Dr. Matthias Ludewig SS 2020
Universität Regensburg

Exercises C^*-algebras and K-theory
Sheet 7

Let H be a Hilbert space and let $T \in \mathbb{B}(H)$. A number $\lambda \in \mathbb{C}$ is in the *essential spectrum* of T, denoted by $\lambda \in \sigma_{\text{ess}}(T) \subseteq \sigma(T)$, if the operator $\lambda - T$ is not a Fredholm operator. Remember here that an operator $T \in \mathbb{B}(H)$ is a *Fredholm operator* if is has closed range and finite-dimensional kernel and cokernel; equivalently (by *Atkinson's theorem*), if it admits a *parametrix*, that is an operator $S \in \mathbb{B}(H)$ such that $TS - \text{id}$, $ST - \text{id} \in \mathbb{K}(H)$; clearly the latter just means that $[T]$ is invertible in the *Calkin algebra* $\mathbb{B}(H)/\mathbb{K}(H)$.

Exercise 1. Prove *Weyl's criterion*: Suppose that for $\lambda \in \mathbb{C}$, there exists a sequence $\varphi_1, \varphi_2, \dots$ in H with no accumulation point such that $\|\varphi_n\| = 1$ for all $n \in \mathbb{N}$, and that $\|T\varphi_n - \lambda\varphi_n\| \to 0$. Then $\lambda \in \sigma_{\text{ess}}(T)$.

Denote by $\hat{\cdot} : C(\mathbb{T}) \to C^*_r(\mathbb{Z}) \subset \mathbb{B}(\ell^2(\mathbb{Z}))$ the inverse Gelfand transform. Given $f \in C(\mathbb{T})$, the *Toeplitz operator* T_f is defined by

$$T_f = V\hat{f}V^*,$$

where $V : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{N})$ is the orthogonal projection. The C^*-subalgebra $\mathcal{T} \subset \mathbb{B}(\ell^2(\mathbb{N}))$ generated by all Toeplitz operators is called the *Toeplitz algebra*.

Exercise 2. Show that the *shift operator* $S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$, given by

$$(S\alpha)_n = \alpha_{n+1} \qquad \text{for} \qquad \alpha = (\alpha_n)_{n\in\mathbb{N}},$$

is contained in $\mathcal{T}$ and that it is in fact a generator, that is, any C^*-subalgebra of $\mathcal{T}$ containing S is equal to $\mathcal{T}$. Compute $\sigma(S)$ and $\sigma_{\text{ess}}(S)$.

Exercise 3. Show that $\mathbb{K} := \mathbb{K}(\ell^2(\mathbb{N})) \subseteq \mathcal{T}$.

Exercise 4. Show that for all $f, g \in C(\mathbb{T})$, we have $T_f T_g - T_{fg} \in \mathbb{K}$ and conclude that for invertible $f \in C(\mathbb{T})$, T_f is a Fredholm operator.

Exercise 5. Show that there exists a unique $*$-homomorphism $\pi : \mathcal{T} \to C(\mathbb{T})$ such that $\pi(T_f) = f$ for all $f \in C(\mathbb{T})$, and that $\ker(\pi) = \mathbb{K}$.
Hint: Observe that $\mathcal{T}/\mathbb{K}$ is a commutative C^*-algebra and that the map $C(\mathbb{T}) \to \mathcal{T}/\mathbb{K}$, $f \mapsto [T_f]$ is a surjective $*$-homomorphism.