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Exercises C^*-algebras and K-theory
Sheet 10

Exercise 1. Let A be a C^*-algebra.

(a) Show that elements $x, y \in K_1(A)$ coincide if and only if there exists $n \in \mathbb{N}$ and a homotopy $(u_t)_{t\in[0,1]}$ of unitaries $u_t \in \mathcal{U}_n^+(A)$ with $x = [u_0]$ and $y = [u_1]$.

(b) Show that $K_1(A) = \lim_{n\to\infty} \mathcal{U}_n^+(A)/U_n^+(A)_0$.

(c) Show that $K_1(A) \cong \mathcal{U}^+(A \otimes \mathbb{K})/\mathcal{U}^+(A \otimes \mathbb{K})_0$.

Exercise 2. Let A be a C^*-algebra. Let $m \geq n$ and let $w \in M_{n+m}(A)$ be unitary such that

$$w \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} w^* = \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} \in M_{n+m}(A).$$

Then there exist unitaries $u \in M_n(A)$, $v \in M_m(A)$, such that $w = \operatorname{diag}(u, v)$.

Exercise 3. Consider the C^*-algebra

$$DD_n = \{f \in C([0,1], M_n(\mathbb{C})) \mid f(0) = \lambda\mathbf{1}_n, f(1) = \mu\mathbf{1}_n \text{ for some } \lambda, \mu \in \mathbb{C}\}.$$

Show that the class $[\mathbf{1}_n] \in K_1(DD_n)$ is torsion.
Hint: Consider the element $u \in \mathcal{U}(DD_n)$ given by $u(t) = \operatorname{diag}(e^{2\pi i t}, 1, \ldots, 1)$.

Exercise 4. Compute $K_i(C_0(\mathbb{R}^n))$ and $K_i(C(S^n))$ for $i = 0, 1$, $n \in \mathbb{N}$.

Exercise 5. The *Cuntz algebra* is the C^* algebra $\mathcal{O}_n$ generated by partial isometries $S_1, \ldots, S_n$ satisfying $S_i^* S_i = \mathbf{1}$ for each $i = 1, \ldots, n$ and

$$\sum_{i=1}^n S_i S_i^* = \mathbf{1}.$$

(a) Give a concrete realization of $\mathcal{O}_n$, that is, find a Hilbert space H and isometries $S_1, \ldots, S_n \in \mathbb{B}(H)$ satisfying the above relations. (One can show that up to isomorphism, $\mathcal{O}_n$ is independent from the concrete realization).

(b) Show that the class $[\mathbf{1}] \in K_0(\mathcal{O}_n)$ is torsion.