Dr. Matthias Ludewig SS 2020
Universität Regensburg

Exercises C^*-algebras and K-theory
Sheet 3

Exercise 1. Let A be a C^*-algebra. Show that for all $a \in A$,

$$\|a\| = \sup_{\|b\|\leq 1} \|ab\| = \sup_{\|b\|\leq 1} \|ba\|.$$

Exercise 2. Let A be a C^*-algebra. For $T \in \mathbb{B}(A)$, define $T^* \in \mathbb{B}(A)$ by $T^*(a) := (T(a^*))^*$.

(a) Why is $\mathbb{B}(A)$ with the operation $T \mapsto T^*$ *not* a C^*-algebra?

For a C^*-algebra B, denote by B^{op} its *opposite algebra*, which is the algebra with the same underlying Banach space and $*$-operation, but multiplication $a \cdot_{\mathrm{op}} b := ba$.

(b) Show that $\mathbb{B}(A) \oplus \mathbb{B}(A)^{\mathrm{op}}$ is a $*$-algebra with $*$-operation $(T, S)^* := (S^*, T^*)$.

(c) $\mathbb{B}(A) \oplus \mathbb{B}(A)^{\mathrm{op}}$ is a Banach its natural norm $\|(T, S)\| := \max\{\|T\|, \|S\|\}$. Is it a C^*-algebra with the $*$-operation above?

Exercise 3 (The multiplier algebra)**.** Let A be a C^*-algebra. A *double centralizer* is a pair $(L, R) \in \mathbb{B}(A) \oplus \mathbb{B}(A)^{\mathrm{op}}$ such that for all $a, b \in A$,

$$L(ab) = L(a)b, \qquad R(ab) = aR(b), \qquad R(a)b = aL(b).$$

Denote by $\mathcal{M}(A) \subseteq \mathbb{B}(A) \oplus \mathbb{B}(A)^{\mathrm{op}}$ the set of double centralizers.

(a) Show that for all $(L, R) \in \mathcal{M}(A)$, one has $\|L\| = \|R\|$.

(b) Show that $\mathcal{M}(A)$ is a closed subalgebra of $\mathbb{B}(A) \oplus \mathbb{B}(A)^{\mathrm{op}}$, which is a $C*$-algebra with the $*$-operation defined in Exercise 2(b). It is called the *multiplier algebra*.

(c) Show that for any $a \in A$, the pair $(L_a, R_a) \in \mathbb{B}(A) \oplus \mathbb{B}(A)^{\mathrm{op}}$ with $L_a(b) = ab$ and $R_a(b) = ba$ for $b \in A$ is a double centralizer with $\|L_a\| = \|R_a\| = \|a\|$.

(d) Show that the map $\Phi : A \to \mathcal{M}(A)$, $a \mapsto (L_a, R_a)$ is an isometric $*$-homomorphism.

(e) Show that the image of Φ is an ideal in $\mathcal{M}(A)$.

Exercise 4. Let X be a locally compact Hausdorff space. Show that $\mathcal{M}(C_0(X))$ is canonically isomorphic to $C_b(X)$, the algebra of bounded complex-valued functions on X.

Exercise 5. Let H be a Hilbert space. Show that the multiplier algebra $\mathcal{M}(\mathbb{K}(H))$ of the algebra of compact operators on H is canonically isomorphic to $\mathbb{B}(H)$, the algebra of bounded operators on H.