0(S\lambda)$, the suspension of inclusion $\lambda : \mathbb{C} \to \mathbb{K}$ as rank one operators. Conclude that $K_0(S^n\mathcal{T}_0) = 0$ for all $n \in \mathbb{N}$.