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Exercises C^*-algebras and K-theory
Sheet 5

***Exercise 1.** Let $\{A_i, \Phi_{ji}\}_I$ be a direct system of C^*-algebras with direct limit $\{A, \Psi_i\}_I$. Show that the $\{M_n(A), \Psi_i\}_I$ is a direct limit of the direct system $\{M_n(A_i), \Phi_{ji}\}_I$.

***Exercise 2.** Show that for any C^*-algebra A, the spatial tensor product $A \otimes M_n(\mathbb{C})$ is isomorphic to $M_n(A)$.
Hint: The point is to show that $A \otimes_{\mathrm{alg}} M_n(\mathbb{C}) \cong M_n(A)$ is already complete with respect to the spatial norm.

Exercise 3. Let $\{A_i, \Phi_{ji}\}_I$ be a direct system of C^*-algebras with direct limit $\{A, \Phi_i\}_I$ and assume that each of the maps Φ_{ji} is injective. Show that for any C^*-algebra B, the cocone $\{A \otimes B, \Phi_i \otimes \mathrm{id}_B\}_I$ is the direct limit of the direct system $\{A_i \otimes B, \Phi_{ji} \otimes \mathrm{id}_B\}_I$.
Hint: First prove the following property of the spatial tensor product: If $\Phi : A \to A'$ and $\Psi : B \to B'$ are both injective, then so is $\Phi \otimes \Psi : A \otimes A' \to B \otimes B'$.

Exercise 4. Let A be a C^*-algebra and let X be a locally compact Hausdorff space.

(a) Show that the obvious $*$-homomorphism $\Phi : C_0(X) \otimes_{\mathrm{alg}} A \to C_0(X, A)$ has dense image.
Guide: $C_c(X, A) \subseteq C_0(X, A)$ is dense. Given $f \in C_c(X, A)$ with support in a compact K, choose a fine enough finite open cover of K and a subordinate partition of unity to construct an element $T \in C_0(X) \otimes_{\mathrm{alg}} A$ with $\|f - \Phi(T)\|_\infty$ small.

(b) Show that Φ is continuous with respect to the spatial norm. Conclude that $C_0(X) \otimes A \cong C_0(X, A)$.
Hint: For $C_0(X)$, consider the "universal representation" on the Hilbert space $\ell^2(X)$ of functions $\alpha : X \to \mathbb{C}$ with countable support $x_1, x_2, \dots$ such that $\sum_{n=1}^\infty |\alpha(x_n)|^2 < \infty$.

Exercise 5. Compute the direct limit of the direct systems $\{A_n, \Phi_{mn}\}_{\mathbb{N}}$ defined below.

(a) Set $A_n := C([-1, 1])$ and for $n \leq m$, define $*$-homomorphisms $\Phi_{mn} : A_n \to A_m$ by $\Phi(f)(t) = f(2^{n-m}t)$.

(b) Set $A_n := C_0((-1, 1))$. For $n \leq m$, define $*$-homomorphisms $\Phi_{mn} : A_n \to A_m$ by

$$\Phi(f)(t) = \begin{cases} f(2^{m-n}t) & \text{if} |t| \leq 2^{n-m} \\ 0 & \text{otherwise} \end{cases}.$$

Exercise 6. For $n \in \mathbb{N}$, let $A_n := M_{2^n}(\mathbb{C})$. Define embeddings $\Phi_{n+1,n} : A_n \to A_{n+1}$ by

$$X \longmapsto \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}$$

and for $m > n$, set $\Phi_{mn} := \Phi_{m,m-1} \circ \cdots \Phi_{n+1,n}$. Show that $\{A_n, \Phi_{mn}\}_{\mathbb{N}}$ is a direct system of C^*-algebras and calculate $K_0(A)$, where A is the direct limit.

Exercise 7. Inspired by the previous exercise, construct a C^*-algebra B such that for any C^*-algebra A, we have $K_0(A \otimes B) = K_0(A) \otimes_{\mathbb{Z}} \mathbb{Q}$.