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Exercises C^*-algebras and K-theory
Sheet 8

Exercise 1. Let A be a C^*-algebra and let $J \subset A$ be a closed ideal.

(a) Show that for any projection $f \in M_n(S(A/J)^+$, there exists a projection $\tilde{f} \in M_n(CA^+)$ such that $\pi(\tilde{f})(t) = f(t)$ for all $t \in [0, 1]$.

(b) Show the following formula for the boundary map associated to the pair (J, A): Given $x \in K_0(S(A/J))$, represent $x = [f] - [\mathbf{1}_k]$ for some $f \in M_n(S(A/J)^+)$ and choose a lift $\tilde{f}$ as above. Then $\delta(x) = [\tilde{f}(1)] - [\tilde{f}(0)]$.
Hint: Let i and j be the maps from the definition of the boundary map. Compare $K_0(i)(x)$ and $K_0(j)([\tilde{f}(1)] - [\tilde{f}(0)])$ and construct a homotopy between the relevant projections using $\tilde{f}$.

Exercise 2. Let A, B be C^*-algebras and let $J \subset A$ be a closed ideal.

(a) Show that the following sequence of $*$-algebras is exact:

$$0 \longrightarrow J \otimes_{\mathrm{alg}} B \longrightarrow A \otimes_{\mathrm{alg}} B \longrightarrow A/J \otimes_{\mathrm{alg}} B \longrightarrow 0.$$

(b) Show that $\iota \otimes \mathrm{id}_B : J \otimes B \to A \otimes B$ is injective and that $\pi \otimes \mathrm{id}_B : A \otimes B \to A/J \otimes B$ is surjective.

(c) Show that the inclusion

$$A/J \otimes_{\mathrm{alg}} B \cong (A \otimes_{\mathrm{alg}} B)/(J \otimes_{\mathrm{alg}} B) \longrightarrow (A \otimes B)/(J \otimes B)$$

is injective.

(d) Conclude that there exists a unique C^*-norm ν on $A/J \otimes_{\mathrm{alg}} B$ such that

$$0 \longrightarrow J \otimes B \longrightarrow A \otimes B \longrightarrow A/J \otimes_{\nu} B \longrightarrow 0.$$

is exact.

(e) Varying on the above chain of arguments, show the exactness of

$$0 \longrightarrow J \otimes_{\max} B \longrightarrow A \otimes_{\max} B \longrightarrow A/J \otimes_{\max} B \longrightarrow 0.$$

Let A, B be C^*-algebras. A linear map $\varphi : A \to B$ is called *completely positive* (c. p.), if for every $n \in \mathbb{N}$, the map $M_n(\Phi) : M_n(A) \to M_n(A)$ sends positive elements to positive elements. By *Stinesprings theorem*, any c. p. map $\varphi : A \to \mathbb{B}(H)$ has the form $\varphi(a) = V^*\rho(a)V$ for a $*$-representation $\rho : A \to \mathbb{B}(K)$ and a bounded operator $V : K \to H$.

Exercise 3. Show that if $\varphi : A \to B$ and $\varphi' : A' \to B'$ are completely positive, then $\varphi \otimes_{\mathrm{alg}} \varphi'$ extends to a completely positive map $\varphi \otimes \varphi' : A \otimes A' \to B \otimes B'$.
Hint: Represent B faithfully on a Hilbert space and use Stinespring's theorem.

Exercise 4. Let A, B be C^*-algebras and let $J \subset A$ be a closed ideal. Show that if the projection $A \to A/J$ has a c. p. section $s : A/J \to A$ (that is, a c. p. map such that $\pi \circ s = \mathrm{id}_{A/J}$), then the short sequence

$$0 \longrightarrow J \otimes B \longrightarrow A \otimes B \longrightarrow A/J \otimes B \longrightarrow 0$$

is exact. In other words, the norm ν from Exercise 2(d) is the spatial norm.