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# Exercises $C^*$-algebras and $K$-theory
## Sheet 6

We say that a norm on a $*$-algebra is a $C^*$*-norm* if it is submultiplicative and satisfies the $C^*$-identity. Let $A$, $B$ be $C^*$-algebras. A norm $\nu$ on $A \otimes_{\mathrm{alg}} B$ is a *cross norm* if $\nu(a \otimes b) \leq \|a\| \|b\|$ for all $a \in A$, $b \in B$. The *maximal norm* an $A \otimes_{\mathrm{alg}} B$ is defined by

$$\|x\|_{\max} := \sup\{\nu(x) \mid \nu \text{ a } C^*\text{-norm and a cross norm}\}.$$

The completion $A \otimes_{\max} B$ with respect to the maximal norm is called the *maximal tensor product* of $A$ and $B$.

**Exercise 1.** Let $A$ and $B$ be $C^*$-algebras.

(a) Show that $\|x\|_{\max}$ is finite for each $x \in A \otimes_{\mathrm{alg}} B$.

(b) Show the following universal property of the maximal tensor product: For any $C^*$-algebra $C$ and each $*$-homomorphism $\Phi_0 : A \otimes_{\mathrm{alg}} B \to C$, there exists a unique $*$-homomorphism $\Phi : A \otimes_{\max} B \to C$ extending $\Phi_0$.

(c) Show that there exists a unique surjective $*$-homomorphism $A \otimes_{\max} B \to A \otimes B$ that is the identity on $A \otimes_{\mathrm{alg}} B$.

**Exercise 2.** Let $\mathbb{K} = \mathbb{K}(H)$, compact operators on an infinite-dimensional separable Hilbert space. Show that for any $C^*$-algebra $A$, the maximal and the spatial norm on $\mathbb{K} \otimes_{\mathrm{alg}} A$ coincide, in other words, we have $\mathbb{K} \otimes A = \mathbb{K} \otimes_{\max} A$.
*Hint: Show that also $\mathbb{K} \otimes A$ is the direct limit of the direct system formed by the matrix algebras $M_n(A)$ and their inclusions.*

**Exercise 3.** Let $A$ be a $C^*$-algebra.

(a) Show that the cone $CA = \{f \in C([0,1], A) \mid f(0) = 0\}$ is contractible, that is, $\mathrm{id}_{CA}$ is homotopic to the zero map.

(b) For any $*$-homomorphism $\Phi : A \to B$, the mapping cylinder is $Z_\Phi = \{(a, f) \in A \oplus C([0,1], B) \mid f(1) = \Phi(a)\}$. Show that $\mathrm{pr}_1 : Z_\Phi \to A$, $(a, f) \mapsto a$, is a homotopy equivalence.

**Exercise 4.** Let $X$ be a compact Hausdorff space.

(a) Denote by $\mathcal{V}(X)$ the set of isomorphism classes of finite-dimensional vector bundles over $X$. Show that there is a natural homomorphism of semigroups $\mathcal{V}(X) \cong \mathcal{V}(C(X))$.

(b) Conclude that for any locally compact Hausdorff space, we have $K^0(X) \cong K_0(C_0(X))$.

*Hint: Use the following fact on vector bundles: For each vector bundle $V \to X$, there exists a vector bundle $W \to X$ such that $V \oplus W$ is isomorphic to a trivial bundle.*