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Exercises C^*-algebras and K-theory
Sheet 2

Let G be a countable group. For $1 \leq p < \infty$, define the Banach spaces

$$\ell^p(G) := \left\{ \alpha : G \to \mathbb{C} \mid \|\alpha\|_p := \left(\sum_{g \in G} |\alpha(g)|^p \right)^{1/p} < \infty \right\}.$$

For suitable functions $\alpha_1, \alpha_2 : G \to \mathbb{C}$, define their *convolution* $\alpha_1 * \alpha_2$ by

$$(\alpha_1 * \alpha_2)(g) := \sum_{h \in G} \alpha_1(h)\alpha_2(h^{-1}g).$$

For $\alpha_1 \in \ell^p(G)$ and $\alpha_2 \in \ell^q(G)$ with $\frac{1}{p} + \frac{1}{q} \geq 1$, *Young's inequality* states that $\alpha_1 * \alpha_2 \in \ell^r(G)$, with

$$\|\alpha_1 * \alpha_2\|_r \leq \|\alpha_1\|_p \|\alpha_2\|_q, \qquad \text{where} \qquad \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1.$$

In particular, for $p = q = r = 1$, we get that the convolution product turns $\ell^1(G)$ into a Banach algebra.

Exercise 1. Define

$$\alpha^*(g) := \overline{\alpha(g^{-1})}.$$

Show that this turns $\ell^1(G)$ into a $*$-algebra. Does this operation turn $\ell^1(G)$ into a C^*-algebra?

Exercise 2. We consider the case $G = \mathbb{Z}$. For $z \in \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$, define

$$\varphi_z : \ell^1(\mathbb{Z}) \longrightarrow \mathbb{C}, \qquad \varphi_z(\alpha) := \sum_{n \in \mathbb{Z}} \alpha(n) z^n.$$

Show the following.

(a) φ_z is a homomorphism.

(b) All homomorphisms $\varphi : \ell^1(\mathbb{Z}) \to \mathbb{C}$ are of this form, i.e. we have $\varphi = \varphi_z$ for some $z \in \mathbb{T}$.

(c) Show that the map $\mathbb{T} \to \Gamma_{\ell^1(\mathbb{Z})}$, $z \mapsto \varphi_z$ is a homeomorphism, so that the Gelfand transform sends $\ell^1(\mathbb{Z})$ to $C(\mathbb{T})$ under this identification.

(d) Observe that under the "obvious' identification of $C(\mathbb{T})$ with 2π-periodic functions on $\mathbb{R}$, the Gelfand transform is just the inverse of the discrete Fourier transform. Show that under this identification, the homorphism φ_z corresponds to the evaluation map

$$e_z : C(\mathbb{T}) \longrightarrow \mathbb{C}, \qquad f \longmapsto f(z).$$

Another special case of Young's inequality is the inequality $\|f_1 * f_2\|_2 \leq \|f_1\|_1 \|f_2\|_2$. This means that multiplication by elements of $\ell^1(G)$ defines an action on $\ell^2(G)$; in other words, we obtain an algebra homomorphism

$$\Phi : \ell^1(G) \longrightarrow \mathbb{B}(\ell^2(G)).$$

Exercise 3. Show that Φ is an injective $*$-homomorphism, which is *not* isometric.

The closure of $\Phi(\ell^1(G)) \subset \mathbb{B}(\ell^2(G))$ with respect to the operator norm is called the *reduced group C^*-algebra of* G, denoted by $C_r^*(G)$.

Exercise 4. Write $\delta_e \in \ell^2(G)$ for the function such that $\delta_e(e) = 1$ and $\delta_e(g) = 0$ for all $g \neq e$ (here e is the unit in G). Let $F : C_r^*(G) \to \ell^2(G)$ be the map sending $A \in C_r^*(G) \subseteq \mathbb{B}(\ell^2(G))$ to $A(\delta_e) \in \ell^2(G)$.

(a) Via Φ, we can view $\ell^1(G)$ as a subset of $C_r^*(G)$. Show that under this identification, F restricted to $\ell^1(G)$ is just the inclusion of $\ell^1(G)$ into $\ell^2(G)$.

(b) Show that F is bounded and injective.

This shows that elements of $C_r^*(G)$ can still be viewed as functions $\alpha : G \to \mathbb{C}$.

Exercise 5. Consider again the case $G = \mathbb{Z}$. Show that for each $z \in \mathbb{T}$, the homomorphism $\varphi_z : \ell^1(\mathbb{Z}) \to \mathbb{C}$ extends by continuity to a homomorphism $\varphi_z : C_r^*(\mathbb{Z}) \to \mathbb{C}$, and that all $\varphi \in \Gamma_{C_r^*(\mathbb{Z})}$ are of this form. Show that the Gelfand transform $C_r^*(\mathbb{Z}) \to C(\mathbb{T})$ is an isometric isomorphism.