THE GROTHENDIECK TEICHMÜLLER GROUP
(HIOB SEMINAR WS 2015-16)

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• Talk 1: Introduction

GT and the absolute Galois group

• Talk 2: The étale fundamental group. Recall the definition of an étale morphism. Define the étale fundamental group \([10, \S\S 3,5]\). State Riemann’s existence theorem \([12, \S\S 1.5.1]\) and deduce the comparison theorem relating the étale fundamental group and the topological fundamental group \([12, \S\S 1.5.2]\). Deduce the isomorphism \(\hat{\mathbb{F}}_2 \cong \pi_1^{et}(\mathbb{P}^1_\mathbb{Q} \setminus \{0,1,\infty\})\).

• Talk 3: Curves and function fields. Recall the equivalence between branched covers of normal proper curves and extensions of function fields. See \([13, \S\S 4.4–4.6]\), in particular Prop. 4.6.1, as well as \([4, \S\S 6]\).

Establish the short exact sequence of profinite groups \([13, \text{Prop. 4.7.1}]\) (*).

\[ 1 \longrightarrow \pi_1^{et}X_k \longrightarrow \pi_1^{et}X \longrightarrow \text{Gal}(k/k) \longrightarrow 1. \]

• Talk 4: Belyi’s Theorem I. State Belyi’s Theorem and prove the “if” part \([13, \text{see Remark 5.7.8}]\).

• Talk 5: Belyi’s Theorem II. Give the proof of the “only if” part of Belyi’s theorem. Deduce that the monodromy action \(\text{Gal}(\mathbb{Q}/\mathbb{Q}) \longrightarrow \text{Out}(\pi_1^{et}(\mathbb{P}^1_\mathbb{Q} \setminus \{0,1,\infty\}))\) associated to the short exact sequence (*) is injective \([13, \text{4.7.7 and 4.7.8}]\).

• Talk 6: Lifting the action of \(\text{Gal}(\mathbb{Q}/\mathbb{Q})\). The goal of this talk is to lift the monodromy action to give a faithful action of the absolute Galois group on \(\hat{\mathbb{F}}_2 \cong \pi_1^{et}(\mathbb{P}^1_\mathbb{Q} \setminus \{0,1,\infty\})\). Give the construction of Ihara’s homomorphism

\[ \text{Gal}(\mathbb{Q}/\mathbb{Q}) \longrightarrow \hat{\mathbb{Z}}^\times \times \hat{\mathbb{F}}_2' \]

\[ \sigma \longrightarrow (\chi(\sigma), f_\sigma) \]

from \([11, \S\S 3.1]\). Following \([8, \S 1]\), deduce that the monodromy action lifts to give an injective homomorphism \((\dagger)\)

\[ \text{Gal}(\mathbb{Q}/\mathbb{Q}) \longrightarrow \text{Aut}(\hat{\mathbb{F}}_2). \]

• Talk 7: The monomorphism \(\text{Gal}(\mathbb{Q}/\mathbb{Q}) \longrightarrow \hat{\text{GT}}\). The aim of this talk is to identify the image of the homomorphism \((\dagger)\) with the profinite Grothendieck-Teichmüller group \(\hat{\text{GT}}\). This consists in verifying relations (I), (II), and (III) for pairs \((\chi(\sigma), f_\sigma) \in \hat{\mathbb{Z}}^\times \times \hat{\mathbb{F}}_2'\). Sketch the proof that these relations are satisfied, following \([11, \S\S 3]\) and \([8]\). See also the appendix to \([8]\) as well as \([7]\).

THE ACTION OF GT ON THE TEICHMÜLLER TOWER

• Talk 8: Moduli spaces, the Teichmüller tower, and braid groups. Introduce the Teichmüller space \(\mathcal{T}_{g,n}\), the mapping class group \(\Gamma_{g,n}\) and the moduli space \(\mathcal{M}_{g,n} = \mathcal{T}_{g,n}/\Gamma_{g,n}\). Explain what it means that \(\Gamma_{g,n}\) is the “orbifold fundamental group” of \(\mathcal{M}_{g,n}\). Introduce the braid groups \(B_n\), the pure braid groups \(K_n\), the sphere braid groups \(H_n\), and their relation with \(\Gamma_{0,n}\) \([2]\). In the remaining time, sketch the proof of Proposition A3 in the appendix of \([9]\). You should coordinate closely with (or be) the speaker for talk 9.

• Talk 9: \(\hat{\text{GT}}\) and braid groups. Recall the action of \((\lambda, f) \in \hat{\mathbb{Z}}^\times \times \hat{\mathbb{F}}_2'\) on \(\hat{\mathbb{F}}_2\) from Talk 1. Show that the action extends to an automorphism of \(\hat{\mathbb{B}}_3\) if and only if relations (I) and (II) are satisfied, then show that
the action extends to an automorphism of the profinite completion $\hat{M}(0,n)$ of the mapping class group if and only if relations (I) – (III) are satisfied. See [11, Lemmas 1 and 2] for an overview and [9, §3–§4] for full proofs. Reformulate these results in terms of the braid towers $\hat{T}_n$ and prove the profinite case of the main theorem in [9] (see p. 12). The ideas here are in [1] and [3] as well.

**GT and the little discs operad**

- **Talk 10: Operads, discs, braids.** Review the definition of an operad in a symmetric monoidal category. Define the operad of paranthesized braids $\mathcal{PaB}$ in groupoids and explain its relationship with braided monoidal categories (see the paper [1] for a nice discussion of $\mathcal{PaB}$). Define the little $n$-discs operad $E_n$ and show that there is an equivalence of operads in spaces $B\mathcal{PaB} \simeq E_2$ (there is a sketch of this fact in [14, §3.2]). References: [1] [5], [6], [14].

- **Talk 11: Profinite completion.** Introduce the category $\hat{S}$ of profinite spaces and the category $\hat{G}$ of profinite groupoids. Define Quick’s model structure on $\hat{S}$ [15, 16] and Horel’s model structure on $\hat{G}$ [6]. Show that in both instances the profinite completion functor is a left Quillen functor. Discuss the profinite classifying space functor $B: \hat{G} \to \hat{S}$ and show that it is a Quillen right adjoint.

- **Talk 12: $\hat{GT}$ and the braid groups, revisited.** Prove that $\hat{GT}$ is the group of automorphisms of the profinite completion $\mathcal{PaB}$ of the operad of paranthesized braids. This is a direct continuation of Talk 9. There is no reference in the literature for this proof—one approach is to explain enough about the operad $\mathcal{PaB}$ so that the result follows directly from the main theorem in [9]. Another is to prove the theorem directly along the lines of [3].

- **Talk 13: Weak operads.** Review the notion of an algebraic theory. Give the definition of a weak operad and explain its relationship with braided operads in both spaces and in groupoids. Explain the equivalence between weak operads and operads. Explain why $\mathcal{PaB}$ is a cofibrant approximation of $\mathcal{PaB}$. Reference: [6].

- **Talk 14: $\hat{GT} \cong \text{hAut}(E_2)$.** Prove the main theorem in [6].

**References**


