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Exercises C^*-algebras and K-theory
Sheet 4

Exercise 1. Let A be an algebra.

(a) Show that the multiplication $(a, \lambda) \cdot (b, \mu) := (ab + \lambda b + \mu a, \lambda\mu)$ turns the vector space $A \oplus \mathbb{C}$ into a unital algebra, denoted by A^+.

(b) Show that if A is unital, then the algebra A^+ from part (a) is isomorphic to $A \oplus \mathbb{C}$ *as an algebra.*

(c) Suppose additionally that A is a $*$-algebra. Show that the $*$-operation $(a, \lambda)^* := (a^*, \overline{\lambda})$ turns A^+ into a $*$-algebra.

(d) Suppose additionally that A is a C^*-algebra. Show that then A^+ is a C^*-algebra. In other words, there exists a unique complete norm on the $*$-algebra A^+ satisfying the C^*-identity.

Exercise 2. Prove the non-unital version of the Gelfand-Naimark theorem: For any commutative C^*-algebra A, there exists a locally compact Hausdorff space X and an isometric $*$-isomorphism $A \cong C_0(X)$.

Exercise 3. Let A be a C^*-algebra. On the vector space $M := A^n = A \oplus \cdots \oplus A$, define a sesquilinear form $M \times M \to A$ by

$$(a|b) := \sum_{i=1}^{n} a_i^* b_i.$$

(a) Show that for every $T \in \mathbb{B}(M)$, there exists at most one operator $S \in \mathbb{B}(M)$ such that $(Ta|b) = (a|Sb)$ for all $a, b \in M$. If such an S exists, T is called *adjointable*, and we write $S^* = T$.

(b) Show that the set $\mathbb{B}_{\mathrm{ad}}(M) \subseteq \mathbb{B}(M)$ of adjointable operators together with the $*$-operation $T \mapsto T^*$ is a C^*-algebra.

(c) For $a \in M_n(A)$, define an operator L_a by $L_a(b) = ab$ for $b \in M$. Show that L_a is adjointable with $\|L_a\| = \|a\|$.

(d) Show that the map $M_n(A) \to \mathbb{B}_{\mathrm{ad}}(M)$, $a \mapsto L_a$ is an injective $*$-homomorphism with closed image.

Exercise 4. Let $A = \mathbb{C}$ or $\mathbb{K}(H)$ for some Hilbert space H. Show that the map $\tau : K_0(A) \to \mathbb{Z}$ defined by

$$\tau([p] - [q]) = \mathrm{tr}(p) - \mathrm{tr}(q)$$

is a well-defined group isomorphism.

Exercise 5. Compute $K_0(\mathbb{B}(H))$, for H an infinite-dimensional Hillbert space.