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Universität Regensburg

Exercises C^*-algebras and K-theory
Sheet 1

Exercise 1. Let A be a C^*-algebra. Show that $\|a^*\| = \|a\|$ and that $e^* = e$ in the case that A is unital.

Exercise 2. Let A, B be unital C^*-algebras. Show that any unital homomorphism $\Phi : A \to B$ (not assumed to be bounded) is in fact contractive, i.e. $\|\Phi(a)\| \leq \|a\|$ for all $a \in A$.

Exercise 3. Let A be a C^*-algebra.

(1) An element $p \in A$ is called *projection* if $p = p^2 = p^*$. Show that if $p \in A$ is a projection and A is unital, then $\sigma(p) \subseteq \{0, 1\}$.

(2) If A is unital, an element $u \in A$ is called *unitary* if $u^*u = uu^* = \mathbf{1}$. Show that if $u \in A$ is unitary, then $\sigma(u) \subseteq \mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$.

(3) An element $v \in A$ is called *partial isometry* if v^*v is a projection. Show that if $v \in A$ is a partial isometry, then we have the identities

$$v = vv^*v, \qquad v^* = v^*vv^*$$

and that v^* is a partial isometry is well.

Exercise 4. Let A be a unital C^*-algebra and let $a \in A$ be self-adjoint. Show that $\sigma(a) \subset \mathbb{R}$.

Hint: If $\mu \in \sigma(a)$, then also $\mu + i\lambda \in \sigma(a + i\lambda)$ for all $\lambda \in \mathbb{R}$. Observe now that $a + i\lambda$ is normal, so that $|\mu + i\lambda| \leq \|a + i\lambda\|$ for all $\lambda \in \mathbb{R}$.

***Exercise 5.** Review the proofs of the basic results of spectral theory for $\mathbb{B}(H)$ (see e.g. Werner, *Funktionalanalysis*, VI.1.3. & VI.1.7) and convince yourself that the proof carries over immediately to general Banach algebras, respectively C^*-algebras.